SURJECTIVE ENDOMORPHISMS OF PROJECTIVE SURFACES – THE EXISTENCE OF INFINITELY MANY DENSE ORBITS

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Abstract. Let $f\colon X \to X$ be a surjective endomorphism of a normal projective surface. When $\deg f \geq 2$, applying an (iteration of) f-equivariant minimal model program (EMMP), we determine the geometric structure of X. Using this, we extend the second author's result to singular surfaces to the extent that either X has an f-invariant non-constant rational function, or f has infinitely many (disjoint) Zariski-dense forward orbits; this result is also extended to adelic topology (which is finer than Zariski topology).

Contents

1. Introduction

We work over an algebraically closed field $\mathbf{k}$ of characteristic zero. We first give a structure theorem for non-isomorphic surjective endomorphisms.

Theorem 1.1. *Let $f\colon X \to X$ be a non-isomorphic surjective endomorphism of a normal projective surface. Then X has only log canonical (lc) singularities. If the canonical divisor K_X is pseudo-effective, then Case 1.1(1) occurs. If K_X is not pseudo-effective, replacing f by an iteration, we may run an f-equivariant minimal model program (EMMP)*

$$X = X_1 \longrightarrow \cdots \longrightarrow X_j \longrightarrow \cdots \longrightarrow X_r \longrightarrow Y,$$

contracting K_{X_j}-negative extremal rays, with $X_j \to X_{j+1}$ $(j < r)$ being divisorial and $X_r \to Y$ being Fano contraction, such that one of the following cases occurs.

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(1) f *is quasi-étale, i.e., étale in codimension* 1*; there exists an* f*-equivariant quasi-étale finite Galois cover* $\nu\colon V \to X$ *as in Theorem 2.14.*

(2) Y *is a smooth projective curve of genus* $g(Y) \geq 1$*;* f *descends to an automorphism of finite order on the curve* Y*.*

(3) Y *is an elliptic curve;* $X \to Y$ *is a* $\mathbb{P}^1$*-bundle.*

(4) $Y \cong \mathbb{P}^1$*; there is an* $f|_{X_r}$*-equivariant finite surjective morphism* $X_r \to Y \times \mathbb{P}^1$*.*

(5) $Y \cong \mathbb{P}^1$*;* f *is polarized;* $K_X + S \sim_{\mathbb{Q}} 0$ *for an* f^{-1}*-stable reduced divisor* S*; there is an* f*-equivariant quasi-étale finite Galois cover* $\nu\colon V \to X$ *as in Theorem 2.13(2).*

(6) $Y \cong \mathbb{P}^1$*;* f *is polarized; there exists an equivariant commutative diagram:*

$$\begin{array}{ccc} \widetilde{f} \circlearrowright \widetilde{X} & \xrightarrow{\mu_X} & X \circlearrowleft f \\ {\scriptstyle \widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ g_E \circlearrowright E & \longrightarrow & Y \circlearrowleft g \end{array}$$

here E *is an elliptic curve;* $\widetilde{X}$ *is the normalisation of* $X \times_Y E$*;* $\widetilde{\pi}$ *is a* $\mathbb{P}^1$*-bundle;* π *is a* $\mathbb{P}^1$*-fibration;* $\widetilde{f}$ *and* g_E *are finite surjective endomorphisms;* μ_X *is quasi-étale.*

(7) $Y \cong \mathbb{P}^1$*;* X_r *is of Fano type; there is an* $f|_{X_r}$*-equivariant birational morphism* $X_r \to \overline{X}$ *to a klt Fano surface with Picard number* $\rho(\overline{X}) = 1$*; every* X_j $(1 \leq j \leq r)$ *is a rational surface whose singularities are klt (hence* $\mathbb{Q}$*-factorial).*

(8) Y *is a point;* X_r *is a projective cone over an elliptic curve* E*; the normalisation* Γ *of the graph of* $X \dashrightarrow E$ *is a* $\mathbb{P}^1$*-bundle, and* f *lifts to* $f|_\Gamma$ *such that* $(f|_\Gamma)^*|_{\mathrm{N}^1(\Gamma)} = \delta_f \,\mathrm{id}$*.*

(9) Y *is a point and hence* $\rho(X_r) = 1$*;* $-K_{X_r}$ *is ample; every* X_j $(1 \leq j \leq r)$ *is a rational surface whose singularities are lc and rational (hence* $\mathbb{Q}$*-factorial).*

Remark 1.2. By a result of Nakayama [Nak20b, Theorem 1.1], all the cases (1), (5) and (8) occur. Note that in case (8) above, the surface X_r is singular. Taking X to be the product of $\mathbb{P}^1$ with a curve Y as in cases (2) – (4) and (6), and a suitable non-isomorphic endomorphism f on X, we see that cases (2) – (4) and (6) are possible. A Hirzebruch surface $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ with $n \geq 1$ is an example of case (7) (cf. [Nak02, Theorem 3]).

Consider the weighted projective space $X = \mathbb{P}(a_0, a_1, a_2)$ with coordinate functions x_0, x_1, x_2 and the endomorphism $f\colon X \to X$ defined by

$$(x_0 : x_1 : x_2) \longmapsto (F_0(x_0, x_1, x_2) : F_1(x_0, x_1, x_2) : F_2(x_0, x_1, x_2))$$

where F_j's are homogeneous polynomials with respect to the weights such that $(\deg F_0 : \deg F_1 : \deg F_2) = (a_0 : a_1 : a_2)$ and F_j's have no common zero. When $\deg F_j \geq 2$ for some $0 \leq j \leq 2$, the pair (X, f) gives an instance of case (9) above, for example, taking the singular surface $X = \mathbb{P}(1, 2, 3)$ and $f(x_0 : x_1 : x_2) = (x_0^2 : x_1^2 : x_2^2)$.

The main ingredients of the proof for Theorem 1.1 are an equivariant minimal model program, [Nak20, Theorem A] and [Nak20b, Theorem 3.11] (= Theorems 2.13 and 2.14), and some analysis of Fano fibration in [MZ, Theorem 5.4]. Theorem 1.1 implies:

Corollary 1.3. *Let $f: X \to X$ be a non-isomorphic surjective endomorphism of a normal projective surface. Then, replacing f by an iteration, one of the following holds.*

(1) X has an f-invariant non-constant rational function.
(2) f lifts to an endomorphism on X' via a generically finite surjective morphism, where X' is an abelian surface or a $\mathbb{P}^1$-bundle over an elliptic curve.
(3) f descends to an endomorphism on $\mathbb{P}^1 \times \mathbb{P}^1$ (commuting with both projections) via a generically finite surjective morphism.
(4) f descends to an endomorphism on a normal projective surface $\overline{X}$ via a birational morphism such that
  (a) The Picard number $\rho(\overline{X}) = 1$, so $f|_{\overline{X}}$ and hence f are polarized;
  (b) The anti-canonical divisor $-K_{\overline{X}}$ is an ample $\mathbb{Q}$-Cartier divisor; and
  (c) $\overline{X}$ is a rational surface whose singularities are log canonical (lc) and rational.

Below is the motivation of our paper where Conjecture 1.4(2) is invariant under birational conjugation which is stronger than the long-standing Conjecture 1.4(1). In fact, Conjecture 1.4(1) and (2) are equivalent modulo the Dynamical Mordell-Lang Conjecture as shown in [Xie22, Proposition 2.6] and also reminded by Professor Ghioca.

Conjecture 1.4. *Let X be an (irreducible) projective variety over $\mathbf{k}$ and $f: X \dashrightarrow X$ a dominant rational self-map such that $\mathbf{k}(X)^f = \mathbf{k}$. Then:*

(1) there is a point $x \in X(\mathbf{k})$ such that the (forward) orbit $\mathcal{O}_f(x) := \{f^s(x) : s \geq 0\}$ is well-defined, i.e., f is defined at $f^n(x)$ for any $n \geq 0$, and Zariski-dense in X.
(2) for every Zariski-dense open subset U of X, there exists a point $x \in X(\mathbf{k})$ whose orbit $O_f(x)$ under f is well-defined, contained in U and Zariski-dense in X.

To wit, a rational function $\psi \in \mathbf{k}(X)$ is *f-invariant* if $f^*(\psi) := \psi \circ f = \psi$. Denote by $\mathbf{k}(X)^f$ the field of f-invariant rational functions on X. We have

$$\mathbf{k} \subseteq \mathbf{k}(X)^f \subseteq \mathbf{k}(X).$$

We will extend the classical Conjecture 1.4 (Zariski-topology version) to a stronger Conjecture 1.6 (adelic-topology version). We begin with:

1.5 (Adelic topology). Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. In [Xie22, § 3], the second author has proposed the adelic topology on $X(\mathbf{k})$. The adelic topology has the following basic properties (cf. [Xie22, Proposition 3.16]).

(1) It is stronger than the Zariski topology.
(2) It is T_1, i.e., for any two distinct points $x, y \in X(\mathbf{k})$ there are adelic open subsets U, V of $X(\mathbf{k})$ such that $x \in U, y \notin U$ and $y \in V, x \notin V$.
(3) Morphisms between algebraic varieties over $\mathbf{k}$ are continuous for the adelic topology.
(4) Flat morphisms are open with respect to the adelic topology.
(5) The irreducible components of $X(\mathbf{k})$ in the Zariski topology are the irreducible components of $X(\mathbf{k})$ in the adelic topology.
(6) Let K be any subfield of $\mathbf{k}$ which is finitely generated over $\mathbb{Q}$ and such that X is defined over K and $\overline{K} = \mathbf{k}$. Then the action

$$\mathrm{Gal}(\mathbf{k}/K) \times X(\mathbf{k}) \to X(\mathbf{k})$$

is continuous with respect to the adelic topology.

When X is irreducible, (5) implies that the intersection of finitely many nonempty adelic open subsets of $X(\mathbf{k})$ is nonempty. So, if $\dim X \geq 1$, the adelic topology is not Hausdorff. In general, the adelic topology is strictly stronger than the Zariski topology.

The adelic version of the Zariski-dense orbit conjecture was proposed in [Xie22].

Conjecture 1.6. *Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let X be an irreducible variety over $\mathbf{k}$ and $f: X \dashrightarrow X$ a dominant rational map. If $\mathbf{k}(X)^f = \mathbf{k}$, then there exists a nonempty adelic open subset $A \subseteq X(\mathbf{k})$ such that the orbit of every point $x \in A$ is well-defined and Zariski-dense in X.*

Definition 1.7. Let X be an (irreducible) projective variety over $\mathbf{k}$ and $f: X \dashrightarrow X$ a dominant rational map. We say that a pair (X, f) satisfies

(1) *ZD-property*, if Conjecture 1.4(1) holds true;
(2) *strong ZD-property*, if Conjecture 1.4(2) holds true;
(3) *AZD-property*, if Conjecture 1.6 holds true; and
(4) *SAZD-property*, if there is a nonempty adelic open subset A of $X(\mathbf{k})$ such that for every point $x \in A$, its orbit $O_f(x)$ under f is well-defined and Zariski-dense in X.

Remark 1.8. Conjecture 1.6 implies Conjecture 1.4. Precisely, we have:

(1) SAZD-property implies AZD-property.

(2) Conjecture 1.6 (adelic-topology version) is stronger than the classical Conjecture 1.4 (Zariski-topology version). Indeed, even the hypothesis on $\mathbf{k}$ in Conjecture 1.6 does not cause any problem. To be precise, for every pair (X, f) over $\mathbf{k}$, there exists an algebraically closed subfield K of $\mathbf{k}$ whose transcendence degree over $\mathbb{Q}$ is finite and such that (X, f) is defined over K, i.e., there exists a pair (X_K, f_K) such that (X, f) is its base change by $\mathbf{k}$. By [Xie22, Corollary 3.29], if (X_K, f_K) satisfies AZD-property, then (X, f) satisfies strong ZD-property.

We will prove Conjecture 1.6 (and hence Conjecture 1.4) for surjective endomorphisms of (possibly singular) projective surfaces, extending the smooth case in [Xie22].

Theorem 1.9. *Let $f\colon X \to X$ be a surjective endomorphism of a projective surface X defined over the algebraically closed field $\mathbf{k}$. Assume that $\mathbf{k}$ has finite transcendence degree over $\mathbb{Q}$. Then Conjecture 1.6 holds for (X, f). Precisely, either $\mathbf{k}(X)^f \neq \mathbf{k}$; or there is a nonempty adelic open subset $A \subseteq X(\mathbf{k})$ such that the forward orbit $O_f(x)$ of every point $x \in A$ is Zariski-dense in X.*

Proposition 1.10. *Assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Let X be an irreducible variety over $\mathbf{k}$ of positive dimension and $f\colon X \dashrightarrow X$ a dominant rational map. If (X, f) has SAZD-property, then f has infinitely many Zariski-dense orbits which are disjoint.*

Theorem 1.9 and Proposition 1.10 imply the following (cf. Remark 1.8).

Theorem 1.11. *Let $f\colon X \to X$ be a surjective endomorphism of a projective surface X defined over the algebraically closed field $\mathbf{k}$. Then Conjecture 1.4 holds for (X, f). Precisely, either $\mathbf{k}(X)^f \neq \mathbf{k}$; or for every Zariski-dense open subset U of X, there is a point $x \in X(\mathbf{k})$ whose forward orbit $O_f(x)$ under f is contained in U and Zariski-dense in X. In the latter case, f has infinitely many Zariski-dense orbits which are disjoint.*

Ingredient of the proof. Due to the possible occurrence of Case (4) in Corollary 1.3, we cannot always reduce Conjecture 1.6 to the smooth case. Therefore, the following result for amplified endomorphisms is indispensable when proving Theorem 1.9. Since our X might be singular, extra care is taken in the last section in proving it, extending the smooth case in [Xie22].

Theorem 1.12. *Let X be a projective surface over $\mathbf{k}$. Let $f\colon X \to X$ be an amplified endomorphism such that $\deg f > \delta_f$. Then the pair (X, f) satisfies SAZD-property.*

In particular, if f is int-amplified or polarized, the pair (X, f) satisfies SAZD-property.

Remark 1.13. Below are some histories of Conjecture 1.4(1) (Zariski-topology version). It was proved by Amerik and Campana [AC08, Theorem 4.1] when the field $\mathbf{k}$ is uncountable. If $\mathbf{k}$ is countable, it has been open even for the case of singular surfaces (with f a well-defined morphism). Below are some confirmed cases.

(1) In [Ame11, Corollary 9], in arbitrary dimension, Amerik proved the existence of non-preperiodic algebraic point when f is of infinite order.
(2) Conjecture 1.6 and hence Conjecture 1.4 are proved for $f = (f_1, \cdots, f_n) \colon (\mathbb{P}^1)^n \to (\mathbb{P}^1)^n$, where the f_i's are endomorphisms of $\mathbb{P}^1$, in [Xie22, Appendix B, joint work with Thomas Tucker]; see also [BGT16, Theorem 14.3.4.2], when f_i's are not post-critically finite, and Medvedev and Scanlon [MS14, Theorem 7.16] for endomorphism $f \colon \mathbb{A}^n \to \mathbb{A}^n$ with $f(x_1, \cdots, x_n) = (f_1(x_1), \cdots, f_n(x_n))$, $f_i(x_i) \in \mathbf{k}[x_i]$.
(3) In [Xie17, Theorem 1.1], the second author has proved Conjecture 1.4(1) for dominant polynomial endomorphism $f \colon \mathbb{A}^2 \to \mathbb{A}^2$.
(4) If X is a (semi-) abelian variety and f is a dominant self-map, Conjecture 1.4(1) has been proved in [GS17, Theorem 1.2] and [GS19, Theorem 1.1]; in this abelian variety case, Conjecture 1.6 is proved in [Xie22, Theorem 1.14].
(5) When X is an algebraic surface and f is a birational self-map, Conjecture 1.6 (and hence Conjecture 1.4) has been proved in [Xie22, Corollary 3.31]; see also [BGT14, Theorem 1.3] when X is quasi-projective over $\overline{\mathbb{Q}}$ and f is an automorphism.
(6) Conjecture 1.6 and hence Conjecture 1.4 have been proved when X is a *smooth* projective surface and f is a surjective endomorphism in [Xie22, Theorem 1.15].

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2. Preliminaries

In this section, we collect together some definitions and preliminary results.

Notation and Terminology. We use the following notation throughout the paper, with X a projective variety.

$\operatorname{Pic} X$	the group of Cartier divisors of X modulo linear equivalence $\sim$
$\operatorname{Pic}^0 X$	the group of Cartier divisors of X algebraically equivalent to 0
$\operatorname{NS}(X)$	$\operatorname{Pic} X/\operatorname{Pic}^0 X$, the Néron-Severi group
$\mathrm{N}^1(X)$	$\operatorname{NS}(X)\otimes_{\mathbb{Z}}\mathbb{R}$, the space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence $\equiv$
$\mathrm{N}_1(X)$	the space of 1-cycles with coefficients in $\mathbb{R}$, modulo numerical equivalence $\equiv$
$\operatorname{Nef}(X)$	the cone of nef classes in $\mathrm{N}^1(X)$
$\operatorname{NE}(X)$	the cone of pseudo-effective classes in $\mathrm{N}_1(X)$
$\kappa(X,D)$	the Iitaka D-dimension of a $\mathbb{Q}$-Cartier divisor D (cf. [Iit82, §10.1])
$\rho(X)$	Picard number of X, which is $\dim_{\mathbb{R}}\mathrm{N}^1(X)$
$q(X)$	the irregularity of X, which is $h^1(X,\mathcal{O}_X):=\dim H^1(X,\mathcal{O}_X)$
$\operatorname{Supp} D$	the support of an effective Weil divisor $D=\sum a_iD_i$ on X, which is $\cup_i D_i$, where $a_i>0$ and D_i's are prime divisors
$f\vert_Y$	the lifted (resp. descended) endomorphism on Y of an endomorphism f on X via an equivariant morphism $Y\to X$ (resp. $X\to Y$).

An algebraic variety X is called $\mathbb{Q}$-*factorial* if every Weil $\mathbb{Q}$-divisor is $\mathbb{Q}$-Cartier.

An algebraic variety X is said to have *rational singularities* if X is normal and if $R^i\pi_*\mathcal{O}_Y=0$ $(i\ge 1)$ for one (and hence every) resolution $\pi\colon Y\to X$ of singularities.

A *pair* (X,Δ) consists of a normal variety X and an effective Weil $\mathbb{R}$-divisor $\Delta=\sum b_iD_i$ such that $K_X+\Delta$ is $\mathbb{R}$-Cartier. Let $\pi\colon Y\to X$ be a log resolution of the pair (X,Δ). There are uniquely defined numbers $a(E_j,X,\Delta)$, called the *discrepancy* of E_j with respect to (X,Δ), such that

$$K_Y+\pi_*^{-1}\Delta=\pi^*(K_X+\Delta)+\sum_{E_j:\text{exceptional}}a(E_j,X,\Delta)E_j.$$

We say that the pair (X,Δ) is *Kawamata log terminal* (klt) (resp. log canonical (lc)) if all $b_i<1$ (resp. ≤ 1), and, for one (and hence every) log resolution $\pi\colon Y\to X$ of (X,Δ), we have $a(E_j,X,\Delta)>-1$ (resp. ≥ -1) for every π-exceptional prime divisor E_j (cf. [KM98, Definition 2.34, Corollary 2.32]). We say that X is klt or lc if so is $(X,0)$.

Let $f\colon X\to X$ be a surjective endomorphism of a normal projective variety. Then f is finite and $f_*f^*=(\deg f)\,\mathrm{id}$ on $\operatorname{Pic}_{\mathbb{Q}}X$ (cf. [MZ, Proposition 3.7]).

Let $\pi\colon X_1\to X_2$ be a finite surjective morphism between normal varieties, which automatically restricts to a finite morphism $\pi':=\pi|_{X_1'}\colon X_1'\to X_2'$ between smooth varieties such that $X_i \setminus X_i'$ is a closed subvariety of X_i of codimension ≥ 2. For a Weil divisor D on X_2, we define π^*D as the Zariski closure of $(\pi')^*(D|_{X_2'})$ in X_1.

Denote by R_π the ramification divisor of π. By definition, it is an effective Weil divisor and satisfies the *ramification divisor formula:*

$$K_{X_1} = \pi^* K_{X_2} + R_\pi. \tag{2.1}$$

We say that π is *quasi-étale* if it is étale in codimension 1, i.e., if $R_\pi = 0$.

More generally, suppose $\pi^{-1}(D_2) = D_1$ for reduced effective Weil divisors $D_j \subset X_j$. Then we have the *logarithmic ramification divisor formula*

$$K_{X_1} + D_1 = \pi^*(K_{X_2} + D_2) + R'_\pi, \tag{2.2}$$

where the *log ramification divisor* R'_π is an effective Weil divisor having no common irreducible component with D_1 (cf. [Iit82, §11.4]). If $\pi^*D_2 = qD_1$, then $R_\pi = (q-1)D_1 + R'_\pi$.

Let X be a normal projective surface which is $\mathbb{Q}$-factorial, and C an irreducible curve on X. Then C is called a *negative curve* if the self-intersection $C^2 < 0$.

A surjective morphism $f\colon X \to Y$ of algebraic varieties is a *fibration* if f has connected fibres. Such an f is a $\mathbb{P}^1$-*fibration* if the general fibre of f is isomorphic to $\mathbb{P}^1$. A *cross-section* of a fibration $f\colon X \to Y$ is an irreducible subvariety $C \subset X$ such that the restriction $f|_C\colon C \to Y$ is an isomorphism onto Y.

Definition 2.1. Let X be a normal projective variety. Then X is of *Fano type* if there is an effective Weil $\mathbb{Q}$-divisor Δ such that (X, Δ) is klt and $-(K_X + \Delta)$ is ample.

Definition 2.2. Let $f\colon X \to X$ be a surjective endomorphism of a projective variety.

(1) The *first dynamical degree* δ_f is the spectral radius of the endomorphism $f^*|_{\mathrm{N}^1(X)}\colon \mathrm{N}^1(X) \to \mathrm{N}^1(X)$, i.e., the maximum of moduli of eigenvalues of $f^*|_{\mathrm{N}^1(X)}$. It is known that $\deg f \leq \delta_f^{\dim X}$ (so $\deg f > 1$ implies $\delta_f > 1$). For the case of the surface, see e.g., [Nak20, Proposition 3.3]. When $\dim X = 1$, we have $\delta_f = \deg f$.

(2) The map f is δ-*polarized* if $f^*H \sim \delta H$ for some integer $\delta > 1$ and ample Cartier divisor H, or equivalently $f^*B \equiv \delta B$ for some rational number $\delta > 1$ and big $\mathbb{R}$-Cartier divisor B, or equivalently $f^*B \equiv \delta B$ for some $\delta > 1$ and big $\mathbb{Q}$-Cartier divisor B (indeed such δ is in $\mathbb{Q}$) (cf. [MZ18, Proposition 3.6]).

For the following, see e.g., [MZ, Lemma 2.4], [MZ18, Proposition2.9, Lemma 3.1, Corollary 3.12] and [Zha10, Lemma 2.2].

Lemma 2.3. *Let $f_i\colon X_i \to X_i$ $(i = 1, 2)$ be surjective morphisms of projective varieties, $\pi\colon X_1 \dashrightarrow X_2$ a generically finite dominant rational map such that $\pi \circ f_1 = f_2 \circ \pi$. Then:*

(1) $\delta_{f_1} = \delta_{f_2}$.

(2) Suppose that $f_1^*B \equiv \delta B$ *for some nef and big* $\mathbb{R}$*-Cartier divisor* B *and* $\delta > 0$. *Then* $\deg f_1 = \delta^{\dim X_1}$. *In particular, if* f_1 *is* δ*-polarized then* $\deg f_1 = \delta^{\dim X_1}$.

(3) f_1 *is* δ*-polarized if and only if so is* f_2.

(4) If f_1 *is* δ*-polarized, then* $f_1^*|_{\mathrm{N}^1(X)}$ *is diagonalisable over* $\mathbb{C}$ *and every eigenvalue of* $f_1^*|_{\mathrm{N}^1(X)}$ *has modulus* δ (> 1). *In partially,* $\delta_{f_1} = \delta$.

Definition 2.4. Let $f\colon X \to X$ be a surjective endomorphism of a projective variety.

(1) f is *amplified* if $f^*L - L = H$ for some Cartier divisor L and ample divisor H.

(2) f is *int-amplified* if $f^*L - L = H$ for some ample Cartier divisors L and H, or equivalently, if all the eigenvalues of $f^*|_{\mathrm{N}^1(X)}$ are of modulus greater than 1 (cf. [Men20, Theorem 1.1]).

Remark 2.5. Let $f\colon X \to X$ be a surjective endomorphism of a projective variety.

(1) If f is polarized then f is int-amplified.

(2) If f is int-amplified and $\dim X \geq 2$, then $\deg f > \delta_f$ (cf. [Men20, Lemma 3.6]).

The following results are frequently used.

Lemma 2.6. *(cf.* [Wah90, Theorems 2.8, 2.9]*,* [Nak20, Theorem E]*,* [BdFF12, Theorem B]*) Let* $f\colon X \to X$ *be a non-isomorphic surjective endomorphism of a normal variety of dimension two. Then we have:*

(1) X *has at worst lc singularities. In particular,* K_X *is* $\mathbb{Q}$*-Cartier.*

(2) If $x_0 \in X$ *is a closed point such that* $f^{-1}(x_0) = x_0$ *and* $x_0 \in R_f$*, then* X *has only klt singularity at* x_0.

Lemma 2.7. *Let* f *be a surjective endomorphism of a normal projective surface* X. *Assume that there exists an* f*-equivariant fibration* $\pi\colon X \to Y$ *to a nonsingular projective curve such that* f *descends to* g *on* Y *with* $\deg g > 1$ *(this last part,* $\deg g > 1$*, automatically holds when* f *is polarized by Lemma 2.3). Then* X *has only klt singularities and hence is* $\mathbb{Q}$*-factorial.*

Proof. By Lemma 2.6, X has at worst lc singularities. Assume X is not klt (but is lc) at x_0. Then $f^{-1}(x_0) = x_0$ after iterating f (cf. [BH14, Lemma 2.10]). Also $x_0 \notin \operatorname{Supp} R_f$ by Lemma 2.6. Now $\pi \circ f = g \circ \pi$ implies $g^{-1}(\pi(x_0)) = \pi(x_0)$. Denote by $F_0 := \pi^*(\pi(x_0))$ the fibre of π passing through x_0. It follows that $f^{-1}(\operatorname{Supp} F_0) = \operatorname{Supp} F_0$ and thus $f^*F_0 = (\deg g)F_0$. But this yields $x_0 \in \operatorname{Supp} F_0 \subseteq \operatorname{Supp} R_f$, a contradiction. □

The general result below is a direct consequence of the cone theorem.

Proposition 2.8. *Let $\pi\colon X \to Y$ be a morphism from a normal projective surface to a smooth projective curve with general fibres smooth rational curves. Then we have:*

(1) X has only rational singularities (and hence is $\mathbb{Q}$-factorial). If X has no negative curve, then $\mathrm{Nef}(X) = \mathrm{NE}(X)$.

(2) Suppose that $\pi\colon X \to Y$ is a Fano contraction. Then all the fibres are irreducible.

Proof. The first part of (1) is from [Nak17, Proposition 2.33]. By [Zha16, Lemma 2.3], we have a natural embedding $\mathrm{N}^1(X) \subseteq \mathrm{N}_1(X)$. Since X is $\mathbb{Q}$-factorial, we may identify $\mathrm{N}^1(X)$ and $\mathrm{N}_1(X)$. The second part follows from the same proof as [Kol96, II, Lemma 4.12].

For (2), suppose that F_1, F_2 are two distinct irreducible components of a fibre F. The K_X-negative extremal ray R contracted by π is $\mathbb{R}_{\geq 0}[F]$. Then $F_i \cdot F = 0$ implies $F_i \cdot R = 0$ and hence $F_i = \pi^* L_i$ for some $\mathbb{Q}$-Cartier divisors L_i on Y by the cone theorem [Fuj11, Theorem 1.1(4) iii]. Since in our case $\mathrm{Supp}(L_i) = \pi(F_i) = \pi(F)$, it follows that $\mathrm{Supp}\, F_1 = \pi^{-1}(\pi(F)) = \mathrm{Supp}\, F_2$ as sets, a contradiction. □

Next, we deal with surface fibrations to curves of higher genus.

Proposition 2.9. *Let $f\colon X \to X$ be a surjective morphism of a normal projective surface. Suppose $\pi\colon X \to E$ is an f-equivariant fibration, with connected fibres, to an elliptic curve. We also assume that X is $\mathbb{Q}$-factorial (this is the case when π is a $\mathbb{P}^1$-fibration; cf. Proposition 2.8). Let $g = f|_E$ and Σ_1 (resp. Σ_2) be the set of points $e \in E$ such that the scheme-theoretic fibre $\pi^*(e)$ is reducible (resp. nonreduced). Then $g^{-1}(\Sigma_i) = \Sigma_i$ for $i = 1, 2$.*

Proof. The result is clear when f is an automorphism. Suppose $\deg f \geq 2$. Since general fibres of π are smooth, the Σ_i's are finite sets. It suffices to show $g^{-1}(\Sigma_i) \subseteq \Sigma_i$ for $i = 1, 2$. A fibre is reducible if and only if all its irreducible components are negative curves. After an iteration, f^{-1} preserves negative curves (cf. [Nak02, Lemma 9], or [MZ, Lemma 4.3]), so f^{-1} takes a negative curve to another one. Thus f^{-1} takes reducible fibres to reducible fibres, and $g^{-1}(\Sigma_1) \subseteq \Sigma_1$. Since E is an elliptic curve and hence $g\colon E \to E$ is étale, the reducedness of $\pi^*(e)$ implies that of $\pi^*(g(e))$, so $g^{-1}(\Sigma_2) \subseteq \Sigma_2$ (cf. [CMZ20, Lemma 7.3]). □

Proposition 2.10. *Suppose that $f\colon X \to X$ is a non-isomorphic surjective endomorphism of a normal projective surface. Let $\pi\colon X \to Y$ be a $\mathbb{P}^1$-fibration to a nonsingular projective curve with genus $g(Y) \geq 1$ such that f descends to an endomorphism h on Y. Then either h is an automorphism of finite order; or $g(Y) = 1$ and π is a $\mathbb{P}^1$-bundle.*

Proof. If the genus $g(Y) \geq 2$, then Y is of general type, and the endomorphism h is an automorphism of finite order.

Suppose $g(Y) = 1$. Since X has only lc singularities (cf. Lemma 2.6) and $\deg f \geq 2$, we may run a relative EMMP over Y (cf. [MZ, Theorem 4.7])

$$X = X_1 \to X_2 \to \cdots \to X_r \to Y',$$

where $X_j \to X_{j+1}$ are divisorial contractions for $j < r$ and $X_r \to Y'$ is a Fano contraction. Since both $X \to Y$ and $X_r \to Y'$ have connected fibres, we have $Y' = Y$. Assume $r \geq 2$ and let E be the exceptional divisor of $X_1 \to X_2$. Then f^{-1} fixes E as a set and thus h^{-1} fixes P, where $P := \pi(E)$ is a point in Y (cf. [CMZ20, Lemma 7.3]). Since h is étale, it must be an automorphism. Then h is of finite order since it has at least one fixed point (cf. [Har77, IV, Corollary 4.7]).

Assume that h is not an automorphism of finite order. Then $r = 1$, i.e., π is a Fano contraction. Keep the notation in Proposition 2.9. Proposition 2.8 implies $\Sigma_1 = \emptyset$. By Proposition 2.9, one has $h^{-1}(\Sigma_2) = \Sigma_2$ and hence $\deg g \geq 2$ implies $\Sigma_2 = \emptyset$. So every fibre of π is irreducible and reduced. Then π is a smooth morphism (cf. [Nak17, Proposition 2.33]), and hence a $\mathbb{P}^1$-bundle (cf. [Har77, V. Proposition 2.2]). □

For Lemma 2.11 below, $\mathfrak{Alb}(X)$ as defined in [CMZ20, Section 5], is characterised by the rational map $\mathfrak{alb}_X \colon X \dashrightarrow \mathfrak{Alb}(X)$ as the one such that every rational map from X to an abelian variety factors through it.

Lemma 2.11. *Let X_r be a projective cone over an elliptic curve E. Let $X \to X_r$ be a birational morphism with X a normal projective surface. Suppose $f \colon X \to X$ is a non-isomorphic surjective endomorphism and it descends to an endomorphism $f_r \colon X_r \to X_r$. Let Γ be the normalisation of the graph of $\mathfrak{alb}_X \colon X \dashrightarrow E = \mathfrak{Alb}(X)$. Then $\Gamma \to E$ is a $\mathbb{P}^1$-bundle, $(f|_\Gamma)^* = \delta_f \operatorname{id}$ and $\Gamma \to X_r$ is the contraction of a cross-section. Further, either $X = \Gamma$, or $X = X_r$.*

Proof. Note that f descends to E and lifts to Γ. Since $\rho(X_r) = 1$, f_r, and hence f, $f|_\Gamma$ and $f|_E$ are all polarized (cf. Lemma 2.3). In particular, $f|_E$ is non-isomorphic. Applying Proposition 2.10, $\Gamma \to E$ is a $\mathbb{P}^1$-bundle. Notice that $\rho(\Gamma) = 2$ and $\Gamma \to X_r$ is $f|_\Gamma$-equivariant. Let C be the exceptional divisor for $\Gamma \to X_r$, which is a cross-section of $\Gamma \to E$. The exceptional divisor C is $(f|_\Gamma)^{-1}$-invariant, so $(f|_\Gamma)^* C = \delta_f C$ (cf. Lemma 2.3). Note that $\rho(X_r) = 1$ and hence $(f|_r)^*|_{\mathrm{N}^1(X_r)} = \delta_f \operatorname{id}$ (cf. Lemma 2.3). Since the pullback of $\mathrm{N}^1(X_r)$ is $(f|_\Gamma)^*$-invariant and $\mathrm{N}^1(\Gamma)$ is the direct sum of $\mathbb{R}[C]$ and the pullback of $\mathrm{N}^1(X_r)$, we conclude that $(f|_\Gamma)^*$ is diagonalisable and thus $(f|_\Gamma)^*|_{\mathrm{N}^1(\Gamma)} = \delta_f \operatorname{id}$. Since

$2 = \rho(\Gamma) \ge \rho(X) \ge \rho(X_r) = 1$, Zariski main theorem and normality of Γ, X and X_r imply the last assertion. □

Lemma 2.12 below is known to Iitaka, Sommese, Fujimoto, Nakayama, $\cdots$.

Lemma 2.12. *Let f be a non-isomorphic surjective endomorphism of a normal projective variety X of dimension n. If K_X is $\mathbb{Q}$-Cartier and pseudo-effective, then $R_f = 0$.*

Proof. Assume $R_f \neq 0$. The ramification divisor formula (2.1) for f^s is given by $K_X = (f^s)^*K_X + \sum_{i=0}^{s-1}(f^i)^*R_f$. Pick an ample Cartier divisor H on X. Since R_f is an integral Weil divisor and K_X is pseudo-effective, we get a contradiction by letting $s \to +\infty$:

$$K_X \cdot H^{n-1} = (f^s)^*K_X \cdot H^{n-1} + \sum_{i=0}^{s-1}(f^i)^*R_f \cdot H^{n-1} \ge s.$$ □

Theorems 2.13 and 2.14 of Nakayama are crucial for the proof of Theorem 1.1.

Theorem 2.13 (cf. [Nak20, Theorem A])**.** *Let X be a normal projective surface admitting a non-isomorphic surjective endomorphism f. Assume that $K_X + S \sim_{\mathbb{Q}} 0$ for an f^{-1}-stable reduced divisor S. Then f is quasi-étale outside S, and there exists a quasi-étale finite Galois covering $\nu\colon V \to X$ such that $\nu \circ f_V = f^\ell \circ \nu$ for a non-isomorphic surjective endomorphism f_V of V and a positive integer ℓ, and that V and ν satisfy exactly one of the following conditions.*

(1) V is an abelian surface and $S = 0$.
(2) V is a $\mathbb{P}^1$-bundle over an elliptic curve such that ν^*S is a disjoint union of two cross-sections.
(3) V is a projective cone over an elliptic curve and ν^*S is a cross-section.
(4) V is a toric surface and ν^*S is the boundary divisor.

Theorem 2.14 (cf. [Nak20b, Theorem 3.11])**.** *Let X be a normal projective surface. Then X admits a non-isomorphic quasi-étale surjective endomorphism f if and only if there exists a quasi-étale finite Galois covering $\nu\colon V \to X$ satisfying one of the following conditions.*

(1) V is an abelian surface.
(2) $V \cong E \times T$ for an elliptic curve E and a curve T of genus at least two.
(3) $V \cong \mathbb{P}^1 \times E$ for an elliptic curve E.
(4) V is a $\mathbb{P}^1$-bundle over an elliptic curve associated with an indecomposable locally free sheaf of rank two and degree zero.

Moreover, f^ℓ lifts to V for some positive integer ℓ.

We need the following results of [Xie22] in proving Theorem 1.9.

Proposition 2.15 (cf. [Xie22, Proposition 3.27]). *Let X be an (irreducible) variety over $\mathbf{k}$, and $f\colon X \dashrightarrow X$ a dominant rational map. Then the following statements are equivalent.*

(1) (X, f) satisfies AZD-property (resp. SAZD-property).

(2) (X, f^m) satisfies AZD-property (resp. SAZD-property) for some $m \geq 1$.

(3) There exists a pair (Y, g) which is birational to the pair (X, f), and (Y, g) satisfies AZD-property (resp. SAZD-property).

Lemma 2.16 (cf. [Xie22, Lemma 3.28]). *Let X and X' be (irreducible) varieties over $\mathbf{k}$, $f\colon X \dashrightarrow X$ and $f'\colon X' \dashrightarrow X'$ dominant rational maps. Let $\pi\colon X' \dashrightarrow X$ be a generically finite dominant rational map such that $\pi \circ f' = f \circ \pi$. Then (X', f') satisfies AZD-property (resp. SAZD-property) if and only if (X, f) satisfies AZD-property (resp. SAZD-property).*

Proposition 2.17 (cf. [Xie22, Proposition 3.30]). *Let X be an (irreducible) surface over $\mathbf{k}$, and $f\colon X \dashrightarrow X$ a dominant rational map. Suppose the pair (X, f) does not satisfy SAZD-property. Then there exists some $m \geq 1$ such that there are infinitely many irreducible curves C on X satisfying $f^m(C) \subseteq C$.*

As a consequence of the above, we have the following.

Corollary 2.18. *(cf. [Xie22, Corollary 3.31]) Let X be an (irreducible) projective surface over $\mathbf{k}$, and $f\colon X \dashrightarrow X$ a birational map. Then the pair (X, f) satisfies AZD-property.*

3. Proof of Theorem 1.1 and Corollary 1.3

In this section, we prove Theorem 3.1 for the case of non-pseudo effective canonical divisor K_X. Theorem 1.1 will follow from it and Theorem 2.13 for the case with K_X being pseudo-effective.

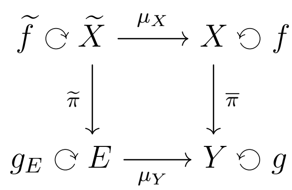
Theorem 3.1. *Let $f\colon X \to X$ be a non-isomorphic surjective endomorphism of a normal projective surface with K_X not being pseudo-effective. Then X has only lc singularities. Replacing f by an iteration we may run an f-equivariant minimal model program*

$$\begin{array}{ccccccccccccc} X & = & X_1 & \xrightarrow{\pi_1} & \cdots & \xrightarrow{\pi_{j-1}} & X_j & \xrightarrow{\pi_j} & \cdots & \xrightarrow{\pi_{r-1}} & X_r & \xrightarrow{\pi} & Y \\ {\scriptstyle f}\downarrow & & {\scriptstyle f_1}\downarrow & & & & {\scriptstyle f_j}\downarrow & & & & {\scriptstyle f_r}\downarrow & & \downarrow{\scriptstyle g} \\ X & = & X_1 & \xrightarrow[\pi_1]{} & \cdots & \xrightarrow[\pi_{j-1}]{} & X_j & \xrightarrow[\pi_j]{} & \cdots & \xrightarrow[\pi_{r-1}]{} & X_r & \xrightarrow[\pi]{} & Y \end{array}$$

contracting K_{X_j}-negative extremal rays, with $X_j \to X_{j+1}$ $(j < r)$ being divisorial and $X_r \to Y$ being Fano contraction (hence every fibre of π is irreducible). If $\dim Y = 1$, then

X_r has Picard number $\rho(X_r) = 2$ and all X_j $(1 \le j \le r)$ have only rational singularities (hence are $\mathbb{Q}$-factorial). Moreover, exactly one of the following cases occurs.

(1) *Y is a smooth projective curve of genus $g(Y) \ge 2$, and g is an automorphism of finite order (hence no f_j is polarized by Lemma 2.3, and $r = 1$ by Theorem 3.3).*
(2) *Y is an elliptic curve. If π has a nonreduced fibre, then g is an automorphism of finite order; otherwise $\pi\colon X_r \to Y$ is a $\mathbb{P}^1$-bundle.*
(3) *$Y \cong \mathbb{P}^1$, and f is not polarized (hence $r = 1$ by Theorem 3.3, so X does not contain negative curves by Lemma 3.6, and $\mathrm{Nef}(X) = \mathrm{NE}(X)$ by Proposition 2.8).*
    (a) *Either $\delta_f > \delta_g$; or $\delta_f = \delta_g$, with $-K_X$ being ample or $R_f \ne 0$. There exist a finite surjective morphism $\tau\colon X \to Y \times \mathbb{P}^1$ and a surjective endomorphism $h\colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $(g \times h) \circ \tau = \tau \circ f$.*
    (b) *$\delta_f = \delta_g$, $-K_X$ is nef but not ample, and $R_f = 0$. There is an f-equivariant quasi-étale finite Galois cover $\nu\colon V \to X$ as in Theorem 2.14(3) or (4).*
(4) *$Y \cong \mathbb{P}^1$, and f_r is polarized (hence for all $1 \le j \le r$, $\delta_{f_j} = \delta_{f_r} = \delta_g$, the f_j is polarized by Lemma 2.3, and the X_j has only klt singularities by Lemma 2.7).*
    If X_r does not contain negative curves, then X_r satisfies Proposition 3.8 and hence 3.8(1) or 3.8(3) occurs.
    Suppose that X_r contains a negative curve C. Then $\mathrm{NE}(X_r) = \langle [C], [F] \rangle$ with F a general fibre of π; $f_r^*C = \delta_f C$, so $R_{f_r} \ge (\delta_f - 1)C$; also $\kappa(X_r, -K_{X_r}) = 2$; further, one of the following is true.
    (c) *Either $-K_{X_r}$ is ample, or C intersects $R_{f_r} - (\delta_f - 1)C$. Then X_r is of Fano type; the contraction $\sigma\colon X_r \to \overline{X}$ of C gives a klt Fano surface $\overline{X}$ with $\rho(\overline{X}) = 1$. Moreover, f_r descends to an endomorphism $\bar{f}$ on $\overline{X}$.*
    (d) *$-K_{X_r}$ is not ample, and C does not meet $R_{f_r} - (\delta_f - 1)C$. Then C is a cross-section of π; every irreducible component of R_{f_r} dominates Y; there exists an equivariant commutative diagram, where E is an elliptic curve, and $\widetilde{X}$ is the normalisation of (the main component of) $X \times_Y E$, $\widetilde{\pi}$ is a $\mathbb{P}^1$-bundle; $\widetilde{f}$ and g_E are finite surjective endomorphisms; μ_X is quasi-étale, μ_Y is finite surjective; $\overline{\pi}$ is the composition $X \to \cdots \to Y$.*



(5) Y is a point, $-K_{X_r}$ is ample, $\rho(X_r) = 1$, and f_r and hence all f_j $(1 \le j \le r)$ are polarized (cf. Lemma 2.3). Either X_r is a projective cone over an elliptic curve, or a rational surface with only rational singularities.

Remark 3.2. The X_r has at most one negative curve when $\dim Y = 1$: if C is a negative curve, then the class $[C]$ and fibre class are the only two extremal rays in $\mathrm{NE}(X_r)$.

We need the following for the proof of Theorem 3.1.

Theorem 3.3 (cf. [MZ, Theorem 5.4])**.** *Let f be a non-isomorphic surjective endomorphism of a normal projective surface with K_X not being pseudo-effective. Then X has only lc singularities. Replacing f by an iteration, we may run an f-equivariant MMP*

$$X = X_1 \to \cdots \to X_j \to \cdots \to X_r \to Y,$$

contracting K_{X_j}-negative extremal rays, with $X_j \to X_{j+1}$ $(j < r)$ being divisorial and $X_r \to Y$ being Fano contraction, such that one of the following cases occurs (with $f_j = f|_{X_j}$).

(1) $\dim Y = 0$*, so $\rho(X_r) = 1$, also f_r and hence all f_j $(1 \le j \le r)$ are polarized.*

(2) $\dim Y = 1$*, f_r and hence all f_j $(1 \le j \le r)$ are polarized; $\rho(X_r) = 2$ and $\delta_f = \delta_{f|Y}$.*

(3) $\dim Y = 1$ *and f_r is not polarized. Further $r = 1$, $\rho(X) = 2$ and one of the following cases occurs.*

(a) $\delta_f = \delta_{f|Y}$.

(b) $\delta_f > \delta_{f|Y}$; there exist a finite surjective morphism $\tau \colon X \to Y \times \mathbb{P}^1$ and a surjective endomorphism $h \colon \mathbb{P}^1 \to \mathbb{P}^1$ such that $(f|_Y \times h) \circ \tau = \tau \circ f$.

Remark 3.4. Let $f \colon X \to X$ be a surjective endomorphism of a normal projective variety, and $\pi \colon X \to Y$ an f-equivariant fibration, with connected fibres, to a smooth projective curve. Let F be a general fibre of π. Then $f^*F \equiv \deg(f|_Y)F = (\delta_{f|Y})F$. In particular, one of the eigenvalues of $f^*|_{\mathrm{N}^1(X)}$ is a positive integer.

We will apply the Lemma 3.5 – Proposition 3.8 below to $X = X_r$ in Theorem 3.1.

Lemma 3.5. *Let $f \colon X \to X$ be a non-isomorphic surjective endomorphism of a normal projective surface. Let $\pi \colon X \to Y$ be a Fano contraction such that f descends to an endomorphism g on Y. Assume either* $\dim Y = 0$*; or* $\dim Y = 1$ *and* $\delta_g = \deg g > 1$*. Then $-K_X$ is pseudo-effective but not numerically trivial.*

Proof. First, by Lemma 2.6, X has only lc singularities and K_X is $\mathbb{Q}$-Cartier. If $\dim Y = 0$, then the Picard number $\rho(X) = 1$; hence $-K_X$ is ample, whence K_X is not pseudo-effective.

Suppose $\dim Y = 1$ and $\deg g > 1$. Then $\rho(X) = \rho(Y) + 1 = 2$. Write $\mathrm{NE}(X) = \langle [C], [F] \rangle$, where F is a general fibre of π, and $[C]$ is another extremal divisor class. Since $K_X \cdot F = \deg(K_X|_F) = \deg K_F = -2$, we may assume $-K_X \equiv C + bF$ with $C \cdot F = 2$. Suppose to the contrary that $b = -b_1 < 0$. Write $f^*C \equiv \delta C$ in $\mathrm{NE}(X)$. We calculate the ramification divisor:

$$R_f = K_X - f^*K_X \equiv (\delta - 1)C - b_1(\delta_g - 1)F,$$

$$(\delta - 1)C \equiv R_f + b_1(\delta_g - 1)F. \tag{3.1}$$

Now we have reached a contradiction from (3.1), because R_f is effective, $b_1(\delta_g - 1) > 0$ (by the assumption) and $[F]$ and $[C]$ are two distinct extremal divisor classes. □

Lemma 3.6. *Let $f\colon X \to X$ be a non-isomorphic surjective endomorphism of a normal projective surface, and $\pi\colon X \to Y$ an f-equivariant Fano contraction with* $\dim Y = 1$*. Let C be a negative curve on X. Then we have:*

(a) f is polarized.

(b) If further, $Y \cong \mathbb{P}^1$, then $\kappa(X, -K_X) = 0$ or 2.

Proof. Note that X is $\mathbb{Q}$-factorial by Proposition 2.8. By assumption, $\mathrm{NE}(X) = \langle [C], [F] \rangle$ where F is a general fibre of π. Note that $f^*C \equiv \delta C$ for some $\delta > 0$ and hence $f^*C = \delta C$, since $C^2 < 0$. Then

$$(\deg f)C \cdot F = f^*C \cdot f^*F = (\delta\, \delta_{f|Y})C \cdot F,$$

$$(\deg f)C^2 = (f^*C)^2 = \delta^2 C^2$$

imply that $\delta = \delta_{f|Y}$ ($= \deg f|_Y$) and f is δ-polarized.

Now suppose $Y \cong \mathbb{P}^1$. Then $q(X) = 0$. Hence, by Lemma 3.5 and its proof, $-K_X$ is numerically and hence $\mathbb{Q}$-linearly equivalent to some effective divisor. Thus $\kappa(X, -K_X) \geq 0$. If $-K_X$ is not big, then its class lies in the boundary of $\mathrm{NE}(X)$. Since $-K_X \cdot F = 2$, $-K_X \sim_{\mathbb{Q}} aC$ for some $a > 0$. Accordingly, $\kappa(X, -K_X) = \kappa(X, C) = 0$. □

Next, we consider the case when the $\mathbb{Q}$-factorial X has no negative curve.

Lemma 3.7. *Let $f\colon X \to X$ be a non-isomorphic surjective endomorphism of a normal projective surface. Suppose that $\pi\colon X \to \mathbb{P}^1$ is an f-equivariant Fano contraction such that $\delta_f = \delta_{f|\mathbb{P}^1}$ (hence X is a rational surface, and has only klt singularities; cf. Lemma 2.7). Write* $\mathrm{Nef}(X) = \langle [D], [F] \rangle$ *with F a general fibre of π, and let $f^*D \equiv \lambda D$. Then:*

(a) λ is a positive integer.

(b) If $D^2 > 0$ then f is polarized and X has at least one negative curve.

(c) *Assume that* $D^2 = 0$ *(hence* D *is a* $\mathbb{Q}$*-divisor after replacing it by a positive multiple) and* $\kappa(X, D) > 0$*. Then* D *is a semi-ample* $\mathbb{Q}$*-Cartier divisor and* $\kappa(X, D) = 1$*.*

Proof. By Lemma 2.7, X is $\mathbb{Q}$-factorial. Since $f^*F \sim_{\mathbb{Q}} \delta_{f|\mathbb{P}^1} F = \delta_f F$, $0 < (\deg f)D \cdot F = f^*D \cdot f^*F = (\lambda\,\delta_f)D \cdot F$, so $\lambda \in \mathbb{Q}_{>0}$ is an algebraic integer and hence an integer.

For (b), if $D^2 > 0$, then $(\lambda\,\delta_f)D^2 = (\deg f)D^2 = (f^*D)^2 = \lambda^2 D^2$ implies $\lambda = \delta_f$ and f is polarized. Assume X has no negative curve. Then $\mathrm{Nef}(X) = \mathrm{NE}(X)$ by Proposition 2.8. Thus $\mathbb{R}_{\ge 0}[D]$ is extremal in $\mathrm{Nef}(X) = \mathrm{NE}(X)$ and $D^2 = 0$, a contradiction.

For (c), we have $\kappa(X, D) = 1$ since $D^2 = 0$. We may assume $h^0(X, D) \ge 2$, and write $|D| = |M| + F_D$ as the moving part and fixed part. Pick two general $D_1, D_2 \in |M|$, so D_1 and D_2 have no common components. Since D and M are nef, $0 = D^2 \ge D \cdot M \ge M^2 = D_1 \cdot D_2 \ge 0$. Thus $M \cdot F_D = 0$ and $D_1 \cdot D_2 = 0$, hence the D_i are semi-ample. Now $0 = D^2 = (M + F_D)^2$ implies $F_D^2 = 0$ $(= M^2 = M \cdot F_D)$. Hodge index theorem (cf. [Nak17, Lemma pp. 302]) implies that F_D is numerically and hence $\mathbb{Q}$-linearly equivalent to a multiple of the movable M. Replacing D by a multiple, we may assume $F_D = 0$ and hence $D = M$ is semi-ample. ☐

Proposition 3.8. *With the same assumptions as in Lemma 3.7, assume* X *has no negative curve. Then* $\mathrm{Nef}(X) = \mathrm{NE}(X) = \langle [D], [F] \rangle$*. Also, one of the following cases occurs.*

(1) *There exist a finite surjective morphism* $\tau\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ *and a surjective endomorphism* $h\colon \mathbb{P}^1 \to \mathbb{P}^1$ *such that* $(f|_{\mathbb{P}^1} \times h) \circ \tau = \tau \circ f$*.*
(2) f *is quasi-étale, but non-polarized. There is an* f^ℓ*-equivariant quasi-étale finite Galois cover* $\nu\colon V \to X$ *for an integer* $\ell > 0$ *as in Theorem 2.14(3) or (4).*
(3) f *is polarized. The (nef non-big) divisor* D *can be chosen to be irreducible, reduced and* f^{-1}*-stable such that* $K_X + D \sim_{\mathbb{Q}} 0$*. There is an* f^ℓ*-equivariant quasi-étale finite Galois cover* $\nu\colon V \to X$ *for an integer* $\ell > 0$ *as in Theorem 2.13(2).*

Proof. By Lemma 3.7, $D^2 = 0$, so D is nef and non-big. By Proposition 2.8, $\mathrm{NE}(X) = \mathrm{Nef}(X) = \langle [D], [F] \rangle$. In particular, $[D]$ generates an extremal ray in $\mathrm{NE}(X)$. Then, by Lemma 3.5, $-K_X \in \mathrm{NE}(X) = \mathrm{Nef}(X)$ is a nef divisor. Note that $-K_X \cdot F = 2$.

Case A. $-K_X \cdot D > 0$, i.e., $-K_X$ is ample. By the cone theorem (cf. [KM98, Theorem 3.7]), D may be chosen to be a rational curve. Since $(aD - K_X) \cdot D > 0$ and $(aD - K_X) \cdot F > 0$ for $a > 0$, Kleiman's ampleness criterion (cf. [KM98, Theorem 1.18]) implies that $aD - K_X$ is ample. Hence D is semi-ample by the basepoint-free theorem (cf. [KM98, Theorem 3.3]), so it gives a fibration $\phi\colon X \to Z$ with connected fibres and normal Z. We have $\dim Z = 1$, since D is not big. Notice that $F \cdot D > 0$, F dominates Z and hence $Z \cong \mathbb{P}^1$.

Let C be an irreducible curve on X. Then C is in a fibre of ϕ if and only if $C \cdot D = 0$. Using the projection formula we obtain $f_*C \cdot D = C \cdot f^*D = \lambda(C \cdot D)$. Consequently, C is in a fibre if and only if so is $f(C)$. Since the fibration ϕ has connected fibres, there exists a surjective endomorphism $h\colon Z \to Z$ such that $h \circ \phi = \phi \circ f$ by the rigidity lemma (cf. [Deb01, Lemma 1.15]). The two distinct fibrations π and ϕ induce a surjective morphism $\tau\colon X \to \mathbb{P}^1 \times Z = \mathbb{P}^1 \times \mathbb{P}^1$ such that $(f|_{\mathbb{P}^1} \times h) \circ \tau = \tau \circ f$. It is finite because $\rho(X) = 2 = \rho(\mathbb{P}^1 \times \mathbb{P}^1)$. Then Case (1) occurs.

Case B. $-K_X \cdot D = 0$, i.e., $-K_X$ is nef but not ample. Since D is also nef, Hodge index theorem (cf. [Nak17, Lemma pp. 302]) implies (the class of) $-K_X \in \mathbb{R}_{\geq 0}[D]$. Hence $f^*K_X \sim_{\mathbb{Q}} \lambda K_X$, because $q(X) = 0$.

If $R_f = 0$ (which is impossible when f is polarized, for otherwise $K_X = f^*K_X \equiv 0$ but $-K_X \cdot F = 2$), then Case (2) occurs by Theorem 2.14. Indeed, only Theorem 2.14(3) - (4) occur since $-K_X$ is nef and not numerically trivial.

Assume $0 \neq R_f$ $(= K_X - f^*K_X \in \mathbb{R}_{\geq 0}[D])$ and write $R_f = \sum a_i D_i$ where $a_i \in \mathbb{Z}_{>0}$ and D_i are irreducible components, automatically with $[D_i] \in \mathbb{R}_{\geq 0}[D]$. If R_f is reducible, then $q(X) = 0$ implies $D_1 \sim_{\mathbb{Q}} tD_2$ for some rational number $t > 0$; or if D_1 is not f^{-1}-stable, then $f^*D_1 \sim_{\mathbb{Q}} \lambda D_1$. In either case, D_1 is $\mathbb{Q}$-Cartier, $D_1^2 = 0$ and $\kappa(X, D_1) > 0$. Hence by Lemma 3.7 we may retake $D := D_1$ and assume it is a semi-ample $\mathbb{Q}$-Cartier divisor and $\kappa(X, D) = 1$. Using the semi-ample divisor D, Case (1) occurs as argued in Case A.

Now we may retake $D := D_1$ (hence D is a $\mathbb{Q}$-divisor) and assume $\operatorname{Supp} R_f = D$ is irreducible (and reduced) and f^{-1}-stable.

Case B-1. f is not polarized. If $\lambda = 1$ we reach a contradiction: $0 \leq R_f = (K_X - f^*K_X) \sim_{\mathbb{Q}} 0$ and hence $R_f = 0$. Thus $1 < \lambda < \delta_f$. By [MZ, Proof of Step 5 in Theorem 5.2] we obtain $\kappa(X, D) > 0$. Arguing as above then Case (1) occurs.

Case B-2. f is polarized. We shall show that Case (3) occurs. Now the log ramification divisor formula (2.2) becomes $K_X + D = f^*(K_X + D)$. So the eigenvector (of f^*) $K_X + D \sim_{\mathbb{Q}} 0$ since f is polarized and $q(X) = 0$. Applying Theorem 2.13, only Cases 2.13(2) - (4) may occur in our situation. Note that X has klt singularities by Lemma 2.7, so does V and hence Case 2.13(3) cannot occur. For Case 2.13(4), ν^*D would be the boundary divisor of a toric surface, so a big divisor. But this violates the fact that $\kappa(V, \nu^*D) = \kappa(X, D) < 2$. So only Case 2.13(2) is possible, i.e., Case (3) occurs. □

Now we can prove the main result of this section.

Proof of Theorem 3.1. The assertions in the first paragraph follow from Lemma 2.6, Theorem 3.3 and Proposition 2.8.

We now apply Theorem 3.3. If Case 3.3(1) occurs, i.e., if $\dim Y = 0$ then 3.1(5) occurs. Indeed, the last assertion there has been proved in [BG17, p. 578]. If $\dim Y = 1$ with $g(Y) \geq 1$, then $q(X_r) = g(Y) \geq 1$ and 3.1(1) – 3.1(2) occur by Proposition 2.10.

Suppose that 3.3(3) occurs with $Y \cong \mathbb{P}^1$. Then f is not polarized and $r = 1$. Lemma 3.6 implies X does not contain negative curves. By Proposition 3.8 and its proof, Case 3.3(3a) leads to 3.1(3): if $-K_X$ is ample or $R_f \neq 0$ then 3.1(3a) occurs; if $-K_X$ is not ample and $R_f = 0$ then 3.1(3b) occurs. Clearly, Case 3.3(3b) implies 3.1(3a).

Now suppose that 3.3(2) occurs with $Y \cong \mathbb{P}^1$. Thus f_r is polarized, and X_r has only klt singularities by Lemma 2.7. The second paragraph of 3.1(4) follows from Case 3.3(2) and Proposition 3.8 when X_r has no negative curve.

We still have to consider Case 3.3(2) with the extra conditions that $Y \cong \mathbb{P}^1$, X_r has only klt singularities and contains a negative curve C. By Lemma 3.6, f_r is δ_f-polarized (cf. Lemma 2.3) and $\kappa(X_r, -K_{X_r}) = 0, 2$. Let F be a general fibre of $\pi\colon X_r \to Y$. Since both $[C]$ and $[F]$ are extremal classes of $\mathrm{NE}(X_r)$ and $\rho(X_r) = 2$, $\mathrm{NE}(X_r) = \langle [C], [F] \rangle$. Note that $-K_{X_r} \cdot F = 2$, $f_r^* C = \delta_f C$. The log ramification divisor formula (2.2) for f_r is

$$K_{X_r} + C = f_r^*(K_{X_r} + C) + R'_{f_r} \tag{3.2}$$

with $R_{f_r} = (\delta_f - 1)C + R'_{f_r}$.

Consider the case $\kappa(X_r, -K_{X_r}) = 0$. We will reach a contradiction. By the proof of Lemma 3.6, this happens if and only if $-K_{X_r} \sim_{\mathbb{Q}} aC$ for some $a > 0$. Then (3.2) gives

$$R'_{f_r} \sim_{\mathbb{Q}} (a-1)(\delta_f - 1)C. \tag{3.3}$$

Since R'_{f_r} is effective and has no common component with C, (3.3) gives $a = 1$ and $R'_{f_r} = 0$. The eigenvector (of f_r^*) $K_{X_r} + C \sim_{\mathbb{Q}} 0$ since f_r is polarized and $q(X_r) = 0$. Now we apply Theorem 2.13. Only Case 2.13(2) may occur in our situation (cf. Proof of Case B-2 in Proposition 3.8). But for Case 2.13(2) we have $0 = K_V^2 = (\deg \nu) K_{X_r}^2 = (\deg \nu)(-C)^2 < 0$, a contradiction. So the case $\kappa(X_r, -K_{X_r}) = 0$ will not occur.

Thus we may assume $\kappa(X_r, -K_{X_r}) = 2$. If $-K_{X_r}$ is ample, i.e., $-K_{X_r} \cdot C > 0$, then X_r is a klt Fano surface. The K_{X_r}-negative extremal contraction $\sigma\colon X_r \to \overline{X}$ of C gives a klt Fano surface $\overline{X}$ with $\rho(\overline{X}) = 1$ (cf. [KM98, Corollary 3.43(1)]). Note that f_r descends to an endomorphism $\bar{f}\colon \overline{X} \to \overline{X}$ since f^{-1} fixes C as a set. Thus Case 3.1(4c) occurs.

Next, we may also assume that $-K_{X_r}$ is not ample, i.e., $-K_{X_r} \cdot C \leq 0$. Let $-K_{X_r} = P + aC$ be the Zariski decomposition (cf. [Sak84, Corollary (7.5)]) with P nef and big.

We claim that $P \cdot C = 0$. Indeed, if $a > 0$, then the 'negative part' C has $P \cdot C = 0$; if $a = 0$, then $0 \le P \cdot C = -K_{X_r} \cdot C \le 0$, and hence $P \cdot C = 0$.

Now we may assume that things like X, f, C etc. are defined over a field K which is finitely generated over $\mathbb{Q}$. Embedding K into $\mathbb{C}$, we may assume that the base field is $\mathbb{C}$ and we may use some techniques from complex-analytic geometry in the following.

Let $\sigma\colon X_r \to \overline{X}$ be the contraction of the negative curve C to a point $\overline{x}$ in the normal Moishezon surface $\overline{X}$ (cf. [Sak84, Theorem (1.2)]). Then f_r descends to $\bar{f}\colon \overline{X} \to \overline{X}$ with $\bar{f}^{-1}(\bar{x}) = \bar{x}$. By Lemma 2.6, $\overline{X}$ has only lc singularities and $K_{\overline{X}}$ is $\mathbb{Q}$-Cartier. We have $P = \sigma^*(-K_{\overline{X}})$ since both sides are perpendicular to (and hence the same modulo) the negative curve C. Hence $\sigma^* K_{\overline{X}} = -P = K_{X_r} + aC$.

Suppose that C intersects R'_{f_r}. We shall show that Case 3.1(4c) occurs. Indeed, then $\overline{x} \in \operatorname{Supp} R_{\bar{f}}$ and hence $(\overline{X}, \overline{x})$ and also $\overline{X}$ have only klt singularities by Lemma 2.6, so $\overline{X}$ is $\mathbb{Q}$-factorial. For a Moishezon surface, being $\mathbb{Q}$-factorial implies the projectivity (cf. [Fuj21, Lemma 4.1]). Then $-K_{\overline{X}}$ is ample, and $\overline{X}$ is a klt Fano surface with $\rho(\overline{X}) = 1$. Since $\overline{X}$ has only klt singularities, $\sigma^* K_{\overline{X}} = K_{X_r} + aC$ implies that the pair (X_r, aC) is klt. For $b \in \mathbb{Q}_{>0}$ with $0 < b - a \ll 1$, the pair (X_r, bC) is still klt (cf. [KM98, Corollary 2.35(2)]); the divisor

$$-(K_{X_r} + bC) = -(K_{X_r} + aC) - (b-a)C = P - (b-a)C$$

has positive intersections with the two generators $[C]$ and $[F]$ of $\mathrm{NE}(X_r)$ and hence is ample. Thus X_r is of Fano type. So Case 3.1(4c) occurs.

Suppose now that C does not intersect R'_{f_r} even if raise f_r to some power. We shall show that Case 3.1(4d) occurs. Note that $R'_{f_r} \ne 0$, otherwise by (3.2) the eigenvector (of the polarized f_r^*) $K_{X_r} + C \sim_{\mathbb{Q}} 0$, reaching a contradiction: $2 = \kappa(X_r, -K_{X_r}) = \kappa(X_r, C) = 0$.

Note that F is nef but not numerically trivial, so $C \cdot F > 0$. It follows that C dominates Y, i.e., $\pi(C) = Y$. Since R'_{f_r} is disjoint with C by assumption and every fibre of $X_r \to Y$ is irreducible by Proposition 2.8, every irreducible component of R'_{f_r} (and hence of R_{f_r}) is horizontal and hence dominates Y. Then Case 3.1(4d) is true for (X_r, f_r) by [MY22, Theorem 4.4] and also for (X, f) by [MY22, Lemma 4.12]. In fact, it follows from Proposition 2.10 that $\widetilde{\pi}$ is a $\mathbb{P}^1$-bundle, since f is polarized and hence g_E cannot be an automorphism. The equality

$$\begin{aligned} -2 + F \cdot C = F \cdot (K_{X_r} + C) &= F \cdot (f_r^*(K_{X_r} + C) + R'_{f_r}) \\ &= \delta_f(-2 + F \cdot C) + F \cdot R'_{f_r}, \end{aligned}$$

and $F \cdot R'_{f_r} > 0$ imply that C is a cross-section of π. $\square$

Lemma 3.9. *Let X be a normal projective surface with lc singularities. Let $\varphi\colon X \to X'$ be a composition of extremal contractions. Suppose that either X' is a rational surface and $H^i(X', \mathcal{O}_{X'}) = 0$ for $i = 1, 2$; or $X' \cong \mathbb{P}^1$. Then X is a rational surface whose singularities are rational.*

Proof. The first assertion is clear. For the second, we prove by induction on the number of contractions in φ. We may assume φ itself is the contraction of a K_X-negative extremal ray. Then $-K_X$ is φ-ample. By the relative Kodaira vanishing theorem (cf. [Fuj11, Theorem 8.1]), $R^i\varphi_*\mathcal{O}_X = 0$ for every $i > 0$. Then the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q\varphi_*\mathcal{O}_X) \Rightarrow H^{p+q}(X', \mathcal{O}_{X'})$$

degenerates and $H^i(X, \mathcal{O}_X) = H^i(X', \mathcal{O}_{X'}) = 0$ for $i = 1, 2$ in both cases. Since X is a rational surface, it has rational singularities by [BG17, Lemma 2.7]. □

Proof of Theorem 1.1. By Lemma 2.6, X has lc singularities and K_X is $\mathbb{Q}$-Cartier. If K_X is pseudo-effective, then Lemma 2.12 implies that f is quasi-étale. Applying Theorem 2.14, only Cases 2.14(1) - (2) are possible since K_X is pseudo-effective and ν is quasi-étale. Therefore Case 1.1(1) holds. If K_X is not pseudo-effective, then we can apply Theorem 3.1. We have the correspondence below.

1.1(1)	2.14, 3.1(3b)
1.1(2)	3.1(1) – 3.1(2), 2.10
1.1(3)	3.1(2), 2.10
1.1(4)	3.1(3a), 3.1(4) (with 3.8(1))
1.1(5)	3.1(4) (with 3.8(3))
1.1(6)	3.1(4d)
1.1(7)	3.1(4c)
1.1(8)	3.1(5), 2.11
1.1(9)	3.1(5)

Set $f_r := f|_{X_r}$. We first show Case 3.1(4) (with 3.8(3)) implies Case 1.1(5). By Proposition 3.8(3), X_r is a rational surface; f_r is polarized; there is an f_r^{-1}-stable reduced divisor $D \neq 0$ on X_r such that f_r is quasi-étale outside D. Let $\pi\colon X \to \cdots \to X_r$ be the composition of the divisorial contractions, $D' := \pi_*^{-1}(D)$ the proper transform of D and $\mathrm{Exc}(\pi)$ the exceptional divisor of π. Set $S := D' + \mathrm{Exc}(\pi)$. Then S is reduced and f^{-1}-stable since π is f-equivariant. Since the divisor D on X_r is not big and $\pi_*(S) = D$, the divisor S is also non-big on X. By the definition of S, f is quasi-étale outside S. Thus

$K_X + S = f^*(K_X + S)$, by the log ramification divisor formula (2.2). Since f is polarized (cf. Lemma 2.3) and $q(X) = 0$, the eigenvector (of f^*) $K_X + S \sim_{\mathbb{Q}} 0$. Note that X has only klt singularities by Lemma 2.7. Now we can apply Theorem 2.13 to say Case 1.1(5) occurs (cf. Proof of Case B-2 in Proposition 3.8).

It remains to prove the last assertions of Cases 1.1(7) and 1.1(9), respectively. Lemma 2.7 implies that X_j has only klt singularities for $1 \le j \le r$ in Case 1.1(7). In Case 1.1(9), we have $H^i(X_r, \mathcal{O}_{X_r}) = 0$ for $i = 1, 2$ by [Fuj11, Theorem 8.1]. These X_j's in Case 1.1(9) have only rational singularities by Lemma 3.9. □

Proof of Corollary 1.3. We apply Theorem 1.1. Case 1.1(1) implies either Case 1.2(2) occurs; or $V = E \times T$ for an elliptic curve E and a curve T of genus $g(T) \ge 2$ and then V has an $(f|_V)^m$-invariant non-constant rational function for some $m \ge 1$, hence X has an f-invariant non-constant rational function by [Xie22, Lemma 2.1], and thus Case 1.2(1) occurs. Case 1.1(2) leads to Case 1.2(1). Cases 1.1(3), 1.1(5) - 1.1(6) and 1.1(8) satisfy Case 1.2(2). Case 1.1(4) implies Case 1.2(3) (replacing f by f^2). Cases 1.1(7) and 1.1(9) satisfy the conditions 1.2(4a) – 1.2(4c) (cf. Lemma 2.3) and thus Case 1.2(4). □

4. Amplified endomorphisms; Proofs of Theorems 1.9 and 1.12 and Proposition 1.10

In this section we assume that the transcendence degree of $\mathbf{k}$ over $\mathbb{Q}$ is finite. Till Proposition 4.5, we fix an (irreducible) projective surface X over $\mathbf{k}$, and a surjective endomorphism $f: X \to X$.

The key is to prove Theorem 1.12 which is essential in proving Theorem 1.9.

Definition 4.1. Let $o \in X(\mathbf{k})$ be a smooth fixed point of f, and λ_1, λ_2 the eigenvalues of the tangent map $df_o := df|_o : T_{X,o} \to T_{X,o}$. The smooth point $o \in X(\mathbf{k})$ is said to be a *repelling* fixed point of f with respect to a norm $|\cdot|$ of $\mathbf{k}$, if $|\lambda_i| > 1$ for $i = 1, 2$. The smooth point $o \in X(\mathbf{k})$ is said to be a *good* fixed point of f, if df_o is invertible and one of the following conditions holds:

(1) λ_1 and λ_2 are multiplicatively independent;

(2) There exist a prime p and an embedding $\tau: \mathbf{k} \hookrightarrow \mathbb{C}_p$ such that

$$|\tau(\lambda_1) + \tau(\lambda_2)| \le 1 \text{ and } |\tau(\lambda_1)||\tau(\lambda_2)| < 1$$

where $|\cdot|$ is the p-adic norm on $\mathbb{C}_p$.

Remark 4.2. Note that Condition (2) just means that both $|\tau(\lambda_1)|$ and $|\tau(\lambda_2)|$ are at most one and $|\tau(\lambda_i)| < 1$ for $i = 1$ or 2.

Definition 4.3. We say that f has *R-property* if there exist a smooth fixed point o of f and an embedding $\sigma\colon \mathbf{k} \hookrightarrow \mathbb{C}$ such that both $|\sigma(\lambda_1)|$ and $|\sigma(\lambda_2)|$ are strictly greater than 1, where λ_1, λ_2 are the eigenvalues of the tangent map $df_o\colon T_{X,o} \to T_{X,o}$.

From now on till Proposition 4.5, we assume that $f\colon X \to X$ is an amplified endomorphism, i.e., $f^*L \otimes L^{-1}$ is an ample line bundle for some line bundle L on X.

We will show the existence of a good fixed point of f. Let R be a finitely generated $\overline{\mathbb{Q}}$-sub-algebra of $\mathbf{k}$, such that $\mathbf{k}$ is the algebraic closure of $\operatorname{Frac} R$, and X, f, L are defined over $\operatorname{Frac} R$. There is a variety $X_{\operatorname{Frac} R}$ over $\operatorname{Frac} R$ and an endomorphism $f_{\operatorname{Frac} R}\colon X_{\operatorname{Frac} R} \to X_{\operatorname{Frac} R}$, such that $X = X_{\operatorname{Frac} R} \times_{\operatorname{Spec} \operatorname{Frac} R} \operatorname{Spec} \mathbf{k}$ and $f = f_{\operatorname{Frac} R} \times_{\operatorname{Spec} \operatorname{Frac} R} \mathrm{id}$.

After shrinking $W := \operatorname{Spec} R$, we may assume that W is smooth, there exists a projective R-scheme $\pi\colon X_R \to W$ whose generic fibre is $X_{\operatorname{Frac} R}$, $f_{\operatorname{Frac} R}$ extends to a finite endomorphism f_R on X_R and there exists a line bundle L_R on X_R such that $f_R^* L_R \otimes L_R^{-1}$ is π-ample. For every point $t \in W(\overline{\mathbb{Q}})$, denote by X_t the special fibre $X_R \times_W \operatorname{Spec} \overline{\mathbb{Q}}$ of X_R over t. Let L_t, f_t be the restrictions of L_R, f_R on X_t. Let X_R^{reg} be the smooth locus of X_R. After shrinking W, we assume that X_t is irreducible for every $t \in W(\overline{\mathbb{Q}})$ and $X_t \cap X_R^{\mathrm{reg}} \neq \emptyset$.

Lemma 4.4. *Assume that there exists some $t \in W(\overline{\mathbb{Q}})$ such that f_t has a good fixed point in $X_t \cap X_R^{\mathrm{reg}}$. Then f has a good fixed point in X.*

Proof. This lemma is shown by the proof of [Xie22, Lemma 6.6]. ☐

Proposition 4.5. *Let $f\colon X \to X$ be an amplified endomorphism of a projective surface X over $\mathbf{k}$. Let o be a smooth fixed point of f such that df_o is invertible. Let C be an irreducible curve in X passing through o such that $f(C) = C$, and every branch of C at o is invariant under f. Denote by $\pi_C\colon \overline{C} \to C$ the normalisation of C and $f|_{\overline{C}}\colon \overline{C} \to \overline{C}$ the endomorphism induced by $f|_C$. Let $q \in \pi_C^{-1}(o)$ and set $\mu := d(f|_{\overline{C}})|_q$. Assume that there exists an embedding $\alpha\colon \mathbf{k} \hookrightarrow \mathbb{C}$ such that $0 < |\alpha(\mu)| < 1$. Then there exists some $n \geq 1$ such that f^n has a good fixed point in X.*

Proof. This proof is a small modification of the proof of [Xie22, Lemma 6.7].

After enlarging R, we may assume o, C, q are defined over $\operatorname{Frac} R$ and $\mu \in R$. After shrinking W, we may assume there is an irreducible subscheme C_R of X_R whose generic fibre is C and a section $o_R \in X_R(R)$ of $\pi\colon X_R \to W$ whose $\mathbf{k}$-extension is o. For every point $t \in W(\overline{\mathbb{Q}})$, denote by C_t and o_t the specializations of C_R and o_R. Because $o \in X_R^{\mathrm{reg}}$, after shrinking W, we may assume that $o_t \in X_R^{\mathrm{reg}}$ and C_t is irreducible for every $t \in W(\overline{\mathbb{Q}})$. There is a projective morphism $\pi_{C_R}\colon \overline{C}_R \to C_R$ over R whose generic fibre is π_C and an

R-point $q_R \in \overline{C}_R(R)$, whose generic fibre is q. After shrinking W, we may assume that for all $t \in W(\overline{\mathbb{Q}})$, the specialization $\pi_{C_t} : \overline{C}_t \to C_t$ of π_{C_R} is the normalisation of C_t.

The embedding $\alpha : R \subseteq \mathbf{k} \hookrightarrow \mathbb{C}$ defines a point $\eta \in W(\mathbb{C})$. We view μ as a function on $W(\mathbb{C})$. We have $|\mu(\eta)| = |\alpha(\mu)| \in (0,1)$. There exists a Euclidean open neighborhood U of η, such that $|\mu(\cdot)| \in (0,1)$ on U. Picking $t \in U \cap W(\overline{\mathbb{Q}})$, we have $0 < |\mu(t)| < 1$. By Lemma 4.4, we only need to prove that there exists some $n \geq 1$ such that f_t^n has a good fixed point in X_t. Thus we have reduced to the case $\mathbf{k} = \overline{\mathbb{Q}}$.

Now we may assume that $\mathbf{k} = \overline{\mathbb{Q}}$*, the surface* X *and the map* f *are defined over a number field* K*, and there exist a variety* X_K *over* K *and an endomorphism* $f_K : X_K \to X_K$*, such that* $X = X_K \times_{\operatorname{Spec} K} \operatorname{Spec} \mathbf{k}$ *and* $f = f_K \times_{\operatorname{Spec} K} \mathrm{id}$.

Let O_K be the ring of integers of K. There exists a projective O_K-scheme X_{O_K} which is flat over $\operatorname{Spec} O_K$ whose generic fibre is X_K. Denote by $\pi_{O_K} : X_{O_K} \to \operatorname{Spec} O_K$ the structure morphism. The endomorphism f_K on the generic fibre extends to a rational self-map f_{O_K} on X_{O_K}. Denote by o_{O_K} the Zariski closure of o in X_{O_K}. Since o is defined over K, o_{O_K} is a section of π_{O_K}.

Denote by $\pi_{\mathbb{Z}}^{O_K} : \operatorname{Spec} O_K \to \operatorname{Spec} \mathbb{Z}$ the morphism induced by the inclusion $\mathbb{Z} \hookrightarrow O_K$. Let $X_{\mathbb{Z}}$ be the $\mathbb{Z}$-scheme which is the same as X_{O_K} as an absolute scheme with the structure morphism $\pi_{\mathbb{Z}} := \pi_{\mathbb{Z}}^{O_K} \circ \pi_{O_K} : X_{O_K} \to \operatorname{Spec} \mathbb{Z}$. Then $X_{\mathbb{Z}}$ is a projective $\mathbb{Z}$-scheme. Denote by $f_{\mathbb{Z}} : X_{\mathbb{Z}} \dashrightarrow X_{\mathbb{Z}}$ the rational self-map induced by f_{O_K}.

Since $f_{\mathbb{Z}}$ is regular on the generic fibre, there exists a finite set $B(f, \mathbb{Z})$ of primes such that $f_{\mathbb{Z}}$ is regular on $\pi_{\mathbb{Z}}^{-1}(\operatorname{Spec} \mathbb{Z} \setminus B)$. Set $B(f, O_K) := (\pi_{\mathbb{Z}}^{O_K})^{-1}(B(f, \mathbb{Z}))$. Set $W := \operatorname{Spec} O_K \setminus B(f, O_K)$, $X_W := \pi_{O_K}^{-1}(\operatorname{Spec} O_K \setminus B(f, O_K))$. Then f_{O_K} is regular on X_W. Set $o_W := o_{O_K} \cap X_W$. Set $\pi_W : X_W \to W$ to be the restriction of π_{O_K} on X_W. Then o_W is a section of π_{O_K}. For every $t \in W$, denote by X_t, f_t and o_t the specializations of X_W, f_W and o_W at t. After enlarging $B(f, \mathbb{Z})$, X_t is irreducible for every $t \in W$. Then for every point $x \in X^{\mathrm{reg}}(\overline{\mathbb{Q}})$, if $f^m(x) = x$ for some $m \geq 1$ and β_1, β_2 are the eigenvalues of the tangent map $d(f^m)|_x$, then for every prime $p \notin B(f, \mathbb{Z})$, and every embedding $\tau : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, we have $|\tau(\beta_1)|, |\tau(\beta_2)| \leq 1$.

Since f is amplified, $\deg(f|_{\overline{C}}) \geq 2$ by [Xie22, Lemma 5.2]. Then $\overline{C}$ is either $\mathbb{P}^1$ or an elliptic curve. Since on a complex elliptic curve, an endomorphism of degree at least 2 is everywhere repelling, $\overline{C}$ could not be an elliptic curve. Then we have $\overline{C} \cong \mathbb{P}^1$. Since $0 < |\alpha(\mu)| < 1$, by [Mil06, Corollary 14.5], $f|_{\overline{C}}$ is not post-critically finite.

Denote by $J(f)$ the union of the critical locus of f and the singular locus of X. Since $o \notin J(f)$ and $o \in C$, we have $C \not\subset J(f)$. Then $C \cap J(f)$ is finite. Let $P(f, C)$ be the union of the orbits of all periodic points in $C \cap J(f)$. Then $P(f, C)$ is finite. Observe that for every $n \geq 1$, $P(f^n, C) = P(f, C)$. By [Xie22, Lemma 6.8], after replacing f by a suitable positive iteration, there is a prime $p \notin B(f, \mathbb{Z})$, an embedding $\tau \colon \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ and some $\overline{x} \in \mathrm{Fix}(f|_{\overline{C}}) \setminus \pi_C^{-1}(P(f, C))$ such that C is smooth at $\pi_C(\overline{x})$ and $|\tau(d(f|_{\overline{C}})|_{\overline{x}})| < 1$. Set $x := \pi_C(\overline{x})$. Since $x \notin P(f, C)$, X is smooth at x and $df|_x$ is invertible. Since $d(f|_{\overline{C}})|_{\overline{x}}$ is an eigenvalue of $df|_x$, x is a good fixed point of f. □

Lemma 4.6. *Assume f is an amplified endomorphism which has R-property. Then either (X, f) satisfies SAZD-property, or there is an $n \geq 1$ such that f^n has a good fixed point.*

Proof. The proof is a small modification of [Xie22, Lemma 6.5]. Indeed, replacing [Xie22, Lemma 6.7] by Proposition 4.5, the proof of [Xie22, Lemma 6.5] works. □

The following result is a singular version of [Xie22, Proposition 6.15].

Proposition 4.7. *Let X be a projective surface over $\mathbf{k}$, and $f \colon X \to X$ an amplified endomorphism. Assume that f satisfies R-property. Then (X, f) satisfies SAZD-property.*

Proof. By Lemma 4.6 and Proposition 2.15, we may assume that f has a good fixed point. Then [Xie22, §6.3], whose proof still works in the singular case, shows that the pair (X, f) satisfies SAZD-property. □

Now we are ready to give the following:

Proof of Theorem 1.12. Pick any embedding $\sigma \colon \mathbf{k} \hookrightarrow \mathbb{C}$. View $X_{\mathbf{k}}(\mathbb{C})$ as a complex surface induced by σ. Let $\pi \colon X' \to X$ be a projective desingularisation of X. Set $f' := \pi^{-1} \circ f \circ \pi \colon X' \dashrightarrow X'$. We have $\delta_{f'} = \delta_f$ and $\deg f' = \deg f$. Let U be a Zariski open subset of X' such that $\pi|_U$ is an isomorphism to its image. By [Gue05, Theorem 3.1, (iv)], [DNT15, Theorem 1.1] and since $\deg f > \delta_f$, there is an $m \geq 1$ and a repelling fixed point o of f'^m in U. Then $\pi(o)$ is a smooth repelling fixed point of f^m. Thus f^m has R-property. By Propositions 2.15 and 4.7, (X, f^m) and hence (X, f) satisfy SAZD-property. The final assertion follows from Remark 2.5. □

The following is borrowed from [Xie22, §7].

Proposition 4.8. *Suppose that $\pi \colon X \to Y$ is a $\mathbb{P}^1$-bundle over a smooth projective curve Y and a non-isomorphic surjective endomorphism $f \colon X \to X$ descends to an endomorphism $g \colon Y \to Y$. Then (X, f) satisfies AZD-property.*

Proof. This is proved in [Xie22, §7]. We sketch it here for the convenience of the reader. Write $\mathrm{NE}(X) = \langle [F], [E] \rangle$, where F is a general fibre of π. Then $f^*F \equiv \delta_g F$. Write

$f^*E \equiv \lambda E$. Then $\delta_f = \max\{\lambda, \delta_g\}$, and $1 < \deg f = \lambda\delta_g$. If both $\lambda > 1$ and $\delta_g > 1$, then f is int-amplified so (X, f) satisfies AZD-property by Theorem 1.12.

Thus we may assume $\lambda = 1$ or $\delta_g = 1$. In particular, $\lambda \neq \delta_g$. Then $\lambda^2 E^2 = (f^*E)^2 = (\deg f)E^2 = (\lambda\delta_g)E^2$ implies that $E^2 = 0$.

By Propositions 2.15 and 2.17 and replacing f by an iteration, we may assume that there are infinitely many (irreducible) curves C_i with $f(C_i) = C_i$. For $C = C_i$, write $C = aF + bE$. Then $a\lambda F + b\delta_g E = f_*C$ (which is proportional to C) and $\lambda \neq \delta_g$ imply that $C = C_i$ is proportional to F or E.

Suppose that infinitely many C_i's are proportional to F. Then $C_i \cdot F = 0$ and hence C_i equals X_{y_i}, a fibre of π over $y_i \in Y$. Now $f(C_i) = C_i$ implies that $g(y_i) = y_i$. Then g has infinitely many fixed points y_i's, so $g = \mathrm{id}_Y$. Hence (X, f) satisfies AZD-property.

Thus we may assume all C_i's are proportional to E. Since $E^2 = 0$, all C_i's are disjoint. Taking base changes by (the normalisation of) $C_i \to Y$ consecutively and by Lemma 2.16, we may assume that C_i $(i = 1, 2, 3)$ are cross-sections of π. Then there is a natural isomorphism $Y \times \mathbb{P}^1 \to X$ mapping $Y \times \{0, 1, \infty\}$ to $C_1 \cup C_2 \cup C_3$. Identifying $X = Y \times \mathbb{P}^1$, our $f = g \times h$ with $h \colon \mathbb{P}^1 \to \mathbb{P}^1$ a morphism. Now the still disjointness of all C_i $(i \geq 2)$ with C_1 implies that C_i equals X_{p_i}, a fibre of the projection $X \to \mathbb{P}^1$ over $p_i \in \mathbb{P}^1$. Also $f(C_i) = C_i$ implies $h(p_i) = p_i$ for infinitely many points p_i's. Hence $h = \mathrm{id}_{\mathbb{P}^1}$. Thus (X, f) satisfies AZD-property. □

Proof of Theorem 1.9. By Corollary 2.18 and Proposition 2.15, we may assume $\deg f \geq 2$ and X is a normal projective surface. By Corollary 1.3, Theorem 1.12 and Proposition 2.15, Lemma 2.16, we may assume either X is an abelian surface, or X is a $\mathbb{P}^1$-bundle and f descends to the base. Then the theorem follows from [Xie22, Theorem 1.14] and Proposition 4.8. □

Proof of Proposition 1.10. There is a finitely generated field extension K over $\mathbb{Q}$ such that $\overline{K} = \mathbf{k}$, and X, f are defined over K. There exists a subring R of K such that R is finitely generated over $\mathbb{Z}$ and $\mathrm{Frac}\, R = K$. Pick a model $\pi \colon X_R \to \mathrm{Spec}\, R$ which is projective over $\mathrm{Spec}\, R$ and whose generic fibre is X_K.

Our f extends to a rational self-map $f_R \colon X_R \dashrightarrow X_R$. Denote by B_R the indeterminacy locus of f_R. By [Xie22, Lemma 3.23], there exists a nonempty, affine open subset U of $\mathrm{Spec}\, R$ such that

(1) U is of finite type over $\mathrm{Spec}\, \mathbb{Z}$;
(2) for every point $y \in U$, the fibre X_y is geometrically irreducible and $\dim_{K(y)} X_y = \dim_K X_K$, where $K(y)$ is the residue field at y; and

(3) for every $y \in U$, the fibre X_y is not contained in B_R and the restriction f_y of f_R to X_y is dominant.

Moreover, after shrinking U, we may assume that for every $y \in U$, f_y is separable. By [BGT16, Proposition 2.5.3.1], there are infinitely many primes $p \geq 3$ such that R can be embedded into $\mathbb{Z}_p \subseteq \mathbb{C}_p^\circ$. This induces an embedding $\operatorname{Spec} \mathbb{Z}_p \to \operatorname{Spec} R$. Denote by $\tau\colon K \hookrightarrow \mathbb{C}_p$ the field embedding. Set $X_{\mathbb{C}_p^\circ} := X_R \times_{\operatorname{Spec} R} \operatorname{Spec} \mathbb{C}_p^\circ$, and $f_{\mathbb{C}_p^\circ} := f_R \times_{\operatorname{Spec} R} \mathrm{id}$. Let $(X_{\mathbb{C}_p}, f_{\mathbb{C}_p})$ and $(X_{\mathbb{F}_p}, f_{\mathbb{F}_p})$ be the generic fibre and special fibre of $(X_{\mathbb{C}_p^\circ}, f_{\mathbb{C}_p^\circ})$. Then $X_{\overline{\mathbb{F}_p}}$ is irreducible and $f_{\mathbb{F}_p}$ is dominant. Denote by $B_{\mathbb{C}_p^\circ}$ the base change of B_R. Then $X_{\overline{\mathbb{F}_p}} \not\subseteq B_{\mathbb{C}_p^\circ}$.

Since (X, f) has SAZD-property, there is a nonempty adelic open subset $A \subseteq X(\mathbf{k})$, such that for every $y \in A$, the orbit of y is well-defined and Zariski dense. We need:

Lemma 4.9. *Let K'/K be a finite extension and $\tau'\colon K' \hookrightarrow \mathbb{C}_p$ a field embedding extending τ. Let V be any nonempty Zariski open subset of $X_{\overline{\mathbb{F}_p}} \setminus B_{\mathbb{C}_p^\circ}$. Then there is $x \in A$ and a field embedding $\overline{\tau'}\colon \mathbf{k} \hookrightarrow \mathbb{C}_p$ extending τ', such that the reduction of $\phi_{\overline{\tau'}}(O_f(x))$ to $X_{\overline{\mathbb{F}_p}}$ is finite and contained in V. Here $\phi_{\overline{\tau'}}\colon X(\mathbf{k}) \hookrightarrow X(\mathbb{C}_p)$ is the embedding induced by $\overline{\tau'}$.*

Assuming Lemma 4.9, we first construct points $x_n \in A$ $(n \geq 1)$, increasing finite extensions K_n $(n \geq 1)$ of K over which x_n is defined, and field embeddings $\overline{\tau_n}\colon \mathbf{k} \hookrightarrow \mathbb{C}_p$ with $\overline{\tau_n}|_{K_{n-1}} = \overline{\tau_{n-1}}|_{K_{n-1}}$, such that the reductions of $\phi_{\overline{\tau_n}}(O_f(x_n))$ to $X_{\overline{\mathbb{F}_p}}$ are finite, contained in $X_{\overline{\mathbb{F}_p}} \setminus B_{\mathbb{C}_p^\circ}$ and disjoint. For these x_n, we have $\phi_{\overline{\tau_n}}(O_f(x_n)) = \phi_{\overline{\tau_m}}(O_f(x_n))$ $(m \geq n)$. Hence the orbits of x_n $(n \geq 1)$ are disjoint, thus, *it proves Proposition 1.10*.

We construct $x_1, \overline{\tau_1}$ by applying Lemma 4.9 to $V = X_{\overline{\mathbb{F}_p}} \setminus B_{\mathbb{C}_p^\circ}$ and $K' = K$. Let K_1 be any finite field extension of K such that x_1 is defined over K_1. Assume that we have constructed $x_n, K_n, \overline{\tau_n}$ $(n = 1, \dots, m)$. Let S_m be the union of the reductions of $\phi_{\overline{\tau_n}}(O_f(x_n)) = \phi_{\overline{\tau_m}}(O_f(x_n))$ $(n = 1, \dots, m)$, which is a finite subset of $X_{\overline{\mathbb{F}_p}} \setminus B_{\mathbb{C}_p^\circ}$. We construct $x_{m+1}, \overline{\tau_{m+1}}$ by applying Lemma 4.9 to $V = X_{\overline{\mathbb{F}_p}} \setminus (B_{\mathbb{C}_p^\circ} \cup S_m)$ and $K' = K_m$. Let K_{m+1} be any finite field extension of K_m such that x_{m+1} is defined over K_{m+1}. □

Proof of Lemma 4.9. Applying [Ame11, Corollary 2] to the rational self-map $f_{\mathbb{F}_p}|_V\colon V \dashrightarrow V$, there exists a periodic point $\bar{x} \in V$ whose orbit under $f_{\mathbb{F}_p}$ is contained in V. Let U be the p-adic open subset of $X(\mathbb{C}_p)$ of points whose reduction is $\bar{x}$. Let $X_{K'}(\tau', U)$ be the basic adelic subset over K' associated to τ' and U as defined in [Xie22, Section 3.11]. It is a nonempty adelic open subset of $X(\mathbf{k})$.

Since X is irreducible, $A \cap X_{K'}(\tau', U) \neq \emptyset$. Pick $x \in A \cap X_{K'}(\tau', U)$. Then the orbit of x is well-defined and Zariski dense. By definition of $X_{K'}(\tau', U)$, some field embedding

$\overline{\tau'}\colon \mathbf{k} \hookrightarrow \mathbb{C}_p$ extends τ' with $\phi_{\overline{\tau'}}(x) \in U$. Since the reduction of $\phi_{\overline{\tau'}}(x)$ to $X_{\overline{\mathbb{F}_p}}$ is $\overline{x}$ and the orbit of $\overline{x}$ is finite and contained in V, this proves Lemma 4.9 and also Proposition 1.10. □

References

[AC08] E. Amerik and F. Campana, *Fibrations méromorphes sur certaines variétés à fibré canonique trivial*, Pure Appl. Math. Q. **4** (2008), no. 2, 509–546.

[Ame11] E. Amerik, *Existence of non-preperiodic algebraic points for a rational self-map of infinite order*, Math. Res. Lett. **18** (2011), no. 02, 251–256.

[BdFF12] S. Boucksom, T. de Fernex, and C. Favre, *The volume of an isolated singularity*, Duke Math. J. **161** (2012), no. 8, 1455–1520.

[BG17] A. Broustet and Y. Gongyo, *Remarks on log Calabi-Yau structure of varieties admitting polarized endomorphisms*, Taiwanese J. Math. **21** (2017), no. 3, 569–582.

[BGT14] J. P. Bell, D. Ghioca, and T. J. Tucker, *Applications of p-adic analysis for bounding periods of subvarieties under étale maps*, Int. Math. Res. Not. IMRN **2015** (2014), no. 11, 3576–3597.

[BGT16] ———, *The dynamical Mordell–Lang conjecture*, Mathematical Surveys and Monographs, vol. 210, American Mathematical Society, 2016.

[BH14] A. Broustet and A. Höring, *Singularities of varieties admitting an endomorphism*, Math. Ann. **360** (2014), no. 1-2, 439–456.

[CMZ20] P. Cascini, S. Meng, and D.-Q. Zhang, *Polarized endomorphisms of normal projective threefolds in arbitrary characteristic*, Math. Ann. **378** (2020), 637–665.

[DNT15] T.-C. Dinh, V.-A. Nguyên, and T. T. Truong, *Equidistribution for meromorphic maps with dominant topological degree*, Indiana Univ. Math. J. **64** (2015), no. 6, 1805–1828.

[Deb01] O. Debarre, *Higher-dimensional algebraic geometry*, Universitext, Springer-Verlag, 2001.

[Fuj11] O. Fujino, *Fundamental theorems for the log minimal model program*, Publ. Res. Inst. Math. Sci. **47** (2011), no. 3, 727–789.

[Fuj21] ———, *Minimal model theory for log surfaces in Fujiki's class*, Nagoya Math. J. **244** (2021), 256–282.

[GS17] D. Ghioca and T. Scanlon, *Density of orbits of endomorphisms of abelian varieties*, Trans. Amer. Math. Soc. **369** (2017), no. 1, 447–466.

[GS19] D. Ghioca and M. Satriano, *Density of orbits of dominant regular self-maps of semiabelian varieties*, Trans. Amer. Math. Soc. **371** (2019), no. 9, 6341–6358.

[Gue05] V. Guedj, *Ergodic properties of rational mappings with large topological degree*, Ann. of Math. **161** (2005), no. 3, 1589–1607.

[Har77] R. Hartshorne, *Algebraic geometry*, Grad. Texts in Math., vol. 52, Springer-Verlag, 1977.

[Iit82] S. Iitaka, *Algebraic geometry – An Introduction to Birational Geometry of Algebraic Varieties*, Grad. Texts in Math., vol. 76, Springer-Verlag, 1982.

[KM98] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, Cambridge University Press, 1998.

[Kol96] J. Kollár, *Rational curves on algebraic varieties*, Vol. 32, Springer-Verlag, 1996.

[Men20] S. Meng, *Building blocks of amplified endomorphisms of normal projective varieties*, Math. Z. **294** (2020), no. 3, 1727–1747.

[Mil06] J. Milnor, *Dynamics in one complex variable. (AM-160): Third Edition*, Princeton University Press, 2006.

[MS14] A. Medvedev and T. Scanlon, *Invariant varieties for polynomial dynamical systems*, Ann. of Math. **179** (2014), 81–177.

[MY22] Y. Matsuzawa and S. Yoshikawa, *Kawaguchi-Silverman conjecture for endomorphisms on rationally connected varieties admitting an int-amplified endomorphism*, Math. Ann. **382** (2022), 1681–1704.

[MZ18] S. Meng and D.-Q. Zhang, *Building blocks of polarized endomorphisms of normal projective varieties*, Adv. Math. **325** (2018), 243–273.

[MZ] ———, *Kawaguchi-Silverman conjecture for surjective endomorphisms*, Documenta Mathematica (to appear). arXiv:**1908.01605**.

[Nak02] N. Nakayama, *Ruled surfaces with non-trivial surjective endomorphisms*, Kyushu J. Math. **56** (2002), no. 2, 433–446.

[Nak17] ———, *A variant of Shokurov's criterion of toric surface*, Algebraic varieties and automorphism groups, Advanced Studies in Pure Mathematics, 2017, pp. 287–392. RIMS preprints **1825**.

[Nak20a] ———, *On normal Moishezon surfaces admitting non-isomorphic surjective endomorphisms*, RIMS Preprints **1923** (2020).

[Nak20b] ———, *On the structure of normal projective surfaces admitting non-isomorphic surjective endomorphisms*, RIMS Preprints **1934** (2020).

[Sak84] F. Sakai, *Weil divisors on normal surfaces*, Duke Mathematical Journal **51** (1984), 877–887.

[Wah90] J. Wahl, *A characteristic number for links of surface singularities*, J. Amer. Math. Soc. **3** (1990), no. 3, 625–637.

[Xie17] J. Xie, *The existence of Zariski dense orbits for polynomial endomorphisms of the affine plane*, Compos. Math. **153** (2017), no. 8, 1658–1672.

[Xie22] ———, *The existence of Zariski dense orbits for endomorphisms of projective surfaces (with an appendix in collaboration with Thomas Tucker)*, J. Amer. Math. Soc. (2022).

[Zha10] D.-Q. Zhang, *Polarized endomorphisms of uniruled varieties. with an appendix by Y. Fujimoto and N. Nakayama*, Compos. Math. **146** (2010), no. 1, 145–168.

[Zha16] ———, *n-dimensional projective varieties with the action of an abelian group of rank $n-1$*, Trans. Amer. Math. Soc. **368** (2016), no. 12, 8849–8872.

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