THE MULTIPLIER SPECTRUM MORPHISM IS GENERICALLY INJECTIVE

ZHUCHAO JI AND JUNYI XIE

Abstract. In this paper, we consider the multiplier spectrum of periodic points, which is a natural morphism defined on the moduli space of rational maps on the projective line. A celebrated theorem of McMullen asserts that aside from the well-understood flexible Lattès family, the multiplier spectrum morphism is quasi-finite. In this paper, we strengthen McMullen's theorem by showing that the multiplier spectrum morphism is generically injective. This answers a question of McMullen and Poonen.

1. Introduction

1.1. **The multiplier spectrum morphism.** For $d \geq 2$, let $\mathrm{Rat}_d(\mathbb{C})$ be the space of degree d rational maps on $\mathbb{P}^1(\mathbb{C})$. It is a smooth irreducible affine variety of dimension $2d+1$ [Sil12]. A rational map is called *Lattès* if it is semi-conjugate to an endomorphism on an elliptic curve. A Lattès map f is called *flexible Lattès* if one can continuously vary the complex structure of the elliptic curve to get a family of Lattès maps passing through f. The structure of flexible Lattès maps is well-understood [Mil06, Lemma 5.5]. Let $\mathrm{FL}_d(\mathbb{C}) \subseteq \mathrm{Rat}_d(\mathbb{C})$ be the locus of flexible Lattès maps, which is Zariski closed in $\mathrm{Rat}_d(\mathbb{C})$. The group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ acts on $\mathrm{Rat}_d(\mathbb{C})$ by conjugacy. The geometric quotient

$$\mathcal{M}_d(\mathbb{C}) := \mathrm{Rat}_d(\mathbb{C})/\mathrm{PGL}_2(\mathbb{C})$$

is the (coarse) *moduli space* of endomorphisms of degree d [Sil12]. The moduli space $\mathcal{M}_d(\mathbb{C}) = \mathrm{Spec}\,(\mathcal{O}(\mathrm{Rat}_d(\mathbb{C})))^{\mathrm{PGL}_2(\mathbb{C})}$ is an irreducible affine variety of dimension $2d-2$ [Sil07, Theorem 4.36(c)], which is also a complex orbifold [Mil93], [Mil11]. Let $\Psi : \mathrm{Rat}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ be the quotient morphism. We set

$$\mathcal{M}_d^*(\mathbb{C}) := \mathcal{M}_d(\mathbb{C}) \setminus \Psi(\mathrm{FL}_d(\mathbb{C})).$$

We note that $\mathrm{FL}_d(\mathbb{C}) = \emptyset$ when d is not a square number, and $\Psi(\mathrm{FL}_d(\mathbb{C}))$ is an algebraic curve when d is a square number.

Date: March 14, 2024.

As $\mathcal{M}_d$ is affine, to understand the geometry of $\mathcal{M}_d$, we only need to understand its function ring. There is a natural dynamically interesting family of morphisms from $\mathcal{M}_d(\mathbb{C})$ to the affine spaces, which we call the *multiplier spectrum morphisms*. These morphisms give a natural system of functions on $\mathcal{M}_d(\mathbb{C})$, which can be viewed as analogies of the theta functions on moduli space of elliptic curves (c.f. [Sil23, Remark 5.9]).

We now recall the construction. For every $f \in \mathrm{Rat}_d(\mathbb{C})$ and $n \geq 1$, f^n has exactly $N_n = d^n + 1$ fixed points counted with multiplicity. The *multiplier* of a f^n-fixed point x is the differential $df^n(x) \in \mathbb{C}$. Using elementary symmetric polynomials, their multipliers define a point $S_n(f) \in \mathbb{C}^{N_n}$. The *multiplier spectrum* of f is the sequence $S_n(f), n \geq 1$. Since S_n take the same value in a conjugacy class of rational maps, each S_n defines a morphism $S_n : \mathcal{M}_d(\mathbb{C}) \to \mathbb{C}^{N_n}$. Let $\tau_{d,n}$ be the morphism

$$\begin{aligned} \tau_{d,n} : \mathcal{M}_d(\mathbb{C}) &\to \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_n}, \\ [f] &\mapsto (S_1(f), \ldots, S_n(f)). \end{aligned}$$

Here for $f \in \mathrm{Rat}_d(\mathbb{C})$, we define $\tau_{d,n}(f) := \tau_{d,n}([f])$.

For each $n \geq 1$, set

$$R_n := \{([f],[g]) \in \mathcal{M}_d(\mathbb{C})^2 |\ \tau_{d,n}([f]) = \tau_{d,n}([g])\}.$$

Then R_n are decreasing Zariski closed subsets of $\mathcal{M}_d(\mathbb{C})^2$. By the Noetherianity, the sequence R_n is stable for n sufficiently large. Hence there exists a minimal positive integer m_d such that $\tau_{d,m_d}(f) = \tau_{d,m_d}(g)$ implies that $\tau_{d,n}(f) = \tau_{d,n}(g)$ for every $n \geq 1$, i.e. f and g have the same multiplier spectrum. We define

$$\tau_d := \tau_{d,m_d}.$$

It is well-known that elements in an irreducible component of $\mathrm{FL}_d(\mathbb{C})$ have the same multiplier spectrum.

The following remarkable theorem of McMullen [McM87] says that outside $\mathrm{FL}_d(\mathbb{C})$, the multiplier spectrum determines the conjugacy class of rational maps up to finitely many choices.

Theorem 1.1 (McMullen)**.** *For every $d \geq 2$, the morphism*

$$\tau_d : \mathcal{M}_d^*(\mathbb{C}) \to \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_{m_d}}$$

is quasi-finite.

1.2. **The multiplier spectrum morphism is generically injective.**

Definition 1.2. For a point $x \in \mathcal{M}_d(\mathbb{C})$, we say that τ_d is injective at x if $\tau_d^{-1}(\tau_d(x)) = \{x\}$. For a subset $X \subset \mathcal{M}_d(\mathbb{C})$, we say that τ_d is injective on X if τ_d is injective at every $x \in X$.

We quote the following question about the injectivity of τ_d from McMullen [McM87, Page 489]:

Noetherian properties imply there are an N and an M such that $E_1(R), \ldots, E_N(R)$ determine R up to at most M choices, ... is R determined uniquely?[1]

It turns out that τ_d is **not** always injective on $\mathcal{M}_d^*(\mathbb{C})$. Silverman showed that if f is a rigid Lattès map defined over a number field K whose class number is larger than 1, then τ_d is not injective at $[f]$ [Sil07, Theorem 6.62].

Another construction was introduced by Pakovich [Pak19a]. Let f be a rational map. Following Pakovich [Pak19a], for any decomposition $f = h_1 \circ h_2$ into a composition of rational maps, where h_1 and h_2 have degree at least 2, we say that the rational map $\tilde{f} := h_2 \circ h_1$ is an *elementary transformation* of f. We say that rational maps f and g are *equivalent* if there exists a chain of elementary transformations between f and g. One can show that if f and g are equivalent then $\tau_d(f) = \tau_d(g)$, see Pakovich [Pak19b, Lemma 2.1].

Even though τ_d is not always injective on $\mathcal{M}_d^*(\mathbb{C})$ as we have seen, one might hope that τ_d is injective at *generic* parameters, i.e. τ_d is injective on a Zariski open subset. By McMullen's theorem (Theorem 1.1), there is an integer m, a non-empty Zariski open subset $U \subset \mathcal{M}_d(\mathbb{C})$ and a Zariski open subset W of the Zariski closure of $\tau_d(U)$ such that $\tau_d^{-1}(W) = U$ and $\tau_d : U \to W$ is a finite map of degree m. The integer m is independent of the choices of U and W, and we call this integer m the degree of τ_d. We denote this degree m by $\deg(\tau_d)$. When $\deg(\tau_d) = 1$, we say τ_d is *generically injective*.

Poonen asked whether τ_d is always generically injective [Sil12, Question 2.43]. The following is our main theorem. It gives an affirmative answer to Poonen's question, hence an affirmative answer to McMullen's question for generic parameters.

Theorem 1.3. *For every $d \geq 2$, the morphism*

$$\tau_d : \mathcal{M}_d(\mathbb{C}) \to \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_{m_d}}$$

is generically injective.

[1] Following McMullen's notations, $E_i(R)$ is the multiplier spectrum of periods i of the rational map R.

As a consequence, the system of functions given by the multiplier spectrum generate the fraction field of $\mathcal{M}_d$.

One can also consider the moduli space of polynomials and the multiplier spectrum morphism on it. For every $d \geq 2$ and $n \geq 1$, we let $\tilde{\tau}_{d,n}$ (resp. $\tilde{\tau}_d$) be the restriction of $\tau_{d,n}$ (resp. τ_d) on the moduli space of polynomials. As the the moduli space of polynomials is a proper Zariski closed subset of $\mathcal{M}_d$, Theorem 1.3 does not implies the injectivity of $\tilde{\tau}_d$ directly. However, our proof of Theorem 1.3 still works in the polynomial setting.

Theorem 1.4. *For every $d \geq 2$, $\tilde{\tau}_d$ is generically injective.*

The proof is of Theorem 1.4 is indeed even easier than the one for Theorem 1.3. To prove Theorem 1.3, a key ingredient is Pakovich's result Theorem 3.3, which works only when $d \geq 4$. So we need to apply Milnor [Mil93] and Gotou [Got23, Theorem 1.2] to treat the $d = 2, 3$ cases. To prove Theorem 1.4, we may replace Theorem 3.3 by [FG22, Theorem 23 and 24], which works for every $d \geq 2$. We leave the proof of Theorem 1.4 for the readers.

While we were preparing this article, we knew from a private communication that Huguin have an independent proof of Theorem 1.4. Indeed Huguin proved the generic injectivity of $\tilde{\tau}_{d,2}$. From the communication with Huguin, we know that Huguin's method is completely different from ours, which is based on Fujimura's result [Fuj07] and on the computations by Gorbovickis [Gor15].

1.3. Ingredients in the proof of Theorem 1.3.

To show Theorem 1.3, we argue by contradiction. Assume that τ_d is not generically injective. We first construct two algebraic families of rational maps f_t and g_t, parametrized by the same algebraic curve V, such that: (1) for every $t \in V$, the images f_t and g_t in $\mathcal{M}_d$ are different; (2) $\tau_d(f_t) = \tau_d(g_t)$ for every $t \in V$. We can further ask that these two families satisfy several geometric and dynamical assumptions, c.f. Lemma 2.5. Our key step is to show that f_t and g_t are *intertwined* (c.f. Definition 3.1) for all but finitely many $t \in V$. The main ingredient of the proof for this step is the following variant of the Dynamical André-Oort (DAO) conjecture on the distribution of *postcritically finite* (PCF) maps.

Theorem 1.5. *Let $d \geq 2$ and let f_V, g_V be two degree d non-isotrivial algebraic family parametrized by the same irreducible algebraic curve V, and f_V, g_V are not family of flexible Lattès maps. Assume that there are infinitely many $t \in V$ such that f_t and g_t are both PCF. Then for all but finitely many $t \in V$, f_t and g_t are intertwined.*

From the above construction we can get a Zariski dense set of *simple* (c.f. Definition 3.2) rational maps $[f]$ in $\mathcal{M}_d(\mathbb{C})$ which is intertwined with a simple rational map g with $[g] \neq [f]$. We then use a result of Pakovich (c.f. Theorem 3.3) to get a contradiction.

We note that the DAO conjecture of Baker and DeMarco [BDM13] is considered an important conjecture in arithmetic dynamics, however not many of its applications are known. Theorem 1.3 is an application of the DAO conjecture.

In a private communication, DeMarco and Mavraki told us that when they preparing lectures in Harvard and Toronto in November 2023 [DM23], they find a simplification of our proof of Theorem 1.3. They observed that to prove Theorem 1.3, one may replace Theorem 1.5 by a weaker lemma (c.f. Lemma 3.8). This lemma can be proved following the same strategy of Theorem 1.5, but there are two important steps become easier. See Section 3.4 for details.

1.4. **Previous results on the degree of the multiplier spectrum morphism.** In [Gor15], Gorbovickis showed that $\tau_{d,n}$ is generically quasi-finite for $d \geq 2$ and $n \geq 3$. A recursive formula for some upper bound of $\tau_{d,n}$ was obtained by Schmitt in the preprint version [Sch16]. An explicit upper bound of $\tau_{d,n}$ was obtained in Gotou's recent work [Got23].

When $d = 2$, Milnor [Mil93] showed that $\tau_{2,1}$ is in fact injective on $\mathcal{M}_d(\mathbb{C})$ (see also [Sil12, Theorem 2.45]). In particular Theorem 1.3 holds when $d = 2$. When $d = 3$, Gotou showed that $\tau_{3,2}$ is generically injective (but not injective) [Got23, Theorem 1.2]. This was mentioned in [HT13a] which is the errata for [HT13b]. Previously there was no result about the generic injectivity of τ_d when $d \geq 4$.

One can also consider the moduli space of polynomials and the multiplier spectrum morphism on it. For every $d \geq 2$ and $n \geq 1$, we let $\tilde{\tau}_{d,n}$ be the restriction of $\tau_{d,n}$ on the moduli space of polynomials. In this case $\tilde{\tau}_{d,1}$ is generically quasi-finite (while $\tau_{d,1}$ is never generically quasi-finite, except when $d = 2$). Fujimura showed that $\deg(\tilde{\tau}_{d,1}) = (d-2)!$ [Fuj07]. The fiber structure of $\tilde{\tau}_{d,1}$ was studied by Sugiyama [Sug17], [Sug20], [Sug23].

1.5. **Further results and problems.**

McMullen's conjecture for the hyperbolicity of the moduli space $\mathcal{M}_f$. For a rational map $f \in \mathrm{Rat}_d(\mathbb{C})$, McMullen and Sulllivan introduced the Teichmüller space $\mathcal{T}_f$ and the moduli space $\mathcal{M}_f$ [MS98]. We refer the readers to [MS98] and [Ast17] for the definitions of $\mathcal{T}_f$ and $\mathcal{M}_f$. The

Teichmüller space $\mathcal{T}_f$ is a contractible complex manifold of dimension not larger than $2d-2$. The *modular group* Mod_f acts properly discontinuously on $\mathcal{T}_f$. The moduli space is the quotient $\mathcal{M}_f := \mathcal{T}_f/\mathrm{Mod}_f$. There is a natural holomorphic injection $\phi : \mathcal{M}_f \to \mathcal{M}_d(\mathbb{C})$, such that $\phi(\mathcal{M}_f)$ is the set containing all quasiconformal conjugacy class of f.

In [McM87, Page 473], McMullen conjectured that for every $f \in \mathrm{Rat}_d(\mathbb{C})$ that are not flexible Lattès, bounded holomorphic functions separate points on $\mathcal{M}_f$.

The above property of complex analytic spaces is sometimes called *Carathéodory hyperbolic* in the literature, which is a strong hyperbolic condition that implies Kobayashi hyperbolicity. In a forthcoming paper [JXar], we solve McMullen's conjecture using the multiplier spectrum. We show a stronger result, namely for every $f \in \mathrm{Rat}_d(\mathbb{C})$ that is not flexible Lattès, we show that there is a holomorphic injection $\tilde{\phi} : \mathcal{M}_f \to X_f$, where X_f is a normal affine complex algebraic variety of dimension $2d-2$, such that $\tilde{\phi}(\mathcal{M}_f)$ is precompact in X_f. Moreover by applying Theorem 1.3, we get a explicit description of X_f when $\dim \mathcal{M}_f = 2d-2$ and $d \geq 4$.

Injective locus of τ_d. We have seen before that there are two mechanisms that can produce rational maps with the same multiplier spectrum, one from Lattès maps, and another one from equivalent rational maps. Pakovich asked [Pak19a, Problem 3.1] whether these are the only obstructions of the injectivity of τ_d. We conjecture that it is indeed the case.

Conjecture 1.6. Let f, g be rational maps of degree $d \geq 2$ such that the conjugacy classes of f and g are different. Assume that $\tau_d(f) = \tau_d(g)$, then one of the followings holds:

(i) f and g are Lattès maps;
(ii) f is equivalent to g.

Generic injectivity of multiplier spectrum of small periods. In Theorem 1.1 and Theorem 1.3, the definition of τ_d requires the use of periodic points of periods not exceeding the number m_d, whose precise value is not effectively-known and is probably very large. It is interesting to know whether we can get generic injectivity using only periodic points of small periods. By dimension counting, $\tau_{d,1}$ is never generically injective except when $d = 2$. However, we believe the following is true.

Conjecture 1.7. For every $d \geq 2$, the morphism

$$\tau_{d,2} : \mathcal{M}_d(\mathbb{C}) \to \mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$$

is generically injective.

Generic injectivity of the length spectrum. Replace the multipliers by their norm in the definition of multiplier spectrum, one gets the definition of the *length spectrum*. More precisely, for every $f \in \mathrm{Rat}_d(\mathbb{C})$ and $n \geq 1$, we denote by $L_n(f) \in \mathbb{R}_{\geq 0}^{N_n}$ the elements corresponding to the values of the elementary symmetric polynomials at the point $(|\lambda_1|, \ldots, |\lambda_{N_n}|) \in \mathbb{R}_{\geq 0}^{N_n}$ where $\lambda_i, i = 1, \ldots, N_n$ are the multipliers of all f^n-fixed points. The length spectrum of f is defined by the sequence $L_n(f), n \geq 1$. A priori, the length spectrum contains less information than the multiplier spectrum. In [JX23b], a parallel result of Theorem 1.1 has been shown, that is outside the flexible Lattès family, the length spectrum determines the conjugacy class of rational maps up to finitely many choices [JX23b, Theorem 1.5]. We believe the following parallel result of Theorem 1.3 for length spectrum is true. Note that $\mathcal{M}_d$ is defined over $\mathbb{Q}$ (hence over $\mathbb{R}$).

We propose the following conjecture as a parallel result of our Theorem 1.3.

Conjecture 1.8. For every $d \geq 2$, there is a Zariski closed proper subset $E_d \subset \mathcal{M}_d$ defined over $\mathbb{R}$, such that for every $[f] \notin E_d$, if there is $g \in \mathrm{Rat}_d(\mathbb{C})$ such that f and g have the same length spectrum, then g or $\overline{g}$ (the complex conjugation of g) is contained in the conjugacy class of f.

More precisely, we propose the following description for rational maps with the same length spectrum.

Conjecture 1.9. Let f, g be non-Lattès rational maps of degree $d \geq 2$. If f and g has the same length spectrum, then $\tau_d(f)$ equals to either $\tau_d(g)$ or $\tau_d(\overline{g})$.

Multiplier spectrum for endomorphisms on $\mathbb{P}^N$. For holomorphic endomorphisms on $\mathbb{P}^N$, $N \geq 2$, one can also construct the corresponding moduli space $\mathcal{M}_d^N(\mathbb{C})$ [Sil12], and the multiplier spectrum morphisms exist on $\mathcal{M}_d^N(\mathbb{C})$ [Sil12]. Unlike in dimension one, essentially nothing is known about the multiplier spectrum morphisms when $N \geq 2$. It is of great interest to extend McMullen's quasi-finiteness theorem (Theorem 1.1) and the generic injectivity theorem (Theorem 1.3) to higher dimension.

Acknowledgement. We thank Rin Gotou for sharing us his work on the generic injectivity of $\tau_{3,2}$ and let us know the result of Schmitt [Sch16]. We thank Fedor Pakovich, Valentin Huguin, Gabriel Vigny and Joseph Silverman for their comments and suggestions. We thanks Valentin Huguin for telling us his forthcoming proof for the generic injectivity of $\tilde{\tau}_{d,2}$. We thank Laura DeMarco and Niki Myrto Mavraki for sharing us their simplification of our proof of Theorem 1.3.

The first-named author would like to thank Beijing International Center for Mathematical Research in Peking University for the invitation. The second-named author Junyi Xie is supported by NSFC Grant (No.12271007).

2. Hyperbolic PCF maps of disjoint type

Definition 2.1. A PCF map $f \in \mathrm{Rat}_d(\mathbb{C})$ is called *hyperbolic of disjoint type* if f has $2d-2$ number of distinct super-attracting cycles. Let $\underline{n} := \{n_1, \dots, n_{2d-2}\} \in (\mathbb{N}^*)^{2d-2}$, where $\mathbb{N}^*$ is the set of positive integers. A hyperbolic PCF map of disjoint type $f \in \mathrm{Rat}_d(\mathbb{C})$ is called of type $\underline{n}$ if f has $2d-2$ distinct super-attracting cycles of respective exact periods $n_1, \dots, n_{2d-2}$.

For every $\underline{n} := \{n_1, \dots, n_{2d-2}\} \in (\mathbb{N}^*)^{2d-2}$, we let $X_{\underline{n}} \subset \mathcal{M}_d(\mathbb{C})$ be the subset of all conjugacy classes $[f]$ such that f is a hyperbolic PCF map of disjoint type $\underline{n}$. The following theorem was proved by Gauthier-Okuyama-Vigny in [GOV19, Theorem F].

Theorem 2.2. *For every sequence $\underline{n}(k) = (n_1(k), \dots, n_{2d-2}(k)) \in (\mathbb{N}^*)^{2d-2}$ satisfying $\min_{1\le j\le 2d-2} n_j(k) \to +\infty$ when $k \to +\infty$, we have that $\cup_k X_{\underline{n}(k)}$ is Zariski dense in $\mathcal{M}_d(\mathbb{C})$.*

Definition 2.3. Let V be a quasi-projective variety over $\mathbb{C}$. An *algebraic family* on V is an endomorphism f_V on $V \times \mathbb{P}^1$ of the following form.

$$\begin{aligned} f_V : V \times \mathbb{P}^1 &\to V \times \mathbb{P}^1 \\ (t, z) &\mapsto (t, f_t(z)). \end{aligned}$$

We say that f_V has degree d if for every $t \in V$, we have $\deg f_t = d$. For a degree d algebraic family f_V on V, let $\Psi_V : V \to \mathcal{M}_d(\mathbb{C})$ be the morphism sending $t \in V$ to the class of f_t in $\mathcal{M}_d(\mathbb{C})$. We say that f_V is *isotrivial* if $\Psi_V : V \to \mathcal{M}_d(\mathbb{C})$ is locally constant.

The following lemma is a combination of [DF08, Proposition 2.4] and [DeM16, Theorem 1.1].

Lemma 2.4. *Let $f_V : V \times \mathbb{P}^1 \to V \times \mathbb{P}^1$ be a degree d non-isotrivial algebraic family, and let $a : V \to \mathbb{P}^1$ be a marked point. Assume that a is not persistently preperiodic, then there exist infinitely many $t \in V$ such that $a(t)$ is periodic for f_t.*

Proof. By [DeM16, Theorem 1.1], the bifurcation locus $\mathrm{Bif}(f_V, a)$ is a non-empty closed set. Moreover $\mathrm{Bif}(f_V, a)$ is the support of a positive closed current with Hölder continuous potential [DS10, Lemma 1.1], hence $\mathrm{Bif}(f_V, a)$ has positive Hausdorff dimension [Sib99, Theorem 1.7.3], in particular $\mathrm{Bif}(f_V, a)$ is an infinite set. By definition, for every $t_0 \in \mathrm{Bif}(f_V, a)$ and every open neighborhood U of t_0, the family of maps $h_n : U \to \mathbb{P}^1$, $t \mapsto f_t^n(a(t))$ does not form a normal family. By [DF08, Proposition 2.4], there exists $t \in U$ such that $a(t)$ is periodic for f_t. Apply the above construction to infinitely many disjoint open subsets U meeting $\mathrm{Bif}(f_V, a)$, we get infinitely many $t \in V$ such that $a(t)$ is periodic for f_t. $\square$

For every $\underline{n} := \{n_1, \ldots, n_{2d-3}\} \in (\mathbb{N}^*)^{2d-3}$ of $2d-3$ tuples, let $Y_{\underline{n}} \subset \mathcal{M}_d(\mathbb{C})$ be the subset of all conjugacy classes $[f]$ such that f has $2d-3$ distinct super-attracting cycles of respective exact periods $n_1, \ldots, n_{2d-3}$.

Lemma 2.5. *The following statements are true:*

(1) The Zariski closure of $Y_{\underline{n}}$ is a (possibly reducible) algebraic curve provided that it is not empty and $Y_{\underline{n}}$ is Zariski open in its Zariski closure.

(2) The set $\cup_{\underline{n}} Y_{\underline{n}}$ is Zariski dense in $\mathcal{M}_d(\mathbb{C})$.

(3) For every $\underline{n}$ such that $Y_{\underline{n}}$ is non-empty, and for every irreducible component Y of $Y_{\underline{n}}$, there are infinitely many $[f] \in Y$ such that f is a hyperbolic PCF map of disjoint type.

Proof. We first show (1). Passing to a finite surjective morphism $\phi : V \to \mathrm{Rat}_d(\mathbb{C})$, $t \mapsto f_t$, we can choose an algebraic family $f_V : V \times \mathbb{P}^1 \to V \times \mathbb{P}^1$ such that all $2d-2$ critical points in this family can be algebraically parametrized by $c_1, \ldots, c_{2d-2} : V \to \mathbb{P}^1$. For every fixed $\underline{n} := \{n_1, \ldots, n_{2d-3}\} \in (\mathbb{N}^*)^{2d-3}$, we define

$$Z := \left\{ t \in V : f_t^{n_j}(c_j) = c_j, \ \text{ for every } 1 \le j \le 2d-3 \right\}.$$

Then Z is a Zariski closed subset in V. Since $\phi : V \to \mathrm{Rat}_d(\mathbb{C})$ is surjective finite, and $\phi(Z) \subset \mathrm{Rat}_d(\mathbb{C})$ is invariant by $\mathrm{PGL}_2(\mathbb{C})$ conjugacy, $\Psi \circ \phi(Z)$ is Zariski closed in $\mathcal{M}_d(\mathbb{C})$, where $\Psi : \mathrm{Rat}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is the quotient morphism. Moreover by the definition of $Y_{\underline{n}}$, $Y_{\underline{n}}$ is Zariski open in $\Psi \circ \phi(Z)$. For each irreducible component $\tilde{Z}$ of Z, it is the intersection of at most $2d-3$ hypersurfaces, so $\tilde{Z}$ has codimension at most $2d-3$ in V. We claim that $\tilde{Z}$ has codimension $2d-3$ in V. Assume by contradiction it is not the case. Since critical orbit relations are constant along the fibers of the projection $\Psi \circ \phi : V \to \mathcal{M}_d(\mathbb{C})$,

$\Psi \circ \phi(\tilde{Z})$ has codimension $k < 2d-3$ in $\mathcal{M}_d(\mathbb{C})$. In particular the algebraic family $f_{\tilde{Z}}$ (the restriction of f_V on $\tilde{Z}$) is non-isotrivial. Apply Lemma 2.4 to $f_{\tilde{Z}}$ and the marked point $c_{2d-2} : V \to \mathbb{P}^1$, there exists a positive integer m such that the Zariski closed subset

$$W := \left\{ t \in \tilde{Z} : f_t^m(c_{2d-2}) = c_{2d-2} \right\}$$

is non-empty. Moreover, W has codimension at most $k+1 < 2d-2$. Then $\Psi \circ \phi(W)$ has codimension at most $k+1 < 2d-2$, which implies that $\Psi \circ \phi(W)$ contains a positive dimensional hyperbolic PCF family, this contradicts Thurston's rigidity theorem [DH93] which says that the only positive dimensional PCF family in $\mathcal{M}_d(\mathbb{C})$ is the flexible Lattès family. Hence $\tilde{Z}$ must have codimension $2d-3$ in V. This implies $\Psi \circ \phi(Z)$ has pure dimension 1. Then (1) holds since $Y_{\underline{n}}$ is Zariski open in $\Psi \circ \phi(Z)$.

Next we show (2). Since $\cup_{\underline{n}} X_{\underline{n}} \subset \cup_{\underline{n}} Y_{\underline{n}}$, by Lemma 2.2 (2), $\cup_{\underline{n}} Y_{\underline{n}}$ is Zariski dense.

Finally we show (3). By Lemma 2.5 (1) and by Lemma 2.4, there exist infinitely many $[f]$ in $Y_{\underline{n}}$ such that every citical point of f is periodic and f has $2d-3$ distinct super-attracting cycles of respective exact periods $n_1, \dots, n_{2d-3}$. Since the set of conjugacy class $[f]$ such that all critical points of f are periodic with bounded periods is finite, there exist infinitely many $[f]$ in $Y_{\underline{n}}$ such that f has a periodic critical point with exact periods $N > \max_{1 \le j \le 2d-3} n_j$, hence these $[f]$ are hyperbolic PCF maps of disjoint type. This implies (3). □

3. Generic injectivity of the multiplier spectrum morphism

3.1. Intertwined Rational maps.

Definition 3.1. Let $d \geq 2$ and $f, g \in \mathrm{Rat}_d(\mathbb{C})$. We say f and g are *intertwined* if there exists a (maybe reducible) algebraic curve $Z \subset \mathbb{P}^1 \times \mathbb{P}^1$ whose projections to both axis are onto, and Z is invariant by the map $f \times g : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$.

Definition 3.2. A rational map $f \in \mathrm{Rat}_d(\mathbb{C})$ is called *simple* if the f has exactly $2d-2$ number of critical values.

Simple rational maps form a Zariski open subset of $\mathrm{Rat}_d(\mathbb{C})$ for every $d \geq 2$.

The following theorem was obtained by Pakovich [Pak21] by combining several results in [Pak21]. We give a proof here for completeness.

Theorem 3.3. *Let $d \geq 4$. Then there exists a Zariski open subset W_d of $\mathrm{Rat}_d(\mathbb{C})$ such that :*

(i) *Elements in W_d are simple;*
(ii) *Assume that f and g are intertwined, where $f \in W_d$ and g is simple, then f and g are in the same conjugacy class.*

Proof. Let $d \geq 4$, by [Pak21, Theorem 1.2 and Lemma 3.7], there exists a Zariski open set $W_d \subset \mathrm{Rat}_d(\mathbb{C})$ such that: (1) W_d is contained in the set of simple rational maps, (2) if $f \in W_d$ and f share the maximal entropy measure for some $g \in \mathrm{Rat}_d(\mathbb{C})$, then $g = f$. We are going to show that the set W_d satisfies Theorem 3.3 (ii).

Assume that $f \in W_d$, and f is intertwined with a simple rational map $g \in \mathrm{Rat}_d(\mathbb{C})$. By [Pak21, Theorem 1.4], there exist $m \geq 1$ and $\phi \in \mathrm{PGL}_2(\mathbb{C})$ such that $(\phi g \phi^{-1})^m = f^m$. This implies f and $\phi g \phi^{-1}$ have the same maximal entropy measure, hence we have $f = \phi g \phi^{-1}$. Hence f and g are in the same conjugacy class. □

3.2. **A variant of the DAO conjecture.** The following theorem is a variant of the main theorem in [JX23a]. In [JX23a, Theorem 1.2], the proof is for the product map $f_V \times f_V$ on $\mathbb{P}^1 \times \mathbb{P}^1$, here we replace this product map by $f_V \times g_V$ on $\mathbb{P}^1 \times \mathbb{P}^1$. The proof of Theorem 3.4 follows the same lines as previous proof of [JX23a, Theorem 1.3]. Indeed the proof of Theorem 3.4 is easier since we only need to construct a dynamical relation between f_V and g_V.

Theorem 3.4 (=Theorem 1.5)**.** *Let $d \geq 2$ and let f_V, g_V be two degree d non-isotrivial algebraic family parametrized by the same irreducible algebraic curve V, and f_V, g_V are not family of flexible Lattès maps. Assume that there are infinitely many $t \in V$ such that f_t and g_t are both PCF. Then for all but finitely many $t \in V$, f_t and g_t are intertwined.*

Proof. Our proof is a modification of the one for [JX23a, Theorem 1.3].

We first reduce Theorem 3.4 to the case such that V, f_V, g_V are defined over $\overline{\mathbb{Q}}$. To show this, let $\Psi_f, \Psi_g : V \to \mathcal{M}_d$ be the morphisms sending t to the conjugacy classes of f_t and g_t respectively. Denote by $\psi : V \to \mathcal{M}_d \times \mathcal{M}_d$ the morphism sending t to $(\Psi_f(t), \Psi_g(t))$ and by Γ the Zariski closure of its image. Since there are infinitely many $t \in V$ such that f_t and g_t are both PCF and f_V, g_V are not family of flexible Lattès maps, by Thurston's rigidity theorem for PCF maps [DH93], Γ is defined over $\overline{\mathbb{Q}}$. Consider the natural morphism $\Psi \times \Psi : \mathrm{Rat}_d \times \mathrm{Rat}_d \to \mathcal{M}_d \times \mathcal{M}_d$. There is an algebraic curve V' in $(\Psi \times \Psi)^{-1}(\Gamma)$ defined over $\overline{\mathbb{Q}}$ such that $\Psi \times \Psi(V')$ is dense in Γ. Then V' defines algebraic families $f'_{V'}, g'_{V'}$ of degree d maps parametrized by V'. Both of them are defined over $\overline{\mathbb{Q}}$. To prove Theorem 3.4 for f_V, g_V, we only need to prove it for $f'_{V'}, g'_{V'}$. So we may assume now that V, f_V, g_V are defined over $\overline{\mathbb{Q}}$.

After replacing V by its normalization and then by some finite ramification cover, we may assume that V is smooth and both f_V and g_V have exactly $2d-2$ marked critical points $a_1, \dots, a_{2d-2}$ and $b_1, \dots, b_{2d-2}$.

Now we follow the notations in [JX23a]. We denote by μ_{f,a_i}, μ_{g,b_i} the bifucation measures on $V(\mathbb{C})$ for pairs (f_V, a_i) and (g_V, b_i) respectively. By [DF08, Theorem 2.5] (and DeMarco [DeM16]), μ_{f_V,a_i} (resp. μ_{g_V,b_i}) is vanishes if and only if a_i (resp. b_i) is preperiodic. In this case, they are called *passive*. Otherwise, they are called *active*. Let $\mu_{f_V,\mathrm{bif}} := \sum_{i=1}^{2d-2} \mu_{f_V,a_i}$ and $\mu_{g_V,\mathrm{bif}} := \sum_{i=1}^{2d-2} \mu_{g_V,b_i}$ be the bifurcation measures for f_V and g_V respectively. By Thurston's rigidity theorem for PCF maps [DH93], both $\mu_{f_V,\mathrm{bif}}$ and $\mu_{g_V,\mathrm{bif}}$ are non-zero. We may assume that a_1 and b_1 are active.

By [JX23a, Corollary 2.4], for every active a_i (resp. b_i), μ_{f_V,a_i} (resp. μ_{g_V,b_i}) and $\mu_{f_V,\mathrm{bif}}$ (resp. $\mu_{g_V,\mathrm{bif}}$) are proportional. Moreover the proof of [JX23a, Corollary 2.4] implies that $\mu_{f_V,\mathrm{bif}}$ and $\mu_{g_V,\mathrm{bif}}$ are proportional. Let μ be the probability measure on $V(\mathbb{C})$ which is proportional to $\mu_{f_V,\mathrm{bif}}$ (hence $\mu_{g_V,\mathrm{bif}}$).

Since aside from the flexible Lattès locus, the exceptional maps [2] are isolated in $\mathcal{M}_d(\mathbb{C})$. As μ has continuous potential, we have the following property:

(1) For μ-a.e. point $t \in V(\mathbb{C})$, both f_t and g_t are non-exceptional.

Let $\mathrm{Corr}(\mathbb{P}^1)_*^{f_t\times g_t}$ be the set of $f_t \times g_t$-invariant Zariski closed subsets $\Gamma_t \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ of pure dimension 1 such that both $\pi_1|_{\Gamma_t}$ and $\pi_2|_{\Gamma_t}$ are finite, where π_1, π_2 are the first and the second projections. Let $\mathrm{Corr}^\flat(\mathbb{P}^1_V)_*^{f_V\times_V g_V}$ be the set of $f_V \times_V g_V$-invariant Zariski closed subsets $\Gamma \subseteq V \times (\mathbb{P}^1 \times \mathbb{P}^1)$ which is flat over V and whose generic fiber is in $\mathrm{Corr}(\mathbb{P}^1_\eta)_*^{f_\eta\times g_\eta}$, where η is the generic point of V. In general, a correspondence $\Gamma_t \in \mathrm{Corr}(\mathbb{P}^1)_*^{f_t\times g_t}$ may not be contained in any correspondence in $\mathrm{Corr}^\flat(\mathbb{P}^1_V)_*^{f_V\times_V g_V}$. On the other hand, by the proof of [JX23a, Proposition 3.11], it is the case if t is *transcendental* i.e. $t \in V(\mathbb{C}) \setminus V(\overline{\mathbb{Q}})$. As μ has continuous potential, μ-a.e. point $t \in V(\mathbb{C})$ is transcendental. We then get the following property:

(2) For μ-a.e. point $t \in V(\mathbb{C})$, every $\Gamma_t \in \mathrm{Corr}(\mathbb{P}^1)_*^{f_t\times g_t}$ is contained in some correspondence in $\mathrm{Corr}^\flat(\mathbb{P}^1_V)_*^{f_V\times_V g_V}$.

[2] As in [JX23b, Section 1.1], we call g *exceptional* if it is a Lattès map or semiconjugates to a monomial map.

Next, we consider some typical non-uniformly hyperbolic conditions. See [JX23a, Definition 4.1, 5.1 and 8.1] for the definitions of such conditions. By [JX23a, Proposition 8.3], there exists $\lambda_0 > 1$ such that

(3) for $\mu-$a.e. point t, f_t, g_t are CE(λ_0) hence TCE(λ_1) for every $1 < \lambda_1 < \lambda_0$, by Przytycki-Rohde [PR98].

By [JX23a, Theorem 4.3], which is essentially due to De Thélin-Gauthier-Vigny [DTGV21],

(4) for $\mu-$a.e. point t, f_t, g_t are PCE(λ) for some $1 < \lambda_2 < d^{1/2}$.

By [JX23a, Theorem 4.6],

(5) for $\mu-$a.e. point t, f_t, g_t are PR(s) for some $s > 1/2$.

As we have the conditions (1), (3), (4), (5), we may apply the proof of [JX23a, Proposition 7.8] for the active marked points a_1, b_2 to show that, for μ-a.e. point $t \in V(\mathbb{C})$, there is $\Gamma_t \in \mathrm{Corr}(\mathbb{P}^1)_*^{f_t \times g_t}$. By condition (2), $\mathrm{Corr}^\flat(\mathbb{P}^1_V)_*^{f_V \times_V g_V} \neq \emptyset$. This implies that for all but finitely many $t \in V$, f_t and g_t are intertwined, which concludes the proof. □

3.3. **Proof of the generic injectivity.**

Proof of Theorem 1.3. The case $d = 2$ was proved by Milnor [Mil93] (see also [Sil12, Theorem 2.45]). The case $d = 3$ was proved by Gotou [Got23, Theorem 1.2] (ee also [HT13a] which is the errata for [HT13b]).

Now we assume that $d \geq 4$. Assume by contradiction that τ_d is not generically injective. By Theorem 3.3, there is a non-empty Zariski open subset U of $\mathcal{M}_d^*$ such that for every $f \in \mathrm{Rat}_d(\mathbb{C})$ with $[f] \in U$, f satisfies the two conditions in Theorem 3.3. There is a Zariski open subset W of the Zariski closure of $\tau_d(U)$ such that $\tau_d^{-1}(W) \subseteq U$ and $\tau_d|_{\tau_d^{-1}(W)} : \tau_d^{-1}(W) \to W$ if finite étale of degree at least 2. After shrinking U, we may assume that $U = \tau_d^{-1}(W)$. We fix this Zariski open subset U.

For an algebraic family of rational maps f, we let $\Psi_f : V \to \mathcal{M}_d$ be the morphism sending t to the conjugacy class of f_t. Now we construct two non-isotrivial algebraic families f_V, g_V of degree d rational maps parametrized by the same irreducible algebraic curve V such that the following holds: There exists $\underline{n} := \{n_1, \ldots, n_{2d-3}\} \in (\mathbb{N}^*)^{2d-3}$ such that we have

$$\Psi_f(V) \subset Y_{\underline{n}} \cap U, \text{ and } \Psi_g(V) \subset U, \tag{3.1}$$

where $Y_{\underline{n}}$ is the algebraic curve in Lemma 2.5, and we have

$$\tau_d \circ \Psi_f = \tau_d \circ \Psi_g, \tag{3.2}$$

finally for every $t \in V$, we have

$$\Psi_f(t) \neq \Psi_g(t). \tag{3.3}$$

We now describe this construction. By Lemma 2.5 (2), there exists $\underline{n} := \{n_1, \ldots, n_{2d-3}\} \in (\mathbb{N}^*)^{2d-3}$ such that $Y_{\underline{n}} \cap U \neq \emptyset$. We pick an irreducible component Z of $Y_{\underline{n}}$ meeting U. By Lemma 2.5 (1), Z is an algebraic curve. Set $Z_1 := Z \cap U$. Then $Z_1' := \tau_d(Z_1)$ is an algebraic curve. Consider the fiber product $X := Z_1 \times_{Z_1'} (\tau_d|_U)^{-1}(Z_1')$. Recall that by the construction of U, $\tau_d|_U : U \to W$ is finite étale of degree at least 2. This implies that there is an irreducible component $\tilde{Z}$ of X which is not the diagonal. We denote by $\pi_1 : X \to Z_1$ and $\pi_2 : X \to (\tau_d|_U)^{-1}(Z_1')$ for the first and the second projections. Both of π_1 and π_2 are finite étale. Set $Z_2 := \pi_2(\tilde{Z})$. For every $z \in \tilde{Z}$, $\pi_1(z) \neq \pi_2(z)$.

Pick an irreducible curve C_1' in $\Psi^{-1}(Z_1) \subseteq \mathrm{Rat}_d(\mathbb{C})$, such that $\Psi|_{C_1'} : C_1' \to Z_1$ is dominant, where $\Psi : \mathrm{Rat}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is the quotient morphism. Pick an irreducible component C_1 of $C_1' \times_{Z_1} \tilde{Z}$. Let ϕ_1, ϕ_2 be the projection to the first and the second coordinates. We then get an algebraic family h_1 of degree d rational maps parametrized by C_1 such that for every $t \in C_1$, the map f_t is the one given by $\phi_1(t) \in \mathrm{Rat}_d(\mathbb{C})$. Then the morphism $\Psi_1 : C_1 \to \mathcal{M}_d(\mathbb{C})$ sending t to the conjugacy class of f_t is the composition of $\phi_2 : C_1 \to \tilde{Z}$ and $\pi_1 : \tilde{Z} \to Z_1 \subseteq \mathcal{M}_d(\mathbb{C})$.

Similarly, we get an algebraic family h_2 of degree d rational maps parametrized by an irreducible curve C_2 and a dominant morphism $\phi_2' : C_2 \to Z_2$ such that $\Psi_2 : C_2 \to \mathcal{M}_d(\mathbb{C})$ sending t to the conjugacy class of f_t is the composition of $\phi_2' : C_2 \to \tilde{Z}$ and $\pi_2 : \tilde{Z} \to Z_2 \subseteq \mathcal{M}_d(\mathbb{C})$.

Pick an irreducible component V of $C_1 \times_{\tilde{Z}} C_2$ and denote by $\beta_i, i = 1, 2$ the projections to the first and the second coordinates. They induce two algebraic families f_V, g_V of degree d maps parametrized by the same irreducible algebraic curve V. The image of the morphism $\Psi_f : V \to \mathcal{M}_d(\mathbb{C})$ sending t to the conjugacy class of f_t is Zariski dense in Z_1 and the image of $\Psi_g : V \to \mathcal{M}_d(\mathbb{C})$ sending t to the conjugacy class of g_t is Zariski dense in Z_2. Hence f_V and g_V are non-isotrivial. By our construction (3.1) automatically holds. Moreover we have

$$\tau_d \circ \Psi_f = \tau_d \circ \Psi_g.$$

As for every $z \in \tilde{Z}$, $\pi_1(z) \neq \pi_2(z)$, for every $t \in V$, we have

$$\Psi_f(t) \neq \Psi_g(t).$$

Thus we complete the construction.

We need the following lemma.

Lemma 3.5. *Let $d \geq 2$ and let $f, g \in \mathrm{Rat}_d(\mathbb{C})$ have the same multiplier spectrum. If f is a hyperbolic PCF map of disjoint type $\underline{n} := \{n_1, \ldots, n_{2d-2}\} \in (\mathbb{N}^*)^{2d-2}$, then g is also a hyperbolic PCF map of disjoint type $\underline{n}$.*

Proof. We will show that for every $f \in \mathrm{Rat}_d(\mathbb{C})$, we can read the number of the critical cycles and length of them. Clearly, this implies Lemma 3.5.

For $n \geq 1$, we view $S_n(f)$ as the multi-set $\{df^n(x) |\ x \in \mathrm{Fix}(f^n)\}$. It has exactly $d^n + 1$ elements. We denote by u_n the number of zeros in $S_n(f)$. It is clear that the sequence u_n is determined by the multiplier spectrum of f. For every $l \geq 1$, denote by m_l the number of critical cycles of length l. It is clear that the sequence $m_l, l \geq 1$ determines the number of the critical cycles and length of them. So we only need to read the sequence $m_l, l \geq 1$ from $u_n, \geq 1$. It is clear that $u_n = \sum_{l|n} m_l$. By Mobius inversion formula, we get $m_n = \sum_{l|n} \mu(l) u_{n/l}$, where $\mu : \mathbb{N}^* \to \{-1, 0, 1\}$ is the Möbius function. This concludes the proof. $\square$

We continue the proof of Theorem 1.3. By Lemma 2.5 (3), there are infinitely many $t \in V$ such that f_t is a hyperbolic PCF map of disjoint type. By (3.2), f_t and g_t have the same multiplier spectrum, hence by Lemma 3.5, for such t, g_t is also a hyperbolic PCF map of disjoint type. By Theorem 3.4, after shrinking V, f_t and g_t are intertwined for every $t \in V$. Since (3.1) holds and f_t and g_t are intertwined for every $t \in V$, by Theorem 3.3, $\Psi_f(t) = \Psi_g(t)$ for every $t \in V$. This contradicts (3.3). We then conclude the proof. $\square$

Remark 3.6. In the proof Theorem 1.3, we do not need the full strength of Lemma 3.5. One can replace it with the following fact: if two rational maps f and g have the same multiplier spectrum such that f is PCF, then g is also PCF. This is a consequence of [JXZ23, Theorem 1.12].

Remark 3.7. There is also a way to prove Theorem 1.3 considering all PCF maps instead of hyperbolic PCF maps of disjoint type. In such a way, we may replace Gauthier-Okuyama-Vigny's result [GOV19, Theorem F] (c.f. Theorem 2.2) by the well-known fact that the PCF parameters are Zariski dense in $\mathcal{M}_d$. On the other hand, Lemma 3.5 is not sufficient for the proof and we need to use [JXZ23, Theorem 1.12] instead. As [JXZ23, Theorem 1.12] relies on Siegel's deep theorem for integer points and the proof of Lemma 3.5 is elementary, we choose the current proof.

3.4. **An alternative approach.** In a private communication, DeMarco and Mavraki told us that when they preparing lectures in Harvard and Toronto in November 2023 [DM23], they find a simplification of our proof of Theorem 1.3.

Their key observation is as follows: A key ingredient of Theorem 1.3 is Theorem 3.4 (=Theorem 1.5) which proves more than what we need. The proof of Theorem 3.4 strongly relies on the results and methods developed in author's previous paper [JX23a].

First, in the proof of Theorem 1.3, we do not need Theorem 3.4 for general families. If we choose the way to prove Theorem 1.3 as in Remark 3.7, we only need to apply Theorem 3.4 for families f_V, g_V having the following additional assumptions:

(A) For every $t \in V(\mathbb{C})$, the multiplier spectrums for f_t and g_t are the same.
(B) The curve V is smooth and both f_V and g_V have exactly $2d-2$ marked critical points $a_1, \dots, a_{2d-2}$ and $b_1, \dots, b_{2d-2}$. Moreover all these marked critical points except a_1, b_1 are strictly preperiodic at every $t \in V$.

Second, in Theorem 3.4, we proved that f_t and g_t are intertwined for all but finitely many $t \in V$. However, to prove Theorem 1.3, we only need one such t. So we may replace Theorem 3.4 to the following weaker result.

Lemma 3.8. *Let f_V, g_V be families as in Theorem 3.4. We further assume (A), (B) holds. Then there is $t_0 \in V(\mathbb{C})$ such that f_{t_0} and g_{t_0} are intertwined.*

To prove Lemma 3.8, one can follow the same strategy of the proof of Theorem 3.4 (hence the proof of [JX23a, Theorem 1.3]). But there are two important steps become easier. After replacing V by its normalization and then by some finite ramification cover, we may assume that V is smooth and both f_V and g_V have exactly $2d-2$ marked critical points $a_1, \dots, a_{2d-2}$ and $b_1, \dots, b_{2d-2}$.

In the first step of the proof of Theorem 3.4, we show that there is a probability measure on $V(\mathbb{C})$ which is proportional to μ_{f_V,a_i} (resp. μ_{g_V,b_i}) for every active a_i (resp. b_i). The proof is based on Yuan-Zhang's deep equidistribution theorem [YZ21, Theorem 6.2.3] on quasi-projective varieties. The corresponding step for Lemma 3.8 is much easier. By assumption (B), a_1, b_1 are the only active marked points for f_V and g_V respectively. Hence μ_{f_V,a_1} (resp. μ_{g_V,b_1}) is proportional to $\mu_{f_V,\mathrm{bif}}$ (resp. $\mu_{g_V,\mathrm{bif}}$). By [Ber10] the Lyapunov exponent $L(h)$ for every rational function h on $\mathbb{P}^1(\mathbb{C})$ can be computed using its mutiplier spectrum. Then (A) implies that $L(f_t) = L(g_t)$ for every $t \in V(\mathbb{C})$. By [DeM01, DeM03], the function $t \in V(\mathbb{C}) \mapsto L(f_t)$ (resp. $t \in V(\mathbb{C}) \mapsto L(g_t)$) is a potential of $\mu_{f_V,\mathrm{bif}}$ (resp. $\mu_{g_V,\mathrm{bif}}$). This gives a logically easier proof of this step for Lemma 3.8.

Another crucial step to prove Theorem 3.4 is to construct similarities between the bifurcation measure on the parameter space $V(\mathbb{C})$ and the maximal entropy measures on the phase spaces. As the conclusion of Theorem 3.4 is for all but finitely many parameters in $V(\mathbb{C})$, we construct such similarities for μ-a.e. $t \in V(\mathbb{C})$. This step is essentially done in [JX23a], which is technically difficult. We first show that for a μ-generic parameter t, f_t and g_t satisfy some Collet-Eckmann-type conditions (which can be thought as some weak hyperbolicity). We then built the similarities under such conditions. As our hyperbolicity assumption is very weak, we can not apply the previous methods as in [Tan90, FG22, Gau22] to construct similarities. Actually, our proof is based on subtle estimates of distortions for non-injective maps and a suitable binding argument. This step becomes much easier for Lemma 3.8, as we only need to construct the similarities at a single parameter t_0. By [Duj13, Theorem 0.1], there is a dense set P of parameters $t \in \mathrm{Supp}\ \mu_{f_V,a_1}$ such that a_1 is transversally pre-repelling at t. By (B), for every $t \in P$, f_t is PCF with no periodic critical point. By [JXZ23, Theorem 1.12], for every $t \in P$, g_t is also PCF. Moreover, the proof of Lemma 3.5 implies that g_t has no critical periodic point. In particular, $b_1(t)$ is pre-repelling for g_t. Pick any point $t_0 \in P$, it is easy to get the similarities at t_0 applying the method in [FG22, Gau22].

References

[Ast17] Matthieu Astorg. The Teichmüller space of a rational map immerses into moduli space. *Advances in Mathematics*, 313:991–1023, 2017.

[BDM13] Matthew Baker and Laura De Marco. Special curves and postcritically finite polynomials. In *Forum of Mathematics, Pi*, volume 1. Cambridge University Press, 2013.

[Ber10] François Berteloot. Lyapunov exponent of a rational map and multipliers of repelling cycles. *Riv. Math. Univ. Parma (N.S.)*, 1(2):263–269, 2010.

[DeM01] Laura DeMarco. Dynamics of rational maps: a current on the bifurcation locus. *Math. Res. Lett.*, 8(1-2):57–66, 2001.

[DeM03] Laura DeMarco. Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity. *Math. Ann.*, 326(1):43–73, 2003.

[DeM16] Laura DeMarco. Bifurcations, intersections, and heights. *Algebra and Number Theory*, 10(5):1031–1056, 2016.

[DF08] Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. *American journal of mathematics*, 130(4):979–1032, 2008.

[DH93] Adrien Douady and John H. Hubbard. A proof of Thurston's topological characterization of rational functions. *Acta Math.*, 171(2):263–297, 1993.

[DM23] Laura DeMarco and Niki Myrto Mavraki. Multiplier spectrum of maps on $\mathbb{P}^1$:theorems of Ji-Xie and Ji-Xie-Zhang. Notes prepared for lectures at Harvard and Toronto in November 2023, 2023.

[DS10] Tien-Cuong Dinh and Nessim Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. *Holomorphic dynamical systems*, pages 165–294, 2010.

[DTGV21] Henry De Thélin, Thomas Gauthier, and Gabriel Vigny. Parametric Lyapunov exponents. *Bulletin of the London Mathematical Society*, 53(3):660–672, 2021.

[Duj13] Romain Dujardin. The supports of higher bifurcation currents. *Ann. Fac. Sci. Toulouse Math. (6)*, 22(3):445–464, 2013.

[FG22] Charles Favre and Thomas Gauthier. The arithmetic of polynomial dynamical pairs:(ams-214). *Annals of Mathematics Studies*, 401, 2022.

[Fuj07] Masayo Fujimura. The moduli space of rational maps and surjectivity of multiplier representation. *Computational Methods and Function Theory*, 7:345–360, 2007.

[Gau22] Thomas Gauthier. Dynamical pairs with an absolutely continuous bifurcation measure. *Ann. Fac. Sci. Toulouse Math*, 2022.

[Gor15] Igors Gorbovickis. Algebraic independence of multipliers of periodic orbits in the space of rational maps of the Riemann sphere. *Mosc. Math. J.*, 15(1):73–87, 181–182, 2015.

[Got23] Rin Gotou. Dynamical Systems of Correspondences on the Projective Line II: Degrees of Multiplier Maps. arXiv:2309.15404, 2023.

[GOV19] Thomas Gauthier, Yûsuke Okuyama, and Gabriel Vigny. Hyperbolic components of rational maps: Quantitative equidistribution and counting. *Commentarii Mathematici Helvetici*, 94(2), 2019.

[HT13a] Benjamin Hutz and Michael Tepper. Errata for "Multiplier spectra and the moduli space of degree 3 morphisms on $\mathbb{P}^1$". https://mathstat.slu.edu/ hutzba/multipliers_errata.pdf, 2013.

[HT13b] Benjamin Hutz and Michael Tepper. Multiplier spectra and the moduli space of degree 3 morphisms on $\mathbb{P}^1$. *JP J. Algebra Number Theory Appl.*, 29(2):189–206, 2013.

[JX23a] Zhuchao Ji and Junyi Xie. DAO for curves. *arXiv preprint arXiv:2302.02583*, 2023.

[JX23b] Zhuchao Ji and Junyi Xie. Homoclinic orbits, multiplier spectrum and rigidity theorems in complex dynamics. *Forum of Mathematics, Pi*, 11:e11, 2023.

[JXar] Zhuchao Ji and Junyi Xie. The moduli space of a rational map is Carathéodory hyperbolic. to appear.

[JXZ23] Zhuchao Ji, Junyi Xie, and Geng-Rui Zhang. Space spanned by characteristic exponents. *arXiv preprint arXiv:2308.00289*, 2023.

[McM87] Curt McMullen. Families of rational maps and iterative root-finding algorithms. *Ann. of Math. (2)*, 125(3):467–493, 1987.

[Mil93] John Milnor. Geometry and dynamics of quadratic rational maps. *Experiment. Math.*, 2(1):37–83, 1993. With an appendix by the author and Lei Tan.

[Mil06] John Milnor. On Lattès maps. *Dynamics on the Riemann Sphere: A Bodil Branner Festschrift*, page 9, 2006.

[Mil11] John Milnor. *Dynamics in One Complex Variable.(AM-160):(AM-160)-*, volume 160. Princeton University Press, 2011.

[MS98] Curt McMullen and Dennis Sullivan. Quasiconformal homeomorphisms and dynamics III. The Teichmüller space of a holomorphic dynamical system. *Advances in Mathematics*, 135(2):351–395, 1998.

[Pak19a] Fedor Pakovich. On mutually semiconjugate rational functions. *Arnold Mathematical Journal*, 5(2-3):339–354, 2019.

[Pak19b] Fedor Pakovich. Recomposing rational functions. *International Mathematics Research Notices*, 2019(7):1921–1935, 2019.

[Pak21] Fedor Pakovich. On iterates of rational functions with maximal number of critical values. *arXiv preprint arXiv:2107.05963*, 2021.

[PR98] Feliks Przytycki and Steffen Rohde. Porosity of Collet-Eckmann Julia sets. *Fundamenta Mathematicae*, 155(2):189–199, 1998.

[Sch16] Johannes Schmitt. A compactification of the moduli space of selfmaps of $\mathbb{CP}^1$ via stable maps. arXiv:1604.08917v1, 2016.

[Sib99] Nessim Sibony. Dynamique des applications rationnelles de P^k. *Dynamique et géométrie complexes (Lyon, 1997)*, 8:97–185, 1999.

[Sil07] Joseph H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2007.

[Sil12] Joseph H. Silverman. *Moduli spaces and arithmetic dynamics*, volume 30 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2012.

[Sil23] Joseph Silverman. *Survey lecture on arithmetic dynamics*, pages 1682–1710. 12 2023.

[Sug17] Toshi Sugiyama. The moduli space of polynomial maps and their fixed-point multipliers. *Advances in Mathematics*, 322:132–185, 2017.

[Sug20] Toshi Sugiyama. The moduli space of polynomial maps and their holomorphic indices: I. generic properties in the case of having multiple fixed points. *arXiv preprint arXiv:2009.11815*, 2020.

[Sug23] Toshi Sugiyama. The moduli space of polynomial maps and their fixed-point multipliers: II. Improvement to the algorithm and monic centered polynomials. *Ergodic Theory and Dynamical Systems*, pages 1–19, 2023.

[Tan90] Lei Tan. Similarity between the Mandelbrot set and Julia sets. *Comm. Math. Phys.*, 134(3):587–617, 1990.

[YZ21] Xinyi Yuan and Shou-Wu Zhang. Adelic line bundles over quasi-projective varieties. arXiv preprint arXiv:2105.13587, 2021.

Institute for Theoretical Sciences, Westlake University, Hangzhou 310030, China

E-mail address: `jizhuchao@westlake.edu.cn`

Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China

E-mail address: `xiejunyi@bicmr.pku.edu.cn`