LOCAL RIGIDITY OF JULIA SETS

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ABSTRACT. We find criteria ensuring that a local (holomorphic, real analytic, C^1) homeomorphism between the Julia sets of two given rational functions comes from an algebraic correspondence. For example, we show that if there is a local C^1-symmetry between the maximal entropy measures of two rational functions, then probably up to a complex conjugation, the two rational functions are dynamically related by an algebraic correspondence. The holomorphic case of our criterion will play an important role in the authors' upcoming proof of the Dynamical André-Oort conjecture for curves.

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1. INTRODUCTION

A classical problem in complex dynamics is to determine when two rational maps of degree at least two have the same Julia set. This problem has been studied by many authors [BE87] [Bea92] [Din00] [Ere89] [Lev90] [LP97] [SS95]. In certain situations (e.g., if the Julia set is a circle or $\mathbb{P}^1(\mathbb{C})$), the shape of the Julia set contains few information. So it is also interesting to ask when two rational maps have the same maximal entropy measure. Note that a rational map of degree at least two has a unique maximal entropy measure. A result of Levin-Przytycki

[LP97] shows that in this case the two rational functions are dynamically related by an algebraic correspondence. Such rigidity issues have recently played important roles in arithmetic dynamics [BD11] [FG22] [GNY19].

The first aim of this paper is to study the following more general question:

Question 1.1. Can we classify the local (holomorphic, real analytic, C^1) homeomorphisms between the Julia sets (or maximal entropy measures) of two given rational functions of degree at least two?

This question is inspired by [DFG22, Problem in page 3] proposed by Dujardin-Favre-Gauthier. We expect that under reasonable conditions, probably up to a complex conjugation, such two endomorphisms are *dynamically related* i.e. for endomorphisms f, g on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2, they are called dynamically related if there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is periodic under (f^a, g^b). By [Xie22, Proposition 3.14.], we may take $(a, b) = (\frac{\text{lcm}\{\deg f, \deg g\}}{\deg f}, \frac{\text{lcm}\{\deg f, \deg g\}}{\deg g})$. Moreover, we expect that h itself comes from the algebraic correspondence Z. This local rigidity question was recently studied by Dujardin-Favre-Gauthier [DFG22]and by Luo [Luo21] in the holomorphic case.

Remark 1.2. We could not expect a rigidity result for local C^0-homeomorphisms. Indeed for two rational maps in the same stable component, by λ-Lemma [McM16, Theorem 4.1], their maximal entropy measures (hence Julia sets) are homeomorphic via a quasiconformal homeomorphism.

To state our main results, we first introduce some notations. Let g be an endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let μ_g be the maximal entropy measure for g and $\mathcal{J}_g$ be the Julia set of g.

Denote by $\sigma_{\text{con}} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ the anti-holomorphic bijection $z \mapsto \overline{z}$. For every endomorphism g on $\mathbb{P}^1(\mathbb{C})$ denote by $\overline{g} : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ the endomorphism $z \mapsto \overline{g(\overline{z})}$. It is clear that $\mathcal{J}_{\overline{g}} = \sigma_{\text{con}}(\mathcal{J}_g)$ and we have $\sigma_{\text{con}} \circ g = \overline{g} \circ \sigma_{\text{con}}$.

Julia sets of special types. Recall the following result of Eremenko [EVS11, Theorem 1 and the first paragraph in page 2].

Theorem 1.3. *There is an open subset V of $\mathbb{P}^1(\mathbb{C})$ such that $\mathcal{J}_g \cap V$ is not empty and contained in a C^1-smooth curve if and only if $\mathcal{J}_g$ is contained in a circle.*

Definition 1.4. Let C_g be the smallest real analytic subset of $\mathbb{P}^1(\mathbb{C})$ containing $\mathcal{J}_f$.

By Theorem 1.3, C_g is $\mathbb{P}^1(\mathbb{C})$ if $\mathcal{J}_g$ is not contained in a circle, otherwise, C_g is the unique circle containing $\mathcal{J}_g$.

Definition 1.5. We say that $\mathcal{J}_g$ is *smooth* if it equals to $\mathbb{P}^1(\mathbb{C})$, a circle, or a segment.

By Theorem 1.3 and [Mil11, Corollary 4.11], $\mathcal{J}_g$ is smooth if and if it is C^1-smooth at some point in $\mathcal{J}_g$.

Definition 1.6. As in [JX22, Section 1.1], we call g *exceptional* if it is a Lattès map or semiconjugates to a monomial map.

It is clear that when g is exceptional, $\mathcal{J}_g$ is smooth and μ_g is equivalent to the Lebesgue measure on $\mathcal{J}_g$.

1.1. **Main results.** In our first result, we answer Question 1.1 under a measure theoretic assumption. The holomorphic case of Theorem 3.1 will play an important role in the authors' upcoming proof of the Dynamical André-Oort conjecture for curves [JX23].

Theorem 1.7 (Theorem 3.1, 6.1 and 7.1)**.** *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of them is non-exceptional. Let μ (resp. ν) be a non-atomic invariant ergodic probability measure with positive Lyapunov exponent of f (resp. g). Let $U \subset \mathbb{P}^1(\mathbb{C})$ be a connected open subset such that $U \cap C_g$ is connected and $U \cap \mathcal{J}_g \neq \emptyset$. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a homeomorphism such that*

(i) *$h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$* [1];
(ii) *$h_*(\nu)$ is equivalent to μ on $h(U)$.*

Then the following statements holds:

Holomorphic case: *If h is holomorphic, then there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(x), x), x \in U\}$ of h.*

Real analytic case: *If h is real analytic and orientation preserving, then there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is*

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[1] For two measures μ_1, μ_2, we write $\mu_1 \propto \mu_2$ if they are proportional, i.e. there exists $c > 0$ such that $\mu_1 = c\mu_2$.

preperiodic under (f^a, g^b) and contains the graph $\{(h(z), z), z \in U \cap C_f\}$ of h. Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is not contained in any circle, then h is holomorphic.

C^1 **case:** *If h is C^1, then up to change f to its complex conjugation $\overline{f}$, there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is periodic under (f^a, g^b). Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is $\mathbb{P}^1(\mathbb{C})$, then h is conformal.*

Here a homeomorphism $h : U \to h(U)$ on a open subset U is called *conformal* if h is holomorphic or antiholomorphic on every connected component of U.

Remark 1.8. One may take μ, ν to be μ_f and μ_g respectively. Then Theorem 1.7 completely answers the measure version of Question 1.1 in the holomorphic and real analytic cases. Our answer in the C^1 case is less precise (c.f. Remark 1.14).

Remark 1.9. The holomorphic case of Theorem 3.1 was proved earlier by Dujardin-Favre-Gauthier [DFG22] under the following additional assumptions: (1) h maps a repelling periodic point of g to a preperiodic point of f; (2) $\mu = \mu_f$ and $\nu = \mu_g$; (3) $h_*(\nu)$ is strongly equivalent to μ on $h(U)$, i.e. there exists $C > 0$ such that $\frac{1}{C}\mu \le h_*(\nu) \le C\mu$.

Our second result answers Question 1.1 under a non-uniform hyperbolic assumption (see condition (ii) in the below theorem) which is satisfied by e.g. Topological Collet-Eckmann maps [PRL07, Section 1.1, Theorem E, Lemma 7.2] and Weakly Hyperbolic maps in the sense of Rivera Letelier-Shen [RLS14, Definition 2.1 and 2.2, Corollary 6.3]. Examples of Weakly Hyperbolic maps in the sense of Rivera Letelier-Shen can be found in [RLS14, Fact 2.4].

Theorem 1.10 (Theorem 3.2, 6.2 and 7.2)**.** *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of them is non-exceptional. Let $U \subset \mathbb{P}^1(\mathbb{C})$ be a connected open subset such that $U \cap C_g$ is connected and $U \cap \mathcal{J}_g \neq \emptyset$. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a homeomorphism such that*

    (i) *$h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$;*

    (ii) *the Hausdorff dimension of non-conical points of g is 0 (c.f. Definition 3.3).*

Then the following statements holds:

Holomorphic case: *If h is holomorphic, then there exist positive integers a and b and an irreducible algebraic curve Z in*

$\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ *such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(x), x), x \in U\}$ of h.*

Real analytic case: *If h is real analytic and orientation preserving, then there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(z), z), z \in U \cap C_f\}$ of h. Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is not contained in any circle, then h is holomorphic.*

C^1 **case:** *If h is C^1, then up to change f to its complex conjugation $\overline{f}$, there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is periodic under (f^a, g^b). Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is $\mathbb{P}^1(\mathbb{C})$, then h is conformal.*

Remark 1.11. Since holomorphic maps are real analytic, the real analytic case of Theorem 1.7 and 1.10 implies the holomorphic case. But actually the proof of the real analytic case is based on the proof of the holomorphic case. That is the reason we state the holomorphic case separately.

Remark 1.12. In Theorem 1.7 and 1.10, the assumption of the non-exceptionalness is necessary (see [DFG22, Remark 2.13]).

Remark 1.13. When $C_g \neq \mathbb{P}^1(\mathbb{C})$, there are infinitely many local real analytic homeomorphisms whose restrictions to C_g are the same as h. So in the real analytic case of Theorem 1.7 and 1.10, it is necessary to restrict h to C_g.

Remark 1.14. If h in the real analytic case of Theorem 1.7 (resp. 1.10), does not preserve the orientation, then $\sigma_{\mathrm{con}} \circ h$ preserves the orientation. So we may apply Theorem 1.7 (resp. 1.10), for $f, \overline{g}$ and $\sigma_{\mathrm{con}} \circ h$.

Remark 1.15. The C^1 case of Theorem 1.7 and 1.10 is less precise than the holomorphic or the real analytic case. We showed that, probably up to a complex conjugation, f and g are dynamically related. However, we can not show that the graph of h has intersection with Z.

Combing the holomorphic case of Theorem 1.7 and [DFG22, Theorem 4.1], we get the following consequence for polynomial maps.

Corollary 1.16. *Let f, g be two polynomial endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of g is non-exceptional and*

    (1) either $\mathcal{J}_g$ is disconnected;
    (2) or $\mathcal{J}_g$ is connected and locally connected.

Let $U \subset \mathbb{P}^1(\mathbb{C})$ be a connected open subset. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a biholomorphic map such that $h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$. Then there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(x), x), x \in U\}$ of h.

Remark 1.17. Corollary 1.16 was proved by Luo [Luo21] in the case where f and g are polynomials with the same degree and $\mathcal{J}_g$ is connected and locally connected, and it was proved by Dujardin-Favre-Gauthier [DFG22] in the case g is a Topological Collet-Eckmann polynomial (Topological Collet-Eckmann maps automatically satisfy (1) or (2)).

Our another aim is to improve local conjugacies to algebraic correspondences. For instance, we prove the rigidity of marked length spectrums of real rational maps. See Section 4 for details.

Theorem 1.18. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2 defined over $\mathbb{R}$ such that f is non-exceptional. Let N_f (resp. N_g) be $\mathbb{R} \cup \{\infty\}$ or be a compact interval contained in $\mathbb{R} \cup \{\infty\}$. Assume*
  (i) *there exists a homeomorphism $h : N_f \to N_g$ such that $h \circ f = g \circ h$ on N_f;*
  (ii) *$f|_{N_f}$ has positive topological entropy and for every periodic point $x \in N_f$ we have $|df^n(x)| = |dg^n(h(x))|$, where n is the period of x.*

Then there exists an irreducible algebraic curve $\Gamma \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is periodic under (f, g), movrover the intersection of V and the graph of h contains a Cantor set.

We also get rigidity results for conformal expanding repeller (CER) (c.f. Theorem 4.3) and horseshoe (c.f. Theorem 6.4).

1.2. **Sketch of the proof.** An important idea in our proofs of Theorem 1.7 and 1.10 is to study the dynamics at a point x which is "general" in a certain sense. This idea has been already used by Levin-Przytycki in their classification of rational maps with identical maximal entropy measures [LP97], and this idea is also crucial in the authors' upcoming proof of the Dynamical André-Oort conjecture for curves [JX23]. On the other hand, the strategy in [DFG22] is rather different. They worked on preperiodic points and the proof involves entire curves and positive currents.

Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of them is non-exceptional. Let $U \subset \mathbb{P}^1(\mathbb{C})$ be a connected

open subset such that $U \cap C_g$ is connected and $U \cap \mathcal{J}_g \neq \emptyset$. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a (holomorphic, real analytic, C^1) homeomorphism such that $h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$.

If one of f, g is exceptional, then $\mathcal{J}_f$ and $\mathcal{J}_g$ are smooth. Then by [Zdu90], both of them are exceptional. So both f and g are non-exceptional.

We define a point in $U \cap \mathcal{J}_g$ to be *bi-conical* for (g, h, f) if it is not $g-$preperiodic, and there are positive constants r, R, K and two sequences of positive integers $n_j \to +\infty$, $m_j \to +\infty$, $j \geq 1$ such that

  (i) $g^{n_j} : W_j \to B(g^{n_j}(x), r)$ is injective and $W_j \subset U$, where W_j is the connected component of $g^{-n_j}(B(g^{n_j}(x), r))$ containing x;

  (ii) the map $h_j := f^{m_j} \circ h \circ g_{n_j} : B(g^{n_j}(x), r) \to \mathbb{P}^1(\mathbb{C})$ is injective and satisfies

$$B(f^{m_j}(h(x)), R/K) \subset h_j(B(g^{n_j}(x), r)) \subset B(f^{m_j}(h(x)), R),$$

where g_{n_j} is the inverse map of $g^{n_j} : W_j \to B(g^{n_j}(x), r)$.

Our first step is to show the existence of bi-conical points for (g, h, f). Under condition (ii) of Theorem 1.10, this is easy to show via the constructions of CERs. Under condition (ii) of Theorem 1.7, the proof is more delicate. In Lemma 3.14, we show that ν-a.e. point in U is bi-conical for (g, h, f) via ergodic theory. Moreover, the set of times $G = \{n_i, i \geq 0\}$ can be asked to have positive lower density. This was done in Section 3.

We first prove the holomorphic case in Section 3, which is the foundation of the real analytic and the C^1 case. Assume the existence of a bi-conical point x for (g, h, f). By Montel's theorem, we can construct a normal family $\{h_i : D \to \mathbb{P}^1(\mathbb{C})\}$ of injective holomorphic maps on some disk D with $D \cap \mathcal{J}_g \neq \emptyset$ and satisfy condition (i) of Theorem 1.7 (and Theorem 1.10). Moreover, every limit of this family is non-constant. Applying Levin's result [Lev90, Theorem 1] (see also Theorem 5.1), we construct two horseshoes in $\mathcal{J}_g$ and $\mathcal{J}_f$ respectively and a holomorphic conjugacy between them. Applying a generalized form (c.f. Theorem 2.2) of a theorem of Inou [Ino11, Theorem 2] based on Eremenko's theorem [Ere89, Theorem 2] (c.f. Theorem 2.3), we show that the conjugacy can be improved to an algebraic correspondence. This implies the holomorphic case of Theorem 1.7 and 1.10. Generalized Inou's theorem was discussed in Section 2.

There are three ingredients in the above argument which not apply to the real analytic case directly. The first one is Montel's theorem.

We replace it by Lemma 6.3. The second one is Levin's result [Lev90, Theorem 1]. For this, we replace it by Theorem 5.3, which generalizes [Lev90, Theorem 1] to the real analytic setting. The third one is Theorem 2.2. We replace it by our rigidity result for horseshoes (c.f. Theorem 6.4). This was done in Section 5 and 6.

To prove the C^1 case, our idea is to reduce it to the real analytic case. From a bi-conical point x for (g, h, f), we construct a sequence $\{h_i : D \to \mathbb{P}^1(\mathbb{C})\}$ of injective C^1-maps on some disk D with $D \cap \mathcal{J}_g \neq \emptyset$ converging to a real analytic map H which satisfyies condition (i) of Theorem 1.7 (and Theorem 1.10). This reduces the C^1 case of Theorem 1.10 to the real analytic case. The C^1 case of Theorem 1.7 is more delicate, since condition (ii) of Theorem 1.7 may not be satisfied by H. To solve this problem, we need the fact that the times set $G = \{n_i, i \geq 0\}$ has positive lower density. We construct the sequence $\{h_i : D \to \mathbb{P}^1(\mathbb{C})\}$ based on a chosen subsequence $n_{i_j}, j \geq 0$ of G. One show that, under this choice, $g^{n_{i,j}}(x)$ converges to a point $p \in D$ which is bi-conical for (g, H, f) (c.f. Lemma 7.5). Then the argument in the real analytic case can be applied to (p, g, H, f), which concludes the proof. This was done in Section 7.

2. A generalized Inou's theorem

The aim of this section is to prove the following theorem, which will be used in the proof of Theorem 3.1.

Theorem 2.1. *Let f, g be two non-Lattès endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that*

(i) *there exists a connected open set $U \subset \mathbb{P}^1(\mathbb{C})$ and an open subset $U' \subset U$ such that $f(U') \subset U$, $f : U' \to U$ is non-injective, and $f : U' \to U$ has a repelling fixed point;*

(ii) *there exists a non-constant holomorphic map $h : U \to \mathbb{P}^1(\mathbb{C})$ such that $h \circ f = g \circ h$ on U'.*

Then there exists an irreducible algebraic curve $\Gamma \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is invariant under (f, g) and contains the graph of h.

In [Ino11], Inou proved the above result when $f : U' \to U$ is polynomial-like. His proof indeed works in the general case. We give a proof of Theorem 2.1 for the convenience of the readers. The main ingredients of the proof are Inou's construction [Ino11, Theorem 2] and Eremenko's theorem [Ere89, Theorem 2] as follows:

Theorem 2.2 (Inou). *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$. Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open set. Let $h : U \to \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map. Let $\Gamma_0 := \{(z, h(z)) : z \in U\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the graph of h. Consider the forward invariant set*

$$\Gamma := \bigcup_{n \geq 0} F^n(\Gamma_0)$$

of the product map $F := (f, g)$. Then there exists a Riemann surface X and holomorphic maps $G : X \to X$ and $\pi : X \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that $\pi(X) = \Gamma$, $\pi \circ G = F \circ \pi$ on X and has the following properties:

   (i) *Let $\phi_i := p_i \circ \pi$, where $p_i : \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is the projection to the i-th coordinate. Then there exists an open subset $V \subset X$ such that $\pi(V) = \Gamma_0$ and $\phi_1 : V \to U$ is biholomorphic.*

  (ii) *The cardinality of the preimages of a point in X by G is not greater than $\deg f \cdot \deg g$.*

 (iii) *If U is connected and there exists $U' \subset U$ such that $f(U') \subset U$, and $h \circ f = g \circ h$ on U', then X is connected, $\phi_1 \circ G = f \circ \phi_1$ on X and $\phi_2 \circ G = g \circ \phi_2$ on X.*

Theorem 2.3 (Eremenko). *Let R, G be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ such that $\deg G \geq 2$. Let ϕ be a meromorphic function in $\mathbb{C}$ or in $\mathbb{C}^*$ with an essential singularity at ∞. If ϕ satisfies $\phi \circ G = R \circ \phi$, then R is a Lattès map and G is exceptional of monomial type.*

Proof of Theorem 2.1. We first construct $\Gamma_0, \Gamma, X, G, \pi_i, \phi$ and V as in Theorem 2.2. We only need to show that $\Gamma = \pi(X)$ is an irreducible algebraic curve in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Once this holds, since Γ contains the graph of h, it is invariant under (f, g).

By the assumptions (i) and (ii) of Theorem 2.1 and by Theorem 2.2, X is connected, and $G : X \to X$ is non-injective and has a repelling fixed point. Hence X can not be a hyperbolic Riemann surface. Therefore X is biholomorphic to $\mathbb{P}^1(\mathbb{C})$, a torus $\mathbb{C}/\Lambda$, $\mathbb{C}$ or $\mathbb{C}^*$.

In the first two cases X is projective, hence ϕ_1 and ϕ_2 are finite morphisms, Then $\Gamma = \pi(X)$ is an irreducible algebraic curve in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. In the case $X = \mathbb{C}$ or $\mathbb{C}^*$, by Theorem 2.2 (ii), G is a polynomial map in the case $X = \mathbb{C}$, and $G(z) = z^m$ for some $m \in \mathbb{Z}$ in a suitable

coordinate in the case $X = \mathbb{C}^*$. By assumption (ii) in Theorem 2.1, since f and G are conjugated by ϕ_1 on U', G is non-injective, i.e. G has degree at least 2. By Theorem 2.2 (iii), we have $\phi_1 \circ G = f \circ \phi_1$ on X and $\phi_2 \circ G = g \circ \phi_2$ on X. Since f and g are not Lattès, by Theorem 2.3, ϕ_1 and ϕ_2 must be finite morphisms. Hence $\Gamma = \pi(X)$ is an irreducible algebraic curve in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This finishes the proof. $\qquad\square$

3. Holomorphic local rigidity of Julia sets

The aim of this section is to prove the following two theorems.

Theorem 3.1. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of them is non-exceptional. Let μ (resp. ν) be a non-atomic invariant ergodic probability measure with positive Lyapunov exponent of f (resp. g). Let $U \subset \mathbb{P}^1(\mathbb{C})$ be a connected open subset such that $U \cap \operatorname{Supp} \nu \neq \emptyset$. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a biholomorphic map such that*

    (i) *$h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$;*

    (ii) *$h_*(\nu)$ is equivalent to μ on $h(U)$.*

Then there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(x), x), x \in U\}$ of h.

Theorem 3.2. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of them is non-exceptional . Let $U \subset \mathbb{P}^1(\mathbb{C})$ be a connected open subset and let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a biholomorphic map. Assume that*

    (i) *$h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$;*

    (ii) *the Hausdorff dimension of non-conical points of g is 0.*

Then there exists positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(x), x), x \in U\}$ of h.

3.1. **Bi-conical points.**

Definition 3.3. Let g be an endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. A point $x \in \mathcal{J}_g$ is called conical if there exists $r > 0$ and a sequence of positive integers $n_j \to +\infty$ such that

$$g^{n_j} : W_j \to B(g^{n_j}(x), r)$$

is injective, where W_j is the connected component of $g^{-n_j}(B(g^{n_j}(x), r))$ containing x.

Lemma 3.4. *In the setting of Definition 3.3 we have*

$$|dg^{n_j}(x)| \to +\infty.$$

Proof. Assume by contradiction that by passing to a subsequence of $\{n_j\}$ there exists $M > 0$ such that $|dg^{n_j}(x)| < M$. Shrink r if necessary such that $2r < \operatorname{diam}(\mathcal{J}_g)$. By Koebe one-quarter theorem W_j contains the disk $D := B(x, r/(4M))$. This implies $g^{n_j}(D) \subset B(g^{n_j}(x), r)$ for every $j \geq 1$, contradicts to the fact $\mathcal{J}_g \subset g^{n_j}(D)$ when n_j large enough. $\square$

Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open subset and let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a homeomorphism.

Definition 3.5. A point $x \in U \cap \mathcal{J}(g)$ is called *bi-conical* for (g, h, f) if it is not g-preperiodic, for which there are positive constants r, R, K and two sequences of positive integers $n_j \to +\infty$, $m_j \to +\infty$, $j \geq 1$ such that

    (i) $g^{n_j} : W_j \to B(g^{n_j}(x), r)$ is injective and $W_j \subset U$, where W_j is the connected component of $g^{-n_j}(B(g^{n_j}(x), r))$ containing x;
    (ii) the map $h_j := f^{m_j} \circ h \circ g_{n_j} : B(g^{n_j}(x), r) \to \mathbb{P}^1(\mathbb{C})$ is injective and satisfies

$$B(f^{m_j}(h(x)), R/K) \subset h_j(B(g^{n_j}(x), r)) \subset B(f^{m_j}(h(x)), R),$$

where g_{n_j} is the inverse map of $g^{n_j} : W_j \to B(g^{n_j}(x), r)$;

It is clear that, if x is bi-conical for (g, h, f), $h(x) \in \mathcal{J}_f$.

Remark 3.6. For $n, m \geq 0$, if $g^n : W \to B(g^n(x), r)$ is injective and $W \subset U$, where W is the connected component of $g^{-n}(B(g^n(x), r))$ containing x and f^m is injective on $h(W)$, then $g^n(x)$ is bi-conical for $f^m \circ h \circ g_n$, where g_n is the inverse map of $g^n : W \to B(g^n(x), r)$.

3.2. **Holomorphic rigidity via bi-conical points.**

Lemma 3.7. *Let f, g be two non-exceptional endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let $U \subset \mathbb{P}^1(\mathbb{C})$ be a connected open subset and let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a biholomorphic map such that $h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$, if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$. Assume that there is a point $x \in U \cap \mathcal{J}_g$ which is bi-conical for (g, h, f), then Then there exists positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(z), z), z \in U\}$ of h.*

Proof. By Lemma 3.4 we have

$$(3.1) \qquad |dg^{n_j}(x)| \to +\infty.$$

Shrink r if necessary, we may further assume that there is $C > 1$ such that for every $j \geq 1$ we have

$$C^{-1} < |dh_j(x)/dh_j(y)| < C$$

for every $y \in B(g^{n_j}(x), r)$. By passing to a subsequence of $\{(n_j, m_j)\}$, we may assume that $g^{n_j}(x)$ converges. Then there exists a disk D' centered at the limit point and with radius $r/4$. We may assume that $g^{n_j}(x) \in D'$ for every $j \geq 1$. Let D be the disk of radius $r/2$ and has the same center with D'. Then we have

$$B(g^{n_j}(x), r/4) \subset D \subset B(g^{n_j}(x), r)$$

for every $j \geq 1$. By Montel's theorem, $\{h_j|_D\}$ is a normal family and every limit map in this family is non-constant. Cover $\mathcal{J}_f$ by finitely many disks of radius $R/(8CK)$. By passing to a subsequence of $\{(n_j, m_j)\}$, there exists a disk V in this finite family such that $f^{m_j}(h(x)) \in V$ for every $j \geq 1$. By (ii) for every $j \geq 1$, we have

$$V \subset B(f^{m_j}(h(x)), R/(4CK)) \subset h_j(B(g^{n_j}(x), r/4)) \subset h_j(D).$$

For every $j \geq 1$ the following maps are well defined

$$\sigma_j := h_j \circ h_1^{-1}|_V : V \to \mathbb{P}^1(\mathbb{C}).$$

By our construction σ_j are injective holomorphic maps with bounded distortion, moreover the diameter of $\sigma_j(V)$ are uniformly bounded form above and from below. Then $\{\sigma_j\}$ is a normal family and every limit map in this family is non-constant. By our construction, we have $\sigma_j(V \cap \mathcal{J}_f) = \sigma_j(V) \cap \mathcal{J}_f$, i.e. σ_j are local symmetries of $\mathcal{J}(f)$. When $\mathcal{J}_f$ is smooth, using the total invariance of maximal entropy measures, for every $j \geq 1$ we have $(h_j)_*(\nu) \propto \mu$ on $h_j(D)$, where ν and μ are maximal entropy measures of g and f. As a consequence we have $(\sigma_j)_*(\mu) \propto \mu$ on $\sigma_j(V)$. Now by Levin's result [Lev90, Theorem 1] (see also Theorem 5.1), since f is non-exceptional, $\{\sigma_j\}$ is a finite set. By passing to a subsequence we may assume that $\sigma_j = \sigma_{j_0}$ on V for every $j \geq j_0$. This implies $h_j = h_{j_0}$ on D for every $j \geq j_0$. Without loss of generality we may assume $j_0 = 1$.

The definition of $h_j, j \geq 1$ shows that

$$f^{m_j} \circ h \circ g_{n_j} = f^{m_1} \circ h \circ g_{n_1}$$

on D. By (3.1), we have $m_j \to +\infty$. By passing to a subsequence, we may assume $n_j > \max(n_1, n_2)$ and $m_j > \max(m_1, m_2)$ for every

$j > 2$. For $j > 2$ let U_j be the connected component of $g^{-(n_j-n_1)}(D)$ containing $g^{n_1}(x)$. Thus we have

$$f^{m_j-m_1} \circ h_1 = h_1 \circ g^{n_j-n_1}$$

on U_j. Similarly for $j > 2$ let U_j' be the connected component of $g^{n_2-n_j}(D)$ containing $g^{n_2}(x)$. Thus we have

$$f^{m_j-m_2} \circ h_2 = h_2 \circ g^{n_j-n_2}$$

on U_j'. Since $h_1 = h_2$ on D we have

$$f^{m_j-m_2} \circ h_1 = h_1 \circ g^{n_j-n_2}$$

on U_j'.

Since x is not a preperiodic point, we have $g^{n_1}(x) \neq g^{n_2}(x)$. By (3.1) we know that the diameter of U_j and U_j' shrink to 0 when $j \to +\infty$. Hence we may choose $p > 2$ such that $\overline{U_p} \cap \overline{U_p'} = \emptyset$ and $\overline{U_p} \cup \overline{U_p'} \subset D$. The two maps $g^{n_p-n_1} : U_p \to D$ and $g^{n_q-n_2} : U_p' \to D$ are both biholomorphic. Let

$$W_1 := g_{n_p-n_1}(U_p') \subset U_p,$$

where $g_{n_p-n_1}$ is the inverse map of $g^{n_p-n_1} : U_p \to D$, and let

$$W_2 := g_{n_p-n_2}(U_p) \subset U_p',$$

where $g_{n_q-n_2}$ is the inverse map of $g^{n_p-n_2} : U_p' \to D$.

Let $W := W_1 \cup W_2 \subset D$, set $a := 2m_p - m_1 - m_2$ and $b := 2n_p - n_1 - n_2$, then we have

$$f^a \circ h_1 = h_1 \circ g^b$$

on W. Moreover $g^b : W \to D$ has two repelling fixed point, and is non-injective.

By Lemma 2.1 there exists an irreducible algebraic curve $\Gamma \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is invariant under (f^a, g^b) and contains the graph of h_1, i.e. the set $\{(h_1(w), w) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) : w \in D\}$.

It remains to show the there exists an irreducible algebraic curve Z containing the graph of h that is preperiodic under (f^a, g^b). We need the following lemma:

Lemma 3.8. *Let $Z \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be an irreducible algebraic curve. Assume there exist $m \geq 0$ and $n \geq 0$ such that $(f^m, g^n)(Z) = \Gamma$. Then Z is preperiodic under (f^a, g^b).*

Proof. Passing to an iteration of (f^a, g^b), we can further assume that $\min\{a, b\} > \max\{n, m\}$. For every $l \geq 1$, since Γ is invariant under (f^a, g^b), we have

$$(f^{al}, g^{bl})(Z) = (f^{a(l-1)+a-m}, g^{b(l-1)+b-n})(\Gamma) = (f^{a-m}, g^{b-n})(\Gamma).$$

This means that the (f^a, g^b)-forward images of Z are all contained in the curve $(f^{a-m}, g^{b-n})(\Gamma)$, which implies that Z is preperiodic under (f^a, g^b). $\square$

Let $Z' := \{(h(w), w) \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) : w \in U\}$ be the graph of h. Then we have

$$(f^{m_1}, g^{n_1})(Z') \subset \Gamma.$$

Let Z be the irreducible component of $(f^{m_1}, g^{n_1})^{-1}(\Gamma)$ containing Z'. Since Γ is irreducible we have

$$(f^{m_1}, g^{n_1})(Z) = \Gamma.$$

By the above lemma, Z is preperiodic under (f^a, g^b) and contains the graph of h. This finishes the proof. $\square$

3.3. Existence of bi-conical points via ergodic theory. We recall the asymptotic density of subsets of $\mathbb{Z}_{>0}$.

Definition 3.9. Let A be a subset of positive integers. The *asymptotic lower/upper density* of A is defined by

$$\underline{d}(A) := \liminf_{n \to \infty} |A \cap [0, n-1]|/n,$$

and

$$\overline{d}(A) := \limsup_{n \to \infty} |A \cap [0, n-1]|/n.$$

If $\underline{d}(A) = \overline{d}(A)$, we set $d(A) := \underline{d}(A) = \overline{d}(A)$ and call it the *asymptotic density* of A.

The proof of the following lemma was implicitly contained in [LP97, Lemma 1]. For completeness we give a proof here.

Lemma 3.10 (Levin-Przytycki). *Let g be an endomorphism on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let ν be a g-invariant ergodic probability measure with positive Lyapunov exponent. Then for every $\epsilon > 0$ there is $r > 0$ such that for ν-a.e. $x \in \mathbb{P}^1(\mathbb{C})$, there exists a subset $A = A_x$ of positive integers such that $d(A) > 1 - \epsilon$, and every $n \in A$ satisfies that if we denote by W the connected component of $g^{-n}(B(g^n(x), r))$ containing x, then the map $g^n : W \to B(g^n(x), r)$ is biholomorphic and has bounded distortion,*

$$1/2 < |dg^n(x)/dg^n(y)| < 2,$$

for all $y \in W$.

Proof. Let $J := \operatorname{Supp} \nu$. Consider the Rohlin's natural extension $(\tilde{J}, \tilde{g}, \tilde{\nu})$ of (J, g, ν). For $n \in \mathbb{Z}$ denote $\pi_n : \tilde{J} \to J$ the projection to the n-th coordinate. Then for $\tilde{\nu}$-a.e. $\tilde{x} \in \tilde{J}$ there exists $r(\tilde{x}) > 0$

such that for every positive integer n, if W is the connected component of $g^{-n}(B(x, r(\tilde{x})))$ containing $\pi_{-n}(\tilde{x})$ where $x := \pi_0(\tilde{x})$, then the map $g^n : W \to B(x, r(\tilde{x}))$ is biholomorphic and has bounded distortion,

$$1/2 < |dg^n(x)/dg^n(y)| < 2,$$

for all $y \in W$. Moreover $r(\tilde{x})$ is a measurable function in $\tilde{x}$. See Przytycki-Urbanski [PU10, Theorem 11.2.3].

For fixed $\epsilon > 0$, there is $r > 0$ such that $\tilde{\nu}(\tilde{F}) > 1-\epsilon$ where $\tilde{F} := \{\tilde{x} \in \tilde{J} : r(\tilde{x}) > r\}$. By Birkhoff ergodic theorem, there is a subset $\tilde{G}$ with $\tilde{\nu}(\tilde{G}) = 1$ such that for every $\tilde{x} \in \tilde{G}$, we have $d(A(\tilde{x})) = \tilde{\nu}(\tilde{F}) > 1 - \epsilon$, where $A(\tilde{x}) := \left\{n \geq 1 : \tilde{g}^n(\tilde{x}) \in \tilde{F}_0\right\}$. Then $\nu(\pi_0(\tilde{G})) = 1$. For $x \in \pi_0(\tilde{G})$, pick $\tilde{x} \in \pi_0^{-1}(x) \cap \tilde{G}$ and set $A_x := A_{\tilde{x}}$. Then $d(A_x) > 1 - \epsilon$ and (x, A_x, r) has the bounded distortion property we need. This concludes our proof. $\square$

Let g be an endomorphism on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. We let L be a constant strictly larger than the maximum of $|dg|$ on $\mathbb{P}^1(\mathbb{C})$ w.r.t. the spherical metric. Let $x \in \mathbb{P}^1(\mathbb{C})$ satisfying $a_n \geq n \log \lambda - Q$, where $a_n := \log |dg^n(x)|$, $\lambda > 1$ and $Q > 0$. We define a function $\alpha_{g,x}$ as follows.

Definition 3.11. The function $\alpha_{g,x} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ is defined as follows: for each $m \in \mathbb{Z}_{\geq 0}$, $\alpha_{g,x}(m)$ is the minimal $n \in \mathbb{Z}_{\geq 0}$ such that $a_n \geq m \log L$.

Lemma 3.12. *The function $\alpha_{g,x}$ has the following properties:*
  (i) *it is strictly increasing, i.e. $\alpha_{g,x}(m + 1) > \alpha_{g,x}(m)$ for every $m \in \mathbb{Z}_{\geq 0}$;*
  (ii) *$m \leq \alpha_{g,x}(m) \leq \lceil (m \log L + Q)/ \log \lambda \rceil$.*

Proof. We first prove (i). By the definition of $\alpha_{g,x}$, we clearly have $\alpha_{g,x}(m + 1) \geq \alpha_{g,x}(m)$. Assume by contradiction that $\alpha_{g,x}(m + 1) = \alpha_{g,x}(m) = n$ for some $n \in \mathbb{Z}_{\geq 1}$. Then $a_n \geq (m + 1) \log L$, hence

$$a_{n-1} = a_n - \log |dg(g^{n-1}(x))| \geq m \log L,$$

contradicts to our assumption that $\alpha_{g,x}(m) = n$.

Next we prove (ii). Since $a_n \geq n \log \lambda - Q$ holds for every $n \in \mathbb{Z}_{\geq 0}$, we get $a_{\lceil (m \log L + Q)/ \log \lambda \rceil} \geq m \log L$. The definition of L implies that $a_m \leq m \log L$. This concludes the proof. $\square$

As a direct corollary of Lemma 3.12 we have:

Corollary 3.13. *Let $A \subset \mathbb{Z}_{\geq 0}$ be a subset such that $d(A) = \delta$, where $0 \leq \delta \leq 1$. Then we have:*

$$(\log \lambda / \log L)\delta \leq \underline{d}(\alpha_{g,x}(A)) \leq \overline{d}(\alpha_{g,x}(A)) \leq \delta.$$

Let U be a connected open subset of $\mathbb{P}^1(\mathbb{C})$. Let $K > 1$, we say a homeomorphism $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ is *K-biLipschitz* if for every distinct $x, y \in U$, $K^{-1}(x, y) \leq d(h(x), h(y)) \leq Kd(x, y)$, where $d(\cdot, \cdot)$ is the distance function on $\mathbb{P}^1(\mathbb{C})$. We say that h has *biLipschitz* if it is K-biLipschitz for some $K > 1$. If h is K-biLipschitz, then for every disk $B(x, r) \subset U$, we have $B(h(x), r/K) \subset h(D) \subset B(h(x), Kr)$ for some $r > 0$.

Lemma 3.14. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let μ (resp. ν) be an invariant ergodic probability measure with positive Lyapunov exponent of f (resp. g). Let χ_μ (resp. χ_ν) be the Lyapunov exponent of μ (resp. ν). Let $U \subset \mathbb{P}^1(\mathbb{C})$ be a connected open subset such that $U \cap \operatorname{Supp} \nu \neq \emptyset$. Let L be a constant larger than the maximum of $|df|$ and $|dg|$ on $\mathbb{P}^1(\mathbb{C})$. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a K-biLipschitz homeomorphism such that $h_*(\nu)$ is equivalent to μ on $h(U)$.*

Then for ν-a.e. point x, it is not $g-$preperiodic and for sufficiently small $\epsilon > 0$, there exist positive constants r, R, C, a subset $G \subset \mathbb{Z}_{\geq 0}$ with

$$\underline{d}(G) \geq \frac{\chi_\mu}{\log L}\left(1 - \frac{\log L}{\chi_\nu}\epsilon\right) - \epsilon > 0$$

and a function $\theta : G \to \mathbb{Z}_{\geq 0}$ such that:

   (i) *$\lim\limits_{n\to\infty} \frac{1}{n} \log |dg^n(x)| \to \chi_\nu$;*

   (ii) *for every $n \in G$, if W is the connected component of the set $g^{-n}(B(g^n(x), r))$ containing x, then the map*

$$g^n : W \to B(g^n(x), r)$$

     is biholomorphic and has bounded distortion,

$$1/2 < |dg^n(x)/dg^n(y)| < 2,$$

     for all $y \in W$;

  (iii) *for every $n \in G$, the map*

$$h_n := f^{\theta(n)} \circ h \circ g_n : B(g^n(x), r) \to \mathbb{P}^1(\mathbb{C})$$

     is injective and satisfies

$$B\left(f^{\theta(n)}(h(x)), \frac{R}{16K^2L^2}\right) \subset f^{\theta(n)}(h(W)) \subset B(f^{\theta(n)}(h(x)), R)$$

     where g_n is the inverse map of $g^n : W \to B(g^n(x), r)$ in (ii);

(iv) θ *is strictly increasing and satisfies*

$$(\chi_\nu / \log L)n - C \leq \theta(n) \leq (\log L / \chi_\mu)n + C$$

for every $n \in G$.

Proof. Let F be the subset $\operatorname{Supp} \nu$ with $\nu(F) = 1$ constructed in Lemma 3.10 for the map g, $\epsilon < \chi_\nu / \log L$ and the radius $r > 0$. For $x \in F$, the subset of $\mathbb{Z}_{\geq 0}$ associated with x is denoted by A_x. Let $H := F \cap U$, then $\nu(H) > 0$. Since $h_*(\nu)$ is equivalent to μ on $h(U)$, $\mu(h(H)) > 0$. Let E be the subset $\operatorname{Supp} \mu$ with $\mu(E) = 1$ constructed in Lemma 3.10 for the map f, ϵ and radius$R > 0$. The subset of $\mathbb{Z}_{\geq 0}$ associated with y is denoted by B_y. We have $E \cap h(H) \neq \emptyset$. By Birkhoff ergodic theorem We can choose a point $x \in h^{-1}(E \cap h(H))$ such that x satisdies (i). Since ν is non-atomic, we can further choose x to not be $g-$preperiodic. Set $A := A_x$ and $B := B_{h(x)}$.

Recall the definition of the function $\alpha_{g,x}$ in Definition 3.11 w.r.t. the constant L. We set $\alpha := \alpha_{g,x}$ and $\alpha' := \alpha_{f,h(x)}$. We set

$$G := A \cap (\alpha \circ \alpha'^{-1}(B)).$$

Then by $\lim_{n \to \infty} \frac{1}{n} \log |dg^n(x)| \to \chi_\nu$ and Corollary 3.13, we have

$$\underline{d}(G) \geq \frac{\chi_\mu}{\log L}\left(1 - \frac{\log L}{\chi_\nu}\epsilon\right) - \epsilon > 0.$$

By our construction the condition (ii) automatically holds.

It remains to cunstruct the function θ, constant C and verify they satisfy (iii) and (iv). For every $n \geq 1$ we set $a_n := \log |dg^n(x)|$ and $b_n := \log |df^n(h(x))|$. By (ii), for every $n \in A$, we have

$$B(x, re^{-a_n}/2) \subset W \subset B(x, 2re^{-a_n}),$$

where W is the connected component of $g^{-n}(B(g^n(x), r))$ containing x. Since h is K-biLipschitz we have

$$B(h(x), e^{-a_n}r/2K) \subset h(W) \subset B(h(x), 2Ke^{-a_n}r).$$

Shrink the constants r and R if necessary we may assume that $2KrL < R < 4KrL$. Then we have

$$(3.2) \qquad B\left(h(x), \frac{Re^{-a_n}}{16K^2L}\right) \subset h(W) \subset B\left(h(x), \frac{Re^{-a_n}}{2L}\right).$$

We define the function θ as $\theta(n) := \alpha' \circ \alpha^{-1}(n)$. Since α^{-1} and α' are strictly increasing, θ is strictly increasing. For every $n \in G$ we have

$$\alpha^{-1}(n) \log L \leq a_n < (\alpha^{-1}(n) + 1) \log L,$$

hence
$$a_n/\log L - 1 < \alpha^{-1}(n) \le a_n/\log L.$$

Apply
$$\alpha'^{-1}(k)\log L \le b_k < (\alpha'^{-1}(k) + 1)\log L$$

to $k := \theta(n)$ we have
$$\alpha^{-1}(n)\log L \le b_{\theta(n)} < (\alpha^{-1}(n) + 1)\log L.$$

Hence

(3.3) $$a_n - \log L < b_{\theta(n)} < a_n + \log L.$$

By Lemma 3.10, (3.2) and (3.3) we have

$$B(f^{\theta(n)}(h(x)), R/(16K^2L^2)) \subset f^{\theta(n)}(h(W)) \subset B(f^{\theta(n)}(h(x)), R),$$

hence (iii) holds.

Finally noticing L is larger than the maximum of $|df|$ and $|dg|$, thus the second part of (iv) holds by Lemma 3.12 (ii). The proof is finished. $\square$

3.4. **Holomorphic rigidities.**

Proof of Theorem 3.1. If one of f, g is exceptional, then $\mathcal{J}_f$ and $\mathcal{J}_g$ are smooth. Then by [Zdu90], both of them are exceptional. So both of them are non-exceptional. When $\mathcal{J}_g$ (hence $\mathcal{J}_f$) is smooth, we may ask $\mu := \mu_f$ and $\nu := \mu_g$.

After shrinking U, we may assume that h is biLipschitz. Then Theorem 3.1 is a simple consequence of Lemma 3.7 and Lemma 3.14. $\square$

Proof of Theorem 3.2. If one of f, g is exceptional, then $\mathcal{J}_f$ and $\mathcal{J}_g$ are smooth. Then by [Zdu90], both of them are exceptional. So both of them are non-exceptional.

Let K be a CER (c.f. [JX22, Definition 7.1]) of f such that $K \subset h(U)$ which is not a periodic orbit. Such K always exists, see for example [JX22, Example 7.4]. It is well known that K has positive Hausdorff dimension, see Przytycki-Urbanski [PU10, Corollary 9.1.7]. Hence $h^{-1}(K)$ also has positive Hausdorff dimension. By our assumption (ii), there exists a point $x \in U \cap \mathcal{J}_g$ such that $h(x) \in K$, x is g-conical and is not g-preperiodic. Hence there exists $r > 0$ and a sequence of positive integers $n_j \to +\infty$ such that

$$g^{n_j} : W_j \to B(g^{n_j}(x), r)$$

is injective and having bounded distortion, here W_j is the connected component of $g^{-n_j}(B(g^{n_j}(x), r))$ containing x. Shrink U if necessary,

we may assume that h is bi-Lipschitz. By Lemma 3.4, there exist $C_1 > 1$ and $r_j > 0$, $r_j \to 0$ such that

$$B(h(x), r_j/C_1) \subset h(W_j) \subset B(h(x), r_j).$$

After replacing f by a suitable iterate, we may assume that $f(K) = K$. Since $f|_K$ is uniformly expanding, we know that $|df^n(y)| \geq C\lambda^n$ for every $n \geq 1$ and $y \in K$, where $C > 0$, $\lambda > 1$ are constants. Moreover $d(K, C(f)) > 0$. Pick $0 < R < d(K, C(f))$. Let m_j be the minimal positivet integer such that $|df^{m_j}(h(x))|r_j \geq R/(2L)$, where L is the supremum of $|df|$ on $\mathbb{P}^1(\mathbb{C})$. Hence we have

$$R/(2Lr_j) \leq |df^{m_j}(h(x))| < R/(2r_j).$$

Let V_j be the connected component of $f^{-m_j}(B(f^{m_j}(h(x)), R))$. Then

$$f^{m_j} : V_j \to B(f^{m_j}(h(x)), R)$$

is biholomorphic for every $j \geq 1$. Shrink R if necessary, by Koebe distortion theorem, we have

$$1 - 99^{-99} < |df^{m_j}(h(x))/df^{m_j}(y)| < 1 + 99^{-99},$$

for all $y \in V_j$.

Let V_j' be the connected component of $f^{-m_j}(B(f^{m_j}(h(x)), R/(100C_1L)))$. Then we have

$$h(W_j) \subset B(h(x), r_j) \subset V_j$$

and

$$V_j' \subset B(h(x), r_j/L) \subset h(W_j).$$

Hence the map $h_j := f^{m_j} \circ h \circ g_{n_j} : B(g^{n_j}(x), r) \to \mathbb{P}^1(\mathbb{C})$ is injective and satisfies

$$B(f^{m_j}(h(x)), R/(100C_1L)) \subset h_j(B(g^{n_j}(x), r)) \subset B(f^{m_j}(h(x)), R),$$

where g_{n_j} is the inverse map of $g^{n_j} : W_j \to B(g^{n_j}(x), r)$. By Lemma 3.7, the conclusion follows. $\square$

4. IMPROVE LOCAL CONJUGACIES TO ALGEBRAIC CORRESPONDENCES

4.1. **Extend a local conjugacy on a CER.** The following definition was introduced by Sullivan [Sul86].

Definition 4.1. Let f be an edomorphism on $\mathbb{P}^1(\mathbb{C})$. An compact set $K \subset \mathbb{P}^1(\mathbb{C})$ is called a CER of f if

    (i) There exists $m \geq 1$ and a neighborhood V of K such that $f^m(K) = K$ and $K = \cap_{n \geq 0} f^{-mn}(V)$.

    (ii) $f^m : K \to K$ is expanding.

(iii) $f^m : K \to K$ is topologically exact, i.e. for every open set $U \subset K$ there exists $n \geq 0$ such that $f^{mn}(U) = K$.

Theorem 4.2. *Let f, g be two non-Lattès endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let K_f (resp. K_g) be an invariant CER of f (resp. g) which is not a periodic orbit. Let U (resp. V) be a neighborhood of K_f (resp. K_g). Let $h : U \to V$ be a biholomorphic map such that $h \circ f = g \circ h$ on K_f. Then for every connected component U_0 of U such that $U_0 \cap K_f \neq \emptyset$, there exists an irreducible algebraic curve $\Gamma \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is periodic under (f, g) and contains the graph of $h|_{U_0}$.*

Proof. Let D be a small disk intersecting K_f such that $D \subset U_0$. Fix a point $x \in K_f \cap D$. Topological exactness of $f|_{K_f}$ implies the preimages of x under $f|_{K_f}$ are dense in K_f. Let $x_n \in K_f$ such that $f^n(x_n) = x$. Let D_n be the connected component of $f^{-n}(D)$ containing x_n. Expandingness of $f|_{K_f}$ implies that $\operatorname{diam} D_n \leq C\lambda^{-n}$ for some $C > 0, \lambda > 1$. Since K_f is not a periodic orbit, for m large enough, we can choose two points $x_m \in K_f, x'_m \in K_f$ such that $f^m(x_m) = x$ and $f^m(x'_m) = x$, moreover $D_m \subset\subset D, D'_m \subset\subset D$, and $\overline{D_m} \cap \overline{D'_m} = \emptyset$, where D_m (resp D'_m) is the connected component of $f^{-m}(D)$ containing x_m (resp. x'_m). Set $W := D_m \cup D'_m$. Then $f^m : W \to D$ has two repelling fixed point and is non-injective. Moreover $h \circ f^m = g^m \circ h$ on $K_f \cap W$. Since h is holomorphic and $K_f \cap D$ is non-isolated, we have $h \circ f^m = g^m \circ h$ on W. By Theorem 2.1, there exists an irreducible algebraic curve $\Gamma \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is invariant under (f^m, g^m) and contains the graph of $h|_D$. Then Γ also contains the graph of $h|_{U_0}$. The proof is finished. $\square$

The following theorem will be used in the proof of Theorem 1.18. It can be deduced from Sullivan's rigidity theorem for non-linear CERs [Sul86] (see [JX22, Theorem 7.7] for the precise statement and see [PU10, Section 10.2] for a proof), together with a characterization theorem of linear CERs [JX22, Theorem 1.1].

Theorem 4.3. *Let f and g be two edomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2 such that f is non-exceptional. Let $(f, K_f), (g, K_g)$ be two CERs, $f(K_f) = K_f, g(K_g) = K_g$. Let $h : K_f \to K_g$ be a homeomorphism such that $h \circ f = g \circ h$ on K_f. Then the following two conditions are equivalent*

(i) *for every periodic point $x \in K_f$ we have $|df^n(x)| = |dg^n(h(x))|$, where n is the period of x;*

(ii) *there exist a neighborhood U of K_f and a neighborhood V of K_g such that h extends to a holomorphic or antiholomorphic map $h : U \to V$.*

Moreover, in case that any (hence every) condition of (1), (2) holds, then we have

(iii) *there is an algebraic curve $\Gamma \subseteq \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ whose irreducible components are all periodic such that the graph of $h' : K_f \to K_{g'}$ is contained in Γ, where (g', h') is either (g, h) or $(\overline{g}, \tau_{\mathrm{con}} \circ h)$.*

4.2. Improve a local conjugacy on an interval.

Definition 4.4. Let M be a smooth manifold and let $X \subset M$ be a compact subset. Let f be a continuous map defined on a neighborhood U of X such that $f(X) \subset X$. Then X is called a *repeller* if there exists an open set $U' \subset U$ such that $X = \cap_{n \geq 0} f^{-n}(U')$.

We need the following lemma. A proof can be found in Przytycki-Urbanski [PU10, Lemma 6.1.2].

Lemma 4.5. *Let M be a smooth manifold and let $X \subset M$ be a compact subset. Let f be a continuous map defined on a neighborhood U of X such that $f(X) \subset X$. Then we have:*

(i) *Assume $f|_U$ is an open map. If X is a repeller then $f|_X$ is an open map.*

(ii) *Conversely if $f : X \to X$ is distance expanding and $f|_X$ is an open map, then X is a repeller.*

Remark 4.6. Let N be a smooth manifold, $M \subset N$ be a submanifold, and $X \subset M$ be a compact subset. Let $f : N \to N$ be a continuous map such that $f(M) \subset M$. Then there are two definitions for X being repellers, depending on the ambient space is M or N. In the following we will emphasize that K is seen as a subset of which ambient space.

Proof of Theorem 1.18. By one-dimensional Katok's theory [Gel10], positive topological entropy of $f : N_f \to N_f$ implies that there exists a repeller K of $f|_{N_f}$ (here the ambient space is N_f), which is topologically exact and uniformly expanding. Passing to an iterate of f and g, we assume K is f-invariant and there exists a repelling f-fixed point $x \in K \cap N_f^\circ$. By assumption (ii), $h(x) \in h(K)$ is a repelling fixed point of g. Let I be a small open interval centered at $h(x)$. Since

$g : h(K) \to h(K)$ is topologically exact, one can construct the following horseshoe map: there are two open intervals $I_1 \subset\subset I$, $I_2 \subset\subset I$, such that $g^m : I_1 \to I$ and $g^m : I_2 \to I$ are bijective for some $m \geq 1$, moreover there exists $L > 1$ such that $|dg^m(y)| > L$ for every $y \in I_1 \cup I_2$. We define

$$K_g := \{y \in I_1 \cup I_2 : f^{mn}(y) \in I_1 \cup I_2 \text{ for every } n \geq 0\}.$$

Then K_g is a repeller of $g|_{N_g}$ (here the ambient space is N_g) which is also topologically exact and unifromly expanding. Since $h(K)$ is also a repeller of $g|_{N_g}$ (here the ambient space is N_g), shrink the interval I if necessary, we have $K_g \subset h(K)$. We define $K_f := h^{-1}(K_g)$. Then K_f is a repeller of $f|_{N_f}$ (here the ambient space is N_f) which is also topologically exact and unifromly expanding.

By Lemma 4.5 (i), since f^m (resp. g^m) is an open map on a neighborhood of K_f (resp. K_g), $f^m : K_f \to K_f$ and $g^m : K_g \to K_g$ are open maps. Again by Lemma 4.5 (ii), K_f (resp. K_g) is a repeller of f (resp. g), where the ambient space is $\mathbb{P}^1(\mathbb{C})$. Hence K_f and K_g are CERs. By Theeorem 4.3, $h|_{K_f}$ can be extended to a holomorphic or antiholomorphic map $\tilde{h}$ on a neighbprhoof U of K_f. Since $K_f \subset \mathbb{R} \cup \infty$, we can actually let $\tilde{h}$ be holomorphic. Thus we have $\tilde{h} \circ f^m = g^m \circ \tilde{h}$ on K_f. By Theorem 4.2, there exists an irreducible algebraic curve $\Gamma \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is periodic under (f, g), and contains the graph of $h|_{K_f}$. We conclude the result since by our construction, the graph of $h|_{K_f}$ is a Cantor set. $\square$

5. Variants of Levin's theorem

Let f be an endomorphism of $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let U be a connected open subset of f with $U \cap \mathcal{J}_f \neq \emptyset$.

We reformulate Levin's theorem [Lev90, Theorem 1] in the following form.

Theorem 5.1. *Let $\sigma_n : U \to \mathbb{P}^1(\mathbb{C}), n \geq 0$ be a a family of injective holomorphic maps satisfying $\sigma_n \to \text{id}$ as $n \to \infty$ and $\sigma_n^{-1}(\mathcal{J}_f) = \mathcal{J}_f \cap U$. If in additional $\mathcal{J}_f$ is smooth, we assume further that $(\sigma_n)^*(\nu_f) \propto \mu_f|_U$. Then either f is exceptional or $\sigma_n = \text{id}$ for n sufficiently large.*

A small modification of Levin's original proof implies the following result for C^1-maps.

Theorem 5.2. *Let $\sigma_n : U \to \mathbb{P}^1(\mathbb{C}), n \geq 0$ be a family of injective C^1-maps satisfying $\sigma_n \to \text{id}$ as $n \to \infty$ in C^1-topology and $\sigma_n^{-1}(\mathcal{J}_f) = \mathcal{J}_f \cap U$. If in additional $\mathcal{J}_f$ is smooth, we assume further that $(\sigma_n)^*(\mu_f) \propto$*

$\mu_f|_U$. *If there is a repelling fixed point o of f such that $\sigma_n(o) \neq o$ for every $n \geq 0$, then f is exceptional.*

Proof. After shrinking U, we may ask U to be a linearization domain of o. In a suitable coordinate z, we may let $U = \{|z| \leq 1\}$, o be the origin and f is $z \mapsto \lambda z$ with $|\lambda| > 1$. Write $\sigma_n = a_n + b_n(z) + \epsilon_n(z)$ where $b_n \in M_{2 \times 2}(\mathbb{R})$, $\epsilon_n(0) = 0$ and $d\epsilon_n(0) = 0$. Our assumption shows that $a_n \neq 0$ for every $n \geq 0$, $a_n \to 0$, $b_n \to$ id and $\epsilon_n \to 0$ in C^1-topology. After shrinking U, we may assume that there are sequence $c_n, d_n > 0$ tending to 0 such that $|\epsilon_n(z)| \leq c_n$ and $|d\epsilon_n(z)| \leq d_n$ on U.

After taking subsequence, we may assume that there is a sequence $l_n \geq 0$ tending to $+\infty$ such that $a_n \lambda^{l_n} \to a$ and $|a| \in (0, 0.1)$. View λ as a matrix in $M_{2 \times 2}(\mathbb{R})$. Define

$$\delta_n(z) := \lambda^{l_n} \sigma_n(\lambda^{-l_n} z) = a_n \lambda^{l_n} + \lambda^{l_n} b_n \lambda^{-l_n}(z) + \lambda^{l_n} \epsilon(\lambda^{-l_n} z).$$

We have $a_n \lambda^{l_n} \to a$, $\mathrm{tr}(\lambda^{l_n} b_n \lambda^{-l_n}) = \mathrm{tr}(b_n) \to 2$, $\det(\lambda^{l_n} b_n \lambda^{-l_n}) = b_n \to 1$ and the conformal index of $\lambda^{l_n} b_n \lambda^{-l_n}$ is the same as b_n, which tends to 1, we have $\lambda^{l_n} b_n \lambda^{-l_n}$ tends to id. We have $|\lambda^{l_n} \epsilon(\lambda^{-l_n} z)| \leq |\lambda^{l_n} \lambda^{-l_n} d_n|$ which tends to 0. So δ_n tends to the map $T_a : z \mapsto z + a$ locally uniformly. Then we have $T_a^{-1}(\mathcal{J}_f) = \mathcal{J}_f \cap U$. If in additional $\mathcal{J}_f$ is smooth, we have that $T_a^*(\mu_f) \propto \mu_f|_U$. Then we may conclude the proof by Levin's original proof of [Lev90, Theorem 1]. $\qquad\square$

In the rest of this section, we prove the following real analytic version of Levin's theorem.

Theorem 5.3. *Assume that $\mathcal{J}_f$ is not contained in any circle. Let $\sigma_n : U \to \mathbb{P}^1(\mathbb{C}), n \geq 0$ be a family of injective real analytic maps satisfying $\sigma_n \to$ id as $n \to \infty$ in C^1-topology and $\sigma_n^{-1}(\mathcal{J}_f) = \mathcal{J}_f \cap U$. If in additional $\mathcal{J}_f$ is $\mathbb{P}^1(\mathbb{C})$, we assume further that $(\sigma_n)^*(\mu_f) \propto \mu_f|_U$. Then f is exceptional or $\sigma_n =$ id for n sufficiently large.*

Proof. Assume $\sigma_n \neq$ id for infinitely many n, we need to show f is exceptional. After taking subsequence we assume $\sigma_n \neq$ id for every $n \geq 0$. After shrinking U, we may ask U to be a linearization domain of o. In a suitable coordinate z, we may let $U = \{|z| \leq 1\}$, o be the origin and f is $z \mapsto \lambda z$ with $|\lambda| > 1$. By Theorem 5.2, we may assume that $\sigma_n(0) = 0$ for all $n \geq 0$.

We first treat the case where there is $i \geq 0$ such that $d\sigma_i(0) =$ id. Let δ_n be the map $z \mapsto \lambda^n \sigma_i(\lambda^{-n} z)$. Then $\delta_n \to$ id in C^1-topology. Since $\sigma_i \neq$ id, we have $\delta_n \neq$ id. For every repelling periodic point $p \in U$, let $N_p \subseteq \mathbb{Z}_{\geq 0}$ be the set of $n \geq 0$ such that $\delta_n(p) = p$. If there is a repelling periodic point $p \in U$ such that $\mathbb{Z}_{\geq 0} \setminus N_p$ is infinite, then we

conclude the proof by Theorem 5.2 for $\delta_n, n \in \mathbb{Z}_{\geq 0} \setminus N_p$ and p. Now we may assume that $\mathbb{Z}_{\geq 0} \setminus N_p$ is finite for every repelling periodic point $p \in U$. For every repelling periodic point $p \in U$ and $n \in N_p$, we have $\lambda^{-n} p \in \mathrm{Fix}(\delta_0)$. We claim that $\delta_0(p) = p$. Since δ_0 is real analytic and $\delta_0 \neq \mathrm{id}$, the set $\mathrm{Fix}(\delta_0)$ is a proper and real analytic closed subset of U, this implies $\lambda^r \in \mathbb{R}$ for some $r \geq 1$. Then $\mathrm{Fix}(\delta_0)$ contains the line $p\mathbb{R}$. Hence $\delta_0(p) = p$. This implies the claim. Since repelling periodic points are dense in $\mathcal{J}_f$, $\mathcal{J}_f \cap U \subseteq \mathrm{Fix}(\delta_0)$. Since $\mathrm{Fix}(\delta_0)$ is a proper and real analytic closed subset of U, we conclude the proof by Theorem 1.3.

We now assume that $d\sigma_i(0) \neq \mathrm{id}$ for every $i \geq 0$. Note that for every $i \geq 0$, there is a sequence $n_j \to \infty$ such that the maps $z \mapsto \lambda^{n_j} \sigma_i(\lambda^{-n_j} z)$ tends to $d\sigma_i(0)$. After replacing σ_i by $d\sigma_i(0)$, we may assume that all σ_i are $\mathbb{R}$-linear. Since $\mathcal{J}_f$ is not contained in a circle, by Theorem 1.3, there are repelling periodic points $p, q \in U$ such that $0, p, q$ are not collinear. Then $\{p, q\} \nsubseteq \mathrm{Fix}(\sigma_n)$ for every $n \geq 0$. We may assume that for infinitely many $n \geq 0$, $p \notin \mathrm{Fix}(\sigma_i)$. We then conclude the proof by Theorem 5.2. $\qquad\square$

6. Real analytic local rigidity of Julia sets

The aim of this section is to prove the following two theorems.

Theorem 6.1. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of them is non-exceptional. Let μ (resp. ν) be a non-atomic invariant ergodic probability measure with positive Lyapunov exponent of f (resp. g). Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open subset such that $U \cap C_g$ is connected and $U \cap \mathrm{Supp}\,\nu \neq \emptyset$. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a real analytic homeomorphism preserving the orientation such that*

  (i) *$h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$;*
  (ii) *$h_*(\nu)$ is equivalent to μ on $h(U)$.*

Then there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(z), z), z \in U \cap C_f\}$ of h. Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is not contained in any circle, then h is holomorphic.

Theorem 6.2. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of them is non-exceptional . Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open subset such that $U \cap C_f$ is connected. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a real analytic homeomorphism preserving the orientation. Assume that*

(i) $h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$;

(ii) the Hausdorff dimension of non-conical points of g is 0.

Then there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(z), z), z \in U \cap C_f\}$ of h. Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is not contained in any circle, then h is holomorphic.

6.1. **Replacement of Montel's theorem.** The proof of the following lemma is given by Donyi Wei.

Lemma 6.3. Let $g_n : \mathbb{D} \to \mathbb{D}, n \geq 0$ be a sequence of holomorphic maps. Assume that there is a sequence $r_n \in (0, 1)$ tending to 0 such that $g_n(\mathbb{D}) \subseteq \mathbb{D}(0, r_n)$. Let $h : \mathbb{D} \to \mathbb{D}$ be a C^1-homeomorphism with $|dh| \leq C$ on $\mathbb{D}$ for some $C > 0$ Let $f_n : \mathbb{D}(0, Cr_n) \to \mathbb{D}, n \geq 0$ be a sequence of holomorphic maps. Set $h_n := f_n \circ h \circ g_n, n \geq 0$. Then there is a subsequence $n_j, j \geq 0$ such that h_{n_j} converges in C^1-topology. Moreover $\lim_{j \to \infty} h_{n_j}$ is real analytic.

Assume further that $dh(0)$ is invertible and f_n, g_n are injective, then any non-constant limit $H' = \lim_{j \to \infty} h_{n_j}$ is a homeomorphism to its image and its conformal index is the same as $dh(0)$ at every point in $\mathbb{D}$.

Proof. For $r \in (0, +\infty)$, define $[r] : \mathbb{C} \to \mathbb{C}$ the map $z \mapsto rz$. Then we have

$$h_n = (f_n \circ [r_n]|_{\mathbb{D}(0,C)}) \circ ([r_n^{-1}] \circ h \circ [r_n]|_{\mathbb{D}}) \circ ([r_n^{-1}] \circ g_n).$$

It is clear that $\{f_n \circ [r_n]|_{\mathbb{D}(0,C)}, n \geq 0\}$ and $\{[r_n^{-1}] \circ g_n, n \geq 0\}$ are normal families. Set $H := dh(0)$. We view H as a $\mathbb{R}$-linear map from $\mathbb{C}$ to $\mathbb{C}$. Then $h(z) = H(z) + e(z)$ for some C^1-map e with $e(0) = 0$ and $de(0) = 0$. There is a continuous function $\epsilon : [0, 1] \to [0, C]$ with $\epsilon(0) = 0$ such that

$$|de(z)| \leq \epsilon(|z|).$$

Then

$$[r_n^{-1}] \circ h \circ [r_n]|_{\mathbb{D}} = [r_n^{-1}] \circ H \circ [r_n]|_{\mathbb{D}} + [r_n^{-1}] \circ e \circ [r_n] = H + [r_n^{-1}] \circ e \circ [r_n].$$

We claim that $[r_n^{-1}] \circ e \circ [r_n] \to 0$ in C^1-topology as $n \to \infty$. For every $z \in \mathbb{D}$, we have

$$|[r_n^{-1}] \circ e \circ [r_n](z)| \leq r_n^{-1} r_n \epsilon(r_n) = \epsilon(r_n)$$

and

$$|d([r_n^{-1}] \circ e \circ [r_n])(z)| = |de(r_n z)| \leq \epsilon(r_n),$$

which concludes the claim. Then there is a subsequence $n_j, j \geq 0$ such that $f_{n_i} \circ [r_{n_i}]|_{\mathbb{D}(0,C)}$ and $[r_{n_i}^{-1}] \circ g_{n_i}$ converges to holomorphic maps F and G respectively. Then $\lim\limits_{j \to \infty} h_{n_j} = F \circ H \circ G$ is real analytic.

Assume further that $dh(0)$ is invertible and f_n, g_n are injective. Assume that $h_{n_j}, j \geq 0$ converges to a non-constant map H'. After taking subsequence, we may assume that $f_{n_i} \circ [r_{n_i}]|_{\mathbb{D}(0,C)}$ and $[r_{n_i}^{-1}] \circ g_{n_i}$ converges to holomorphic maps F and G respectively. By Hurwitz's theorem, F (resp. G) is either constant or injective. Since $H' = F \circ H \circ G$ is non-constant, both F and G are injective. Since H is a homeomorphism, then $H' = F \circ H \circ G$ is a homeomorphism to its image. Since F and G are holomorphic, the conformal index of H' is the same as $dh(0)$ at every point in $\mathbb{D}$. $\qquad\square$

6.2. **Real analytic rigidity of horseshoe.**

Theorem 6.4. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of f and g is non-exceptional. Let D be a simply connected open subset of $\mathbb{P}^1(\mathbb{C})$ and W_1, W_2 be disjoint simply connected open subsets of D compactly contained in D such that $g : W_i \to D$ is biholomorphic, $i = 1, 2$. Let $h : D \to h(D) \subseteq \mathbb{P}^1(\mathbb{C})$ be a real analytic homeomorphism to its image such that $h \circ g|_{W_1 \cup W_2} = f \circ h|_{W_1 \cup W_2}$. Then h is conformal.*

Moreover, there exists an irreducible algebraic curve $\Gamma \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is invariant under (f, g') and contains the graph of h' where (g', h') equals to (g, h) or $(\bar{g}, \sigma_{\text{con}} \circ h)$ depending on whether h is holomorphic or anti-holomorphic.

Proof. If h is conformal, the last statement follows from Theorem 2.1. We only need to we show that h is conformal.

We may assume that g is non-exceptional. Set $W := W_1 \cup W_2$. Set $K_g := \{x \in W | \; g^n(x) \in D \text{ for every } n \geq 0\}$ and $K_f := h(K_g)$. Then (K_g, g^b) and (K_f, f^a) are CERs. By (iii) of Theorem 4.3, f is non-exceptional. If K_g is not contained in any proper real analytic closed subset of D, by (iii) of Theorem 4.3, h is conformal. Then we may assume that K_g is contained in a proper real analytic closed subset of D. By (ii) of Theorem 4.3, there is a conform map h' on some neighborhood D' of K_g such that $h'|_{K_g} = h|_{K_g}$. Then for every connected component D'' of $g^{-1}(D') \cap D'$, if $D'' \cap K_g \neq \emptyset$, we have $h'|_{D''} \circ g = f \circ h'|_{D''}$.

There is a connected open subset U in $W \cap D'$ such that $K_g \cap U \neq \emptyset$ and is contained in a closed smooth connected curve γ in U. Let p be any periodic point in $U \cap K_g$. Then there exists $m \geq 1$ such that $dg^m(p) \in (1, +\infty)$. We fix this m.

There is a connected open neighborhood U_1 of p such that $g^i(U_1) \subseteq W \cap D'$ for every $i = 0, \dots, m$ and the map $g^m : U_1 \to g^m(U_1)$ conjugates to a linear map $z \in \lambda^{-1}\mathbb{D} \mapsto \lambda z \in \mathbb{D}$ where $\lambda = dg^m(p)$ has norm > 1. We may take U_1 sufficiently small such that $\overline{\cup_{i=1}^m g^i(U_1)} \neq K_g$. We have $h \circ g^m|_{U_1} = f^m \circ h|_{U_1}$ and $h' \circ g^m|_{U_1} = f^m \circ h'|_{U_1}$. After replacing f, h, h' by $\overline{f}, \sigma_{\mathrm{con}} \circ h, \sigma_{\mathrm{con}} \circ h'$, we may assume that h' is holomorphic on $g^m|_{U_1}$. Since $\lambda \in \mathbb{R}$, we get $df^m(h(p)) = \lambda$. Fix the coordinate z on $g^m(U_1)$ such that $g^m(U_1) = \{|z| < 1\} = \mathbb{D}$ and $g^m|_{U_1} : U_1 \to g^m(U_1)$ is $z \mapsto \lambda z$. Choose a suitable coordinate on $h_1(U_1)$, we may ask $h' = \mathrm{id}$. Then we have $h \circ (\lambda z) = \lambda \circ h(z)$ in this coordinate. Since $\lambda \in (1, +\infty)$ and h is real analytic, h is $\mathbb{R}$-linear in this coordinate. Then $K_g \cap g^m(U_1) \subseteq \mathrm{Fix}(h')$. Then $\mathrm{Fix}(h')$ is a line or $h' = \mathrm{id}$. If $h' = \mathrm{id}$, then we are done, so we may assume that $\mathrm{Fix}(h')$ is a line. After replacing z by a rotation, we may assume that $K_g \cap g^m(U_1) \subseteq \mathrm{Fix}(h') = g^m(U_1) \cap \mathbb{R}$.

Pick a sequence $p_i, i \geq 0$ such that $p_0 = p$, $p_i \in K_g$, $p_{i-1} = g^m(p_i), i \geq 1$ and $p_i \to p$ as $i \to \infty$. There is $l_0 \geq 1$ such that $p_i \in U_1$ for every $i \geq l_0$. For every $r \in (0, 1)$, let $V_i(r)$ be the connected component of $g^{-im}(\mathbb{D}(0, r))$ containing p_i. For r sufficiently small, we have $g^i(V_{l_0}(r)) \subseteq D' \cap W$ for every $i = 0, \dots, l_0 m$ and $g^{l_0 m}|_{V_{l_0}(r)} : V_{l_0}(r) \to \mathbb{D}(0, r)$ is an isomorphism. Then for $l \geq l_0$ sufficiently large, we have $V_l(r) \subset\subset \mathbb{D}(0, r)$. Since $g^{lm}|_{V_l(r)} : V_l(r) \to \mathbb{D}(0, r)$ is an isomorphism, there is a unique g^{lm}-fixed point $q \in V_l(r)$. Set $\mu := dg^{lm}(q)$. We have $\mu \in (1, +\infty)$. Since $q \in K_g \cap g^m(U_1)$, $q \in \mathbb{R}$ in our coordinate. Let w be the coordinate $w = z - q$. Since $q \in \mathrm{Fix}(h)$ and h is linear in the coordinate z, h is linear in the coordinate w. There is a local holomorphic map ϕ at q, taking form $\phi(w) = w + O(w^2)$ such that $\phi \circ g^{lm} = \mu \phi$. Since $g^{lm} \circ h = h \circ g^{lm}$ locally at q, we get

$$\mu(\phi \circ h \circ \phi^{-1})(w) = (\phi \circ h \circ \phi^{-1})(\mu w).$$

The same argument as in the above paragraph shows that $\phi \circ h \circ \phi^{-1}$ is $\mathbb{R}-$linear in w. Hence in coordinate w, we get $\phi \circ h \circ \phi^{-1} = d(\phi \circ h \circ \phi^{-1})(0) = h$. Then we get $\phi \circ h = h \circ \phi$. Then for w sufficiently closed to 0, we have

$$d\phi(h(w)) \times h = h \times d\phi(w)$$

as (2×2)-matrix. We first assume that $d\phi(w)$ is not a constant. Since $d\phi(w) \in \mathbb{R}$, there is $w \neq 0$ sufficiently closed to 0 such that $\alpha := \arg d\phi(w) \notin \pi\mathbb{Z}$. If h is not conformal, it maps a circle C centered at 0 to an ellipse E which is not a circle. Assume that the major axis of E is contained in the line $e^{i\theta}\mathbb{R}$ with $\theta \in [0, \pi)$. Since $d\phi(w)$ and $d\phi(h(w))$ are conformal, the major axis of $h \times d\phi(C)$ is contained in the line $e^{i\theta}\mathbb{R}$

and the major axis of $d\phi(h(w)) \times h(C)$ is contained in the line $e^{i(\theta+\alpha)}\mathbb{R}$. Since $i(\theta+\alpha) - i\theta \notin \pi\mathbb{Z}$, $h \times d\phi(C) \neq d\phi(h(w)) \times h(C)$, which is a contradiction. So $d\phi(w)$ is a constant. In other words, ϕ is $\mathbb{R}-$linear in w. Hence in the coordinate z, the map $g^{ml} : V_l(r) \to \mathbb{D}(0,r)$ is given be that affine map $z \mapsto \mu z + (\mu - 1)q$. Set $W'_2 := V_l(r)$ and $W'_1 := \lambda^{-l}\mathbb{D}(0,r)$ and $W' := W'_1 \cup W'_2$. Set

$$K' := \{x \in W'|\ g^{nlm}(x) \in \mathbb{D}(0,r) \text{ for every } n \geq 0\}.$$

Then (K', g^{lm}) is a CER. Moreover, since in the coordinate z, both $g^{lm}|_{W'_1}$ and $g^{lm}|_{W'_2}$ are affine, it is a linear CER [JX22, Definition 7.6 (ii)]. By [JX22, Theorem 1.1], g is exceptional, which is a contradiction.
$\square$

6.3. **Real analytic rigidity.** We first prove a real analytic version of Lemma 3.7.

Lemma 6.5. *Let f, g be two non-exceptional endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open subset such that $U \cap C_f$ is connected. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a real analytic homeomorphism preserving the orientation, such that $h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$, if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$. Assume that there exist a point $x \in U \cap \mathcal{J}_g$ which is bi-conical for (g, h, f). Then there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a, g^b) and contains the graph $\{(h(z), z), z \in U \cap C_f\}$ of h. Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is not contained in any circle, then h is holomorphic.*

Proof of Lemma 6.5. If one of f, g is exceptional, then $\mathcal{J}_f$ and $\mathcal{J}_g$ are smooth. By [Zdu90], both of them are exceptional, which contradicts to our assumption. So both of them are non-exceptional.

If J_g is contained in a circle, by Theorem 1.3, J_f is contained in a circle. We may assume that $C_f = C_g = \mathbb{R} \cup \{\infty\}$. After shrinking U, we may assume that there is a holomorphic injective map h' on U such that $h' = h$ on $U \cap (\mathbb{R} \cup \{\infty\})$. After replacing h by h', we may assume that h is holomorphic. Then we concludes the proof by Lemma 3.7.

Now assume that $\mathcal{J}_g$ (hence $\mathcal{J}_f$) is not contained in any circle. We claim that h is conformal. Once this claim holds, we may conclude the proof by Lemma 3.7.

We now prove the claim. After shrinking U, we may assume that h is biLipschitz. Since x is biconical for (g, h, f), it is not g-preperiodic and there are positive constants r, R, K and two sequences of positive integers $n_j \to +\infty$, $m_j \to +\infty$, $j \geq 1$ satisfy the conditions (i), (ii) in Definition 3.5. Same argument as in Lemma 3.7 except replacing

Levin's theorem [Lev90, Theorem 1] (c.f. Theorem 5.1), by Theorem 5.3 and Montel's theorem by Lemma 6.3, after taking subsequence, we get $W := W_1 \cup W_2 \subset D$ and $a, b \geq 1$ as the proof of Theorem 5.3, such that

$$f^a \circ h_1 = h_1 \circ g^b$$

on W. Moreover $g^b : W_i \to D, i = 1, 2$ is are holomorphic homeomorphisms. By Theorem 6.4, h_1 is conformal. Hence h is conformal. Since h preserves the orientation, h is holomorphic, which concludes the proof by Lemma 3.7. $\qquad\square$

Proof of Theorem 6.1. The proof of Theorem 6.1 is the same as the proof of Theorem 3.1 except replacing Lemma 3.7 by its real analytic version Lemma 6.5. $\qquad\square$

Proof of Theorem 6.2. The proof is the same as the proof of Theorem 3.2 except replacing Lemma 3.7 by its real analytic version Lemma 6.5. $\qquad\square$

7. C^1 LOCAL RIGIDITY OF JULIA SETS

The aim of this section is to prove the following two theorems.

Theorem 7.1. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Let μ (resp. ν) be a non-atomic invariant ergodic probability measure with positive Lyapunov exponent of f (resp. g). Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open subset such that $U \cap \mathrm{Supp}\,\nu \neq \emptyset$. Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open subset such that $U \cap \mathrm{Supp}\,\nu \neq \emptyset$. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a C^1-homeomorphism such that*

- (i) $h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; *if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$;*
- (ii) $h_*(\nu)$ *is equivalent to μ on $h(U)$.*

Then up to change f to $\overline{f}$, there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is periodic under (f^a, g^b). Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is $\mathbb{P}^1(\mathbb{C})$, then h is conformal.

Theorem 7.2. *Let f, g be two endomorphisms on $\mathbb{P}^1(\mathbb{C})$ of degree at least 2. Assume that one of them is non-exceptional . Let $U \subset \mathbb{P}^1(\mathbb{C})$ be an open subset such that $U \cap C_f$ is connected. Let $h : U \to h(U) \subseteq \mathbb{P}^1(\mathbb{C})$ be a C^1-homeomorphism. Assume that*

- (i) $h(U \cap \mathcal{J}_g) = h(U) \cap \mathcal{J}_f$; *if $\mathcal{J}_f$ is smooth, we assume further that $h_*(\mu_g) \propto \mu_f$ on $h(U)$;*
- (ii) *the Hausdorff dimension of non-conical points of g is 0.*

Then up to change f to $\bar{f}$, there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is periodic under (f^a, g^b). Moreover, if $\mathcal{J}_f$ or $\mathcal{J}_g$ is $\mathbb{P}^1(\mathbb{C})$, then h is conformal.

7.1. **Distances in set of positive density.** Let $G \subseteq \mathbb{Z}_{\geq 0}$ be a subset and $l_1, \ldots, l_s \in \mathbb{Z}_{\geq 0}$. Let $G(l_1, ..., l_s)$ be the set $n \in G$ such that $n + l_1, \ldots, n + l_s \in G$.

Lemma 7.3. *Let $G \subseteq \mathbb{Z}_{\geq 0}$ be a subset with $\bar{d}(G) > 0$. Then for every $l \geq 0$, there is $l_1 \geq l$ such that $\bar{d}(G(l_1)) > 0$.*

Proof. We may assume that $l \geq 2$. For $j = 0, \ldots, l$, let $G_{j/l}$ be the set of $n \in G$ with $n = j \mod l$. Then there is j such that $\bar{d}(G_{j/l}) > 0$. After replacing G by $G_{j/l}$, we may assume that for distinct $n_1, n_2 \in G$, $|n_1 - n_2| \geq l$. Since $\bar{d}(G) > 0$, there is $q \in \mathbb{Z}_{\geq l}$ and a strictly increasing sequence $N_i, i \geq 0$ such that

$$\#(G \cap \{0, \ldots, N_i - 1\}) \geq N_i/q.$$

We may assume that $N_i/100q$ is an integer r_i. Dividing $\{0, ..., N_i - 1\}$ by r_i segments $I_s = \{100qs, \ldots, 100q(s+1) - 1\}$, $s = 0, \ldots, r_i - 1$. It is clear that at least $(0.99N_i)/100q^2$ segments I_s containing at least 2 elements of G. So

$$\bar{d}(\cup_{i=1}^{100q} G(i)) > 1/(200q^2).$$

Since $G(i) = \emptyset$ for $i = 1, \ldots, l - 1$, there is $i \in \{l, \ldots, 100q\}$ such that $\bar{d}(G(i)) > 0$, which concludes the proof. $\square$

Applying Lemma 7.3 inductively, we get the following consequence.

Corollary 7.4. *Let $G \subseteq \mathbb{Z}_{\geq 0}$ be a subset with $\bar{d}(G) > 0$. Then there is a strictly increasing sequence $l_i \geq 1, i \geq 0$ such that for every $s \geq 0$, $\bar{d}(G(l_1, \ldots, l_s)) > 0$.*

7.2. C^1**-rigidity.**

Proof of Theorem 7.1. If one of f, g is exceptional, we conclude the proof by [Zdu90]. We now assume that both of them are non-exceptional. When $\mathcal{J}_g$ (hence $\mathcal{J}_f$) is smooth, we may ask $\mu := \mu_f$ and $\nu := \mu_g$.

After shrinking U, we may assume that h is C-biLipschitz. By Lemma 3.14, for $\nu|_U$-a.e. point x, it is not g-preperiodic and there are positive constants r, R, K and two strictly increasing sequences of positive integers $n'_j \to +\infty$, $m'_j \to +\infty$, $j \geq 1$ satisfy the conditions (i), (ii) in Definition 3.5 and $\bar{d}(G) > 0$ where $G = \{n'_j, j \geq 0\}$. By Corollary 7.4, there is a strictly increasing sequence $l_i \geq 1, i \geq 1$ such that for every $s \geq 0$, $\bar{d}(G(l_1, \ldots, l_s)) > 0$. Define the function $\theta : G \to \mathbb{Z}_{\geq 0}$ by $\theta(n'_i) = m'_i$. Then θ is strictly increasing. Pick a

sequence $n_j \in \overline{d}(G(l_1, \ldots, l_j)), j \geq 1$ and set $m_j := \theta(n_j)$. After taking subsequence, we may assume that $g^{n_j}(x)$ converges to a point $p \in \mathcal{J}(f)$.

Define $h_j := f^{m_j} \circ h \circ g_{n_j}$ as in Lemma 3.7 and D be the disc centered at p of radius $r/2$ as in the proof of Lemma 3.7. After shrinking D and taking subsequence, by Lemma 6.3, we may assume that $h_j|_D$ converges to an injective real analytic map H. Each h_j satisfies the following condition: $h_j(U \cap \mathcal{J}_g) = h_j(U) \cap \mathcal{J}_f$, if $\mathcal{J}_f$ is smooth, we assume further that $(h_j)_*(\mu_g) \propto \mu_f$ on $h_j(D)$. Hence H satisfies the same condition.

Lemma 7.5. *The point p is bi-conical for (f, H, g).*

By Lemma 6.5, up to change f to $\overline{f}$, there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is periodic under (f^a, g^b). If $\mathcal{J}_g$ is not contained in a circle, by Theorem 6.1, H is conformal. Then $dh(x)$ is conformal by Lemma 6.3. We conclude the proof. $\square$

Proof of Lemma 7.5. For an open subset Ω in $\mathbb{P}^1(\mathbb{C})$ and $y \in \Omega$, define

$$\rho^*(\Omega, y) := \inf\{r \geq 0, \Omega \subseteq B(y, r)\}$$

and

$$\rho_*(\Omega, y) := \sup\{r \geq 0, B(y, r) \subseteq \Omega\}.$$

For every $s_1 \in G$ and $s_2 \in \{0, \ldots, s_1\}$, let W_{s_1/s_2} be the connected component of $g^{-(s_1-s_2)}(B(g^{s_1}(x), r))$ containing $g^{s_2}(x)$. Denote by $g_{s_1/s_2} : B(g^{s_1}(x), r) \to W_{s_1/s_2}$ the inverse of the map $g^{s_1-s_2} : W_{s_1/s_2} \to B(g^{s_1}(x), r)$. For every $u \in G$, define $h'_u := f^{\theta(u)} \circ h \circ g_u$. Set $P := \{n_i, i \geq 1\}$. after shrinking r, we may assume that h'_{n_i} tends to H uniformly on D. By Koebe distortion theorem, after shrinking r, we may assume that for every $s_1 \in G$ and $s_2 \in \{0, \ldots, s_1\}$, the injections g_{s_1/s_2} ad h'_{s_1} has good distortion in the following sense: For every $0 < t_1 \leq t_2 \leq 1$,

$$\frac{\rho^*(g_{s_1/s_2}(B(g^{s_1}(x), t_2)), g^{s_2}(x))}{\rho_*(g_{s_1/s_2}(B(g^{s_1}(x), t_1)), g^{s_2}(x))} \leq (1 + 99^{-999})\frac{t_2}{t_1}$$

$$\frac{\rho_*(g_{s_1/s_2}(B(g^{s_1}(x), t_2)), g^{s_2}(x))}{\rho^*(g_{s_1/s_2}(B(g^{s_1}(x), t_1)), g^{s_2}(x))} \geq (1 - 99^{-999})\frac{t_2}{t_1}$$

$$\frac{\rho^*(g_{s_1}(B(g^{s_1}(x), t_2)), h'_{s_1}(g^{s_1}(x)))}{\rho_*(g_{s_1}(B(g^{s_1}(x), t_1)), h'_{s_1}(g^{s_1}(x)))} \leq (1 + 99^{-999})\frac{t_2}{t_1}$$

and

$$\frac{\rho_*(g_{s_1}(B(g^{s_1}(x), t_2)), h'_{s_1}(g^{s_1}(x)))}{\rho^*(g_{s_1}(B(g^{s_1}(x), t_1)), h'_{s_1}(g^{s_1}(x)))} \geq (1 - 99^{-999})\frac{t_2}{t_1}.$$

For every $i \geq 0$, let W_i be the connected component of $g^{-l_i}(B(g^{l_i}(q), r/2))$ containing q. Let L be a constant larger than the maximum of $|df|$ and $|dg|$ on $\mathbb{P}^1(\mathbb{C})$. Pick $u_i \in P$ sufficiently large, we have $u_i + l_i \in G$,

$$d(q, g^{u_i}(x)) \leq 99^{-99i} \min\{r, R/K\} L^{-l_i}$$

and for every $z \in D$,

$$(7.1) \qquad d(H(g^{u_i}(z)), h'_{u_i}(g^{u_i}z)) \leq 99^{-99i} \min\{r, R/K\} L^{-l_i}.$$

Since $d(g^{l_i}(q), g^{u_i+l_i}(x)) \leq 99^{-99i} \min\{r, R/K\}$, we have

$$B(g^{l_i+u_i}(x), 0.49r) \subseteq B(g^{l_i}(q), r/2) \subseteq B(g^{l_i+u_i}(x), 0.51r).$$

Then $g_{l_i,q} := g_{(u_i+l_i)/u_i}|_{B(g^{l_i}(q),r/2)}$ is injective. By Lemma 3.4 and the assumption that $g_{(u_i+l_i)/u_i}$ has good distortion,

$$\operatorname{diam}\left(g_{(u_i+l_i)/u_i}(B(g^{l_i}(q), r))\right) \to 0.$$

We may assume that for every $i \geq 1$, $g_{(u_i+l_i)/u_i}(B(g^{l_i}(q), r)) \subseteq B(q, 0.1r)$. Set

$$v_i := \theta(u_i + l_i) - \theta(u_i).$$

Observe that

$$\rho_*(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x), 0.49r)), g^{u_i}(x)) \geq 0.49r L^{-l_i}.$$

Then we have

$$\frac{\rho_*(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x), 0.49r))), g^{u_i}(x))}{\rho^*(h_{u_i}(B(g^{u_i}(x), r), 0.49r))), g^{u_i}(x))} \geq 0.48 L^{-l_i}$$

So

$$(7.2) \quad \rho_*(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x), 0.49r))), g^{u_i}(x)) \geq 0.48 L^{-l_i} R/K.$$

By (7.1), we have

$$
\begin{aligned}
&\rho_*(H(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x), 0.49r))), h'_{u_i}(g^{u_i}(x))) \\
(7.3) \quad &\geq \rho_*(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x), 0.49r))), h'_{u_i}(g^{u_i}(x))) \\
&\quad - 2 \times 99^{-99i} \min\{r, R/K\} L^{-l_i}
\end{aligned}
$$

and

$$
\begin{aligned}
&\rho^*(H(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x), 0.51r))), h'_{u_i}(g^{u_i}(x))) \\
(7.4) \quad &\geq \rho^*(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x), 0.51r))), h'_{u_i}(g^{u_i}(x))) \\
&\quad + 2 \times 99^{-99i} \min\{r, R/K\} L^{-l_i}.
\end{aligned}
$$

By (7.2) and (7.3), we get

$$\rho_*(H(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),0.49r))),h'_{u_i}(g^{u_i}(x)))$$

(7.5)
$$\geq 0.99\rho_*(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),0.49r))),h'_{u_i}(g^{u_i}(x)))$$

$$\geq 0.48\rho_*(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),r))),h'_{u_i}(g^{u_i}(x))).$$

By (7.2) and (7.4), we get

$$\rho^*(H(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),0.51r))),h'_{u_i}(g^{u_i}(x)))$$

(7.6)
$$\leq 1.01\rho^*(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),0.51r))),h'_{u_i}(g^{u_i}(x)))$$

$$\leq 0.52\rho^*(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),r))),h'_{u_i}(g^{u_i}(x))).$$

So we get

$$\rho_*(f^{v_i}(H(g_{(u_i+l_i)/u_i}(B(g^{l_i}(q),r/2)))),f^{v_i}(h'_{u_i}(g^{u_i}(x))))$$

(7.7)
$$\geq \rho_*(f^{v_i}(H(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),0.49r)))),f^{v_i}(h'_{u_i}(g^{u_i}(x))))$$

$$\geq 0.47\rho_*(f^{v_i}(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),r)))),f^{v_i}(h'_{u_i}(g^{u_i}(x))))$$

$$\geq 0.47R/K.$$

and

$$\rho^*(f^{v_i}(H(g_{(u_i+l_i)/u_i}(B(g^{l_i}(q),r/2)))),f^{v_i}(h'_{u_i}(g^{u_i}(x))))$$

(7.8)
$$\leq \rho^*(f^{v_i}(H(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),0.51r)))),f^{v_i}(h'_{u_i}(g^{u_i}(x))))$$

$$\leq 0.53\rho^*(f^{v_i}(h'_{u_i}(g_{(u_i+l_i)/u_i}(B(g^{l_i+u_i}(x),r)))),f^{v_i}(h'_{u_i}(g^{u_i}(x))))$$

$$\leq 0.53R.$$

This concludes the proof. □

Proof of Theorem 7.2. We note that if $\mathcal{J}_f$ or $\mathcal{J}_g$ is $\mathbb{P}^1(\mathbb{C})$, then both of them are $\mathbb{P}^1(\mathbb{C})$. The proof of Theorem 3.2 show that there is a bi-conical point x for (g,h,f). Moreover, if $\mathcal{J}_f = \mathbb{P}^1(\mathbb{C})$, the CER K in the proof of Theorem 3.2 con be constructed in any given open subset. So the bi-conical points for (g,h,f) are dense in U. As in the proof of Theorem 7.1, one can construct a real analytic morphism $H : D \to \mathbb{P}^1(\mathbb{C})$ such that $H(D \cap \mathcal{J}_g) = H(D) \cap \mathcal{J}_f$; if $\mathcal{J}_f$ is smooth, we assume further that $H_*(\mu_g) \propto \mu_f$ on $H(U)$. Moreover, H is conformal if and only if $dh(x)$ is conformal.

Apply the argument in the proof of Theorem 3.2 again, we show that there is a bi-conical point y for (g,H,f). By Lemma 6.5, Then up to change f to $\overline{f}$, there exist positive integers a and b and an irreducible algebraic curve Z in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ such that Z is preperiodic under (f^a,g^b). Moreover, if $\mathcal{J}_f = \mathbb{P}^1(\mathbb{C})$, then H is conformal. Then $dh(x)$ is

conformal. Since such x can be chosen in a dense set in U and h is C^1, h is conformal. This concludes the proof. $\square$

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