PERIODIC POINTS OF BIRATIONAL TRANSFORMATIONS ON PROJECTIVE SURFACES

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Abstract. We give a classification of birational transformations on smooth projective surfaces which have a Zariski dense set of non-critical periodic points. In particular, we show that if the first dynamical degree is greater than one, the union of all non-critical periodic orbits is Zariski dense.

1. Introduction

Hrushovski and Fakhruddin [19] recently proved by purely algebraic methods that the set of periodic points of a *polarized* endomorphism of a projective variety over any algebraically closed field is Zariski dense. Recall that a morphism $f : X \to X$ of a projective variety is said to be polarized if there is an ample line bundle $L \to X$ such that $f^*L = dL$ for some $d \geq 2$. In this article, we give a complete classification of birational surface maps whose periodic points are Zariski dense.

In order to state our main result, we first review some basic notions related to birational transformations of surfaces. Let X be a projective surface, L be an ample line bundle on X and $f : X \dashrightarrow X$ be any birational transformation. We set $\deg_L(f) = (f^*L \cdot L)$ and call it the degree of f with respect to L. One can show (see e.g. [4]) that $\deg_L(f^{m+n}) \leq 2 \deg_L(f^m) \deg_L(f^n)$ for all $n, m \geq 0$, so that the limit

$$\lambda_1(f) := \lim_{n\to\infty} \deg_L(f^n)^{1/n} \geq 1$$

is well defined. It is independent on the choice of L and it is called the first dynamical degree of f. It is also constant on the conjugacy class of f in the group of birational transformations of X. It is a fact ([15, 24]) that when $\lambda_1(f) = 1$ and $\deg_L(f^n)$ is unbounded, f preserves either an elliptic or a rational fibration and this invariant fibration is unique.

A point p is said to be periodic *non critical* if its orbit under f meets neither the indeterminacy set of f nor its critical set and is finite.

Theorem 1.1. *Let X be a smooth projective surface over an algebraically closed field of characteristic different from 2 and 3. Let $L \to X$ be an ample line bundle and $f : X \dashrightarrow X$ be a birational transformation of X. Denote by $\mathcal{P}$ the set of non-critical periodic points of f. Then we are in one of the following three cases.*

(i) *If $\lambda_1(f) > 1$, then $\mathcal{P}$ is Zariski dense.*
(ii) *If $\lambda_1(f) = 1$ and $\deg_L(f^n)$ is unbounded then $\mathcal{P}$ is Zariski dense if and only if the action of f on the base of its invariant fibration is periodic.*

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(iii) *If $\lambda_1(f) = 1$, and $\deg_L(f^n)$ is bounded, then $\mathcal{P}$ is Zariski dense if and only if there is an integer $N > 0$ such that $f^N = \mathrm{id}$.*

The most interesting case in the previous theorem is case (i). We actually prove this result over a field of arbitrary characteristic.

Theorem 1.2. *Let X be a projective surface over an algebraically closed field $\mathbf{k}$, and $f : X \dashrightarrow X$ be a birational transformation. If $\lambda_1(f) > 1$ then the set of non-critical periodic points is Zariski dense in X.*

In the case $\mathbf{k} = \mathbb{C}$, this theorem has been proved in many cases using analytic methods. In [3, 18, 14] Diller, Dujardin and Guedj proved it for birational polynomial maps, or more generally for any birational transformation such that the points of indeterminacy of f^{-1} do not cluster too much near the points of indeterminacy of f.

For completeness, we also deduce from the work of Amerik [2] the following result.

Theorem 1.3 ([2])**.** *Let X be a projective surface over an algebraically closed field k of characteristic 0, and $f : X \dashrightarrow X$ be a birational transformation with $\lambda_1(f) > 1$. Then there exists a k-point $x \in X(k)$ such that $f^n(x) \in X \setminus I(f)$ for any $n \in \mathbb{Z}$ and $\{f^n(x) | n \in \mathbb{Z}\}$ is Zariski dense.*

This Theorem is closely related to a question of S.-W.Zhang that asked in [31] whether for any polarized endomorphism on a projective variety defined over an algebraically closed field k of characteristic 0 there exists a k-point with a Zariski dense orbit.

Let us explain now our strategy to prove Theorem 1.2. We follow the original method of Hrushovski and Fakhruddin by reducing our result to the case of finite fields.

For the sake of simplicity, we shall assume that $X = \mathbb{P}^2$ and $f = [f_0 : f_1 : f_2]$ is a birational transformation with $\lambda_1(f) > 1$ and has integral coefficients.

First assume we can find a prime $p > 0$ such that the reduction f_p modulo p of f satisfies $\lambda_1(f_p) > 1$. Then $\mathcal{P}(f_p)$ is Zariski dense in $\mathbb{P}^2(\overline{\mathbb{F}_p})$ by a direct application of Hrushovski's arguments. One then lifts these periodic points to $\mathbb{P}^2(\overline{\mathbb{Q}})$ by combining a result of Cantat [7] proving that most periodic points are isolated together with a simple dimensional argument borrowed from Fakhruddin [19].

The main difficulty thus lies in proving that $\lambda_1(f_p) > 1$ for at least one prime p. Recall that a birational transformation of the projective plane is defined over the integers if it can be represented in homogeneous coordinates by polynomials with integral coefficients.

Theorem 1.4. *Let f be any birational transformation of the projective plane defined over $\mathbb{Z}$. Then for any prime p sufficiently large, f induces a birational transformation $f_p : \mathbb{P}^2_{\mathbb{F}_p} \dashrightarrow \mathbb{P}^2_{\mathbb{F}_p}$ on the special fiber at p, and*

$$\lim_{p\to\infty} \lambda_1(f_p) = \lambda_1(f).$$

We also give an example of a birational transformation f such that $\lambda_1(f_p) < \lambda_1(f)$ for all p (see Section 4.3).

In fact we prove a quite general version of Theorem 1.4 for families of birational transformations of surfaces over integral schemes. It allows us to prove:

Theorem 1.5. *Let* $\mathbf{k}$ *be an algebraically closed field and* $d \geq 2$ *be an integer. Denote by* Bir_d *the space of birational transformations of* $\mathbb{P}^2_{\mathbf{k}}$ *of degree* d*. Then for any* $\lambda < d$*, the set* $U_\lambda = \{f \in \mathrm{Bir}_d | \lambda_1(f) > \lambda\}$ *is open and Zariski dense in* Bir_d*.*

In particular, for a general birational transformation f *of degree* $d > 1$*, we have* $\lambda_1(f) > 1$*.*

In order to prove Theorem 1.4, we need to control $\lambda_1(f)$ in terms of the degree of a fixed iterate of f. This control is given by our Key Lemma.

Key Lemma. *Let* X *be a projective surface over an algebraically closed field, let* L *be an ample bundle on* X *and let* $f : X \dashrightarrow X$ *be a birational transformation. If* $q = \frac{\deg_L(f^2)}{3^{18}\sqrt{2}\deg_L(f)}$ *is greater than one, then we have*

$$\lambda_1(f) > q \geq 1.$$

In particular if $\deg_L(f^2) \geq 3^{18}\sqrt{2}\deg_L(f)$ then $\lambda_1(f) > 1$. This result has been stated in [20] by Favre without proof. To prove this lemma we rely on the natural linear action of f on a suitable hyperbolic space of infinite dimension. This space is constructed as a set of cohomology classes in the Riemann-Zariski space of X and was introduced by Cantat in [8]. See also [4, 9, 20, 29].

The article is organized in 7 sections. In Section 2 we give background informations on intersection theory on surfaces and Riemann-Zariski spaces. In Section 3 we prove our Key Lemma. We apply it in Section 4 to study the behavior of the first dynamical degree in families of birational transformations on surfaces. In Section 4.3 we give an example of a birational transformation f on $\mathbb{P}^2_{\mathbb{Z}}$ such that $\lambda_1(f_p) < \lambda_1(f)$ for all prime p. In Section 5 we prove Theorem 1.2 and Theorem 1.3. In Section 6, we study the Zariski density of periodic points in the case $\lambda_1 = 1$. Finally we combine the results that we obtain in Section 5 and Section 6 to get Theorem 1.1.

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2. Background and Notation

In this paper, a variety is always defined over an algebraically closed field and we use the notation $\mathbf{k}$ to denote an algebraically closed field of arbitrary characteristic except in Subsection 5.4 and Section 6. In Subsection 5.4, $\mathbf{k}$ denotes an algebraically closed field of characteristic 0. In Section 6, $\mathbf{k}$ denotes an algebraically closed field of characteristic different from 2 and 3.

2.1. **Néron-Severi group.** Let us recall the definition and some properties of the Néron-Severi group [13, 28].

Let X be a projective variety over $\mathbf{k}$. We denote by $\mathrm{Pic}(X)$ the Picard group of X. The Néron-Severi group of X is defined as the group of numerical equivalence classes of divisors on X. We denote it by $N^1(X)$, and write $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The group $N^1(X)$ is a free abelian group of finite rank (see [27]). Let $\phi : X \to Y$ be a morphism of projective varieties. It induces a natural map $\phi^* : N^1(Y)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$.

We denote by $N_1(X)_{\mathbb{R}}$ the space of numerical equivalence classes of real one-cycles of X. One has a perfect pairing

$$N^1(X)_{\mathbb{R}} \times N_1(X)_{\mathbb{R}} \to \mathbb{R}, \ \ (\delta, \gamma) \to (\delta \cdot \gamma) \in \mathbb{R}$$

induced by the intersection form for which $N_1(X)_{\mathbb{R}}$ is dual to $N^1(X)_{\mathbb{R}}$. We denote by $\phi_* : N_1(X)_{\mathbb{R}} \to N_1(Y)_{\mathbb{R}}$ the dual operator of ϕ^*.

If X is a projective surface defined over $\mathbf{k}$, for any classes $\alpha_1, \alpha_2 \in N^1(Y)$, we denote by $(\alpha_1 \cdot \alpha_2)$ their intersection number. We recall the following

Proposition 2.1 (Pull-back formula, see [22]). *Let $\pi : X \to Y$ be a surjective morphism between two projective surfaces defined over $\mathbf{k}$. For any classes $\alpha_1, \alpha_2 \in N^1(Y)$, we have*

$$(\pi^*\alpha_1 \cdot \pi^*\alpha_2) = \deg(\pi)(\alpha_1 \cdot \alpha_2).$$

When X is a smooth projective surface, we can (and will) identify $N^1(X)_{\mathbb{R}}$ and $N_1(X)_{\mathbb{R}}$. In particular we get a natural bilinear form on $N^1(X)_{\mathbb{R}}$.

A class $\alpha \in N^1(X)_{\mathbb{R}}$ is said to be nef if and only if $(\alpha \cdot [C]) \geq 0$ for any curve C.

Theorem 2.2 (Hodge index theorem). *Let L and M be two $\mathbb{R}$-divisors on a smooth projective surface, such that $(L^2) \geq 0$ and $(L \cdot M) = 0$. Then we have $(M^2) \leq 0$ and $(M^2) = 0$ if and only if $(L^2) = 0$ and M is numerically equivalent to a multiple of L.*

In other words the signature of the intersection form on $N^1(X)_{\mathbb{R}}$ is equal to $(1, \dim N^1(X)_{\mathbb{R}} - 1)$.

2.2. **Basics on birational maps on surfaces.** Recall that the resolution of singularities of surfaces over any algebraically closed field exists (see [1]).

Let X, Y be two smooth projective surfaces. A birational map $f : X \dashrightarrow Y$ is defined by its graph $\Gamma(f) \subseteq X \times Y$, which is an irreducible subvariety for which the projections $\pi_1 : \Gamma(f) \to X$ and $\pi_2 : \Gamma(f) \to Y$ are birational morphisms. We denote by $I(f) \subseteq X$ the finite set of points where π_1 does not admit a local inverse and call it the *indeterminacy* set of f. We set $\mathcal{E}(f) = \pi_1\pi_2^{-1}(I(f^{-1}))$. Observe that $\mathcal{E}(f)$ is the critical set of f. For any algebraic subset $V \subset X$, we write $f(V) := \overline{f(V \setminus I(f))}$.

If $g : Y \dashrightarrow Z$ is another birational map, the graph $\Gamma(g \circ f)$ of the composite map is the closure of the set

$$\{(x, g(f(x))) \in X \times Z |\ x \in X \setminus I(f), f(x) \in Y \setminus I(g)\}.$$

This is included in the set

$$\Gamma(g) \circ \Gamma(f) = \{(x, z) \in X \times Z |\ (x, y) \in \Gamma(f), (y, z) \in \Gamma(g) \text{ for some } y \in Y\}$$

with equality, if and only if there is no component $V \subseteq \mathcal{E}(f)$ such that $f(V) \subseteq I(g)$.

Let $f : X \dashrightarrow Y$ be a birational map between smooth projective surfaces, and Γ be a desingularization of its graph. Denote by $\pi_1 : \Gamma \to X, \pi_2 : \Gamma \to Y$ the natural projections. Then we define the following linear maps

$$f^* = \pi_{1*}\pi_2^* : N^1(Y)_\mathbb{R} \to N^1(X)_\mathbb{R},$$

and

$$f_* = \pi_{2*}\pi_1^* : N^1(X)_\mathbb{R} \to N^1(Y)_\mathbb{R}.$$

Observe that $f_* = f^{-1*}$.

Proposition 2.3 (see [15])**.** *Let $f : X \dashrightarrow Y$ be a birational map between smooth projective surfaces.*

(i) *The linear map f^* (resp. f_*) is integral in the sense that it maps $N^1(Y)$ (resp. $N^1(X)$) to $N^1(X)$ (resp. $N^1(Y)$).*

(ii) *If $\alpha \in N^1(Y)_\mathbb{R}$ is nef, then $f^*\alpha \in N^1(X)_\mathbb{R}$ is nef.*

(iii) *The maps f^* and f_* are adjoint for the intersection form, i.e.*

$$(f^*\alpha \cdot \beta) = (\alpha \cdot f_*\beta),$$

for any $\alpha \in N^1(Y)_\mathbb{R}$ and $\beta \in N^1(X)_\mathbb{R}$.

It is important to observe that $f \mapsto f^*$ is not functorial in general. In fact, let X, Y, Z be smooth projective surfaces, and let $f : X \dashrightarrow Y$ and $g : Y \dashrightarrow Z$ be two birational maps. For any given ample class $\alpha \in N^1(Z)_\mathbb{R}$, $f^*g^*\alpha = (f \circ g)^*\alpha$ if and only if $I(\mathcal{E}(f) \bigcap I(g)) = \emptyset$.

Fix any euclidean norm $\| \cdot \|$ on $N^1(X)_\mathbb{R}$. It follows from [15, 17] that the sequence of rescaled operator norms $C \,\|(f^n)^*\|$ is sub-multiplicative for a suitable constant $C > 0$. We may thus define the first dynamical degree

$$\lambda_1(f) := \lim_{n\to\infty} \|f^{n*}\|^{1/n}.$$

It is not difficult to check that $\lambda_1(f) \geq 1$ and that it only depends on the conjugacy class of f in the group of all birational transformations of X.

For any class $\omega \in N^1_\mathbb{R}(X)$, we set

$$\deg_\omega(f) := (f^*\omega \cdot \omega).$$

If L is an ample line bundle on X, we also write $\deg_L(f)$ for $\deg_{[L]}(f)$. It is possible to compute the dynamical degree of a map in terms of the degree growth of its iterates as follows.

Proposition 2.4 ([15, 17]). *Let $f : X \dashrightarrow X$ be a birational transformation on a projective smooth surface. Then we have $\lambda_1(f) = \lim_{n\to\infty} \deg_\omega(f^n)^{1/n}$, for any big and nef class $\omega \in N^1(X)$.*

Proposition-Definition 2.5 (see [15, 21]). Let $f : X \dashrightarrow X$ be a birational transformation on a projective smooth surface, and fix any ample class $\omega \in N^1(X)_{\mathbb{R}}$. Then f is said to be *algebraically stable* if and only if one of the following holds:

(i) for every $\alpha \in N^1(X)_{\mathbb{R}}$ and every $n \in \mathbb{N}$, one has $(f^*)^n\alpha = (f^n)^*\alpha$;
(ii) there is no curve $V \subseteq X$ such that $f^n(V) \subseteq I(f)$ for some integer $n \geq 0$;
(iii) for all $n \geq 0$ one has $(f^*)^n\omega = (f^n)^*\omega$.

Observe that in the case $X = \mathbb{P}^2$, f is algebraically stable if and only if $\deg_L(f^n) = (\deg_L f)^n$ for any $n \in \mathbb{N}$ where L is the hyperplane line bundle.

Theorem 2.6 ([15]). *Let $f : X \dashrightarrow X$ be a birational transformation of a projective smooth surface, then there is a proper modification $\pi : \widehat{X} \to X$ such that the lift of f to $\widehat{X}$ is algebraically stable.*

2.3. **Classes on the Riemann-Zariski space.** All facts in this subsection can be found in [4, 8, 9, 29]. Let X be a smooth projective surface over $\mathbf{k}$.

Given any two birational morphisms $\pi : X_\pi \to X$ and $\pi' : X_{\pi'} \to X$, we say that π' dominates π and write $\pi' \geq \pi$ if there exists a birational morphism $\mu : X_{\pi'} \to X_\pi$ such that $\pi' = \pi \circ \mu$. The *Riemann − Zariski space* of X is defined to be the projective limit

$$\mathfrak{X} := \varprojlim_\pi X_\pi.$$

Definition 2.7. The space of Weil classes of $\mathfrak{X}$ is defined to be the projective limit

$$W(\mathfrak{X}) := \varprojlim_\pi N^1(X_\pi)_{\mathbb{R}}$$

with respect to pushforward arrows. The space of Cartier classes on $\mathfrak{X}$ is defined to be the inductive limit

$$C(\mathfrak{X}) := \varinjlim_\pi N^1(X_\pi)_{\mathbb{R}}$$

with respect to pullback arrows.

Concretely, a Weil class $\alpha \in W(\mathfrak{X})$ is given by its *incarnations* $\alpha_\pi \in N^1(X_\pi)_{\mathbb{R}}$, compatible with pushforwards; that is, $\mu_*\alpha_{\pi'} = \alpha_\pi$ as soon as $\pi' = \pi \circ \mu$.

The projection formula shows that there is an embedding $C(\mathfrak{X}) \subseteq W(\mathfrak{X})$, so that a Cartier class is a Weil class.

For each π, the intersection pairing $N^1(X_\pi)_{\mathbb{R}} \times N^1(X_\pi)_{\mathbb{R}} \to \mathbb{R}$ is denoted by $(\alpha\cdot\beta)_{X_\pi}$. By the pull-back formula, it induces a pairing $W(\mathfrak{X})\times C(\mathfrak{X}) \to \mathbb{R}$ which is denoted by $(\alpha \cdot \beta)$.

We define the space

$$\mathbb{L}^2(\mathfrak{X}) := \{\alpha \in W(\mathfrak{X})|\ \inf_\pi\{(\alpha_\pi \cdot \alpha_\pi)\} > -\infty\}.$$

It is an infinite dimensional subspace of $W(\mathfrak{X})$ that contains $C(\mathfrak{X})$. It is endowed with a natural intersection product extending the one on Cartier classes and that is of Minkowski's type. Since this fact is crucial to our proof of Theorem 1.2 we state it as a

Proposition 2.8 ([8])**.** *If α, β are any two non zero classes in $\mathbb{L}^2(\mathfrak{X})$ such that $(\alpha^2) > 0$ and $(\alpha \cdot \beta) = 0$, then we have $(\beta^2) < 0$.*

Definition 2.9. We define $\mathbb{H}(\mathfrak{X})$ to be the unique connected component of $\{\alpha \in \mathbb{L}^2(\mathfrak{X}) | \alpha^2 = 1\}$ that contains all Cartier nef classes of self intersection $+1$.

For any $\alpha, \beta \in \mathbb{H}(\mathfrak{X})$, we define

$$d_{\mathbb{H}(\mathfrak{X})}(\alpha, \beta) = (\cosh)^{-1}(\alpha \cdot \beta),$$

where $\cosh(x) = (e^x + e^{-x})/2$. Recall that the function $d_{\mathbb{H}(\mathfrak{X})}$ induces a distance on the space $\mathbb{H}(\mathfrak{X})$, see [8, 9].

Let $f : X \dashrightarrow Y$ be a birational map between two smooth projective surfaces. For each blowup Y_ϖ of Y, there is a blowup X_π of X such that the induced map $X_\pi \to Y_\varpi$ is regular. The associated pushforward map $N^1(X_\pi)_\mathbb{R} \to N^1(Y_\varpi)_\mathbb{R}$ and pullback map $N^1(Y_\varpi)_\mathbb{R} \to N^1(X_\pi)_\mathbb{R}$ are compatible with the projective and injective systems defined by pushforwards and pullbacks that define Weil and Cartier classes respectively.

Definition 2.10. Let $f : X \dashrightarrow X$ be a birational transformation on a smooth projective surface. We denote by $f_* : W(\mathfrak{X}) \to W(\mathfrak{X})$ the induced pushforward operator and by $f^* : C(\mathfrak{X}) \to C(\mathfrak{X})$ the induced pullback operator.

Proposition 2.11 ([8, 9])**.** *The pullback $f^* : C(\mathfrak{X}) \to C(\mathfrak{X})$ extends to a linear map $f^* : \mathbb{L}^2(\mathfrak{X}) \to \mathbb{L}^2(\mathfrak{X})$, such that*

$$((f^*\alpha)^2) = (\alpha^2)$$

for any $\alpha \in \mathbb{L}^2(\mathfrak{X})$. In particular f^* induces an isometry on $(\mathbb{H}(\mathfrak{X}), d_{\mathbb{H}(\mathfrak{X})})$.

Observe that since f is birational $f_* = (f^{-1})^*$ and the pushforward f_* also induces an isometry on $\mathbb{H}(\mathfrak{X})$. For any $\alpha, \beta \in \mathbb{L}^2(\mathfrak{X})$ we have

$$(f^*\alpha \cdot \beta) = (\alpha \cdot f_*\beta).$$

2.4. **Hyperbolic spaces.** In this subsection, we review some properties of hyperbolic spaces in the sense of Gromov.

Recall that a metric space (M, d) is geodesic if and only if for any two points $x, y \in X$, there exists at least one isometric immersion of a segment of $\mathbb{R}$ with boundary x and y. For any given number $\delta \geq 0$, a metric space (M, d) satisfies the Rips condition of constant δ if it is geodesic, and for any geodesic triangle $\Delta = [x, y] \bigcup [y, z] \bigcup [z, x]$ of M, and any $u \in [y, z]$, we have $d(u, [x, y] \bigcup [z, x]) \leq \delta$. A space M is called *hyperbolic in the sense of Gromov* if there is a number $\delta \geq 0$ such that M satisfies the Rips condition of constant δ.

Lemma 2.12 ([11])**.** *The hyperbolic plane $\mathbb{H}^2$ satisfies the Rips condition of constant* $\log 3$*.*

Since the Rips condition only needs to be tested on geodesic triangles, we have the following

Lemma 2.13. *The space* $(\mathbb{H}(\mathfrak{X}), d_{\mathbb{H}(\mathfrak{X})})$ *satisfies the Rips condition of constant* $\log 3$.

Recall that a topological space is separable if and only if it admits a countable dense subset.

Theorem 2.14 ([23])**.** *Let* (M,d) *be a separable geodesic and hyperbolic metric space which satisfies the Rips condition of constant* δ. *If* $(x_i)_{0\leq i\leq n}$ *is a sequence of points such that*

$$d(x_{i+1},x_{i-1}) \geq \max(d(x_{i+1},x_i), d(x_i,x_{i-1})) + 18\delta + \kappa$$

for some constant $\kappa > 0$ *and* $i = 1, \cdots, n-1$. *Then*

$$d(x_n, x_0) \geq \kappa n.$$

3. Effective bounds on λ_1

We begin with proving our

Key Lemma. *Let* X *be a smooth projective surface over* $\mathbf{k}$, L *be an ample line bundle on* X, *and* $f : X \dashrightarrow X$ *be a birational transformation. If* $q = \frac{\deg_L(f^2)}{3^{18}\sqrt{2}\deg_L(f)} \geq 1$, *then we have*

$$\lambda_1(f) > q \geq 1.$$

Proof. For any $n > 0$, set $\mathcal{L}_n = f^{*n}L \in \mathbb{H}(\mathfrak{X})$. Since f^* is an isometry of $\mathbb{H}(\mathfrak{X})$, we have

$$d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1}, \mathcal{L}_{n-1}) = d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_2, \mathcal{L}) = \cosh^{-1}(\deg_L(f^2)) = \cosh^{-1}(3^{18}\sqrt{2}\deg_L(f)q)$$

for any $n \geq 1$. We claim that for any $u, q \geq 1$,

$$\cosh^{-1}(3^{18}\sqrt{2}uq) > \cosh^{-1}(u) + 18\log 3 + \log(q) \qquad (*).$$

Taking this claim for granted we conclude the proof. First we have

$$\cosh^{-1}(3^{18}\sqrt{2}\deg_L(f)q) > \cosh^{-1}(\deg_L(f)) + 18\log 3 + \log(q)$$

Pick $\kappa > \log(q) \geq 0$ such that

$$\cosh^{-1}(3^{18}\sqrt{2}\deg_L(f)q) > \cosh^{-1}(\deg_L(f)) + 18\log 3 + \kappa.$$

Then we get

$$d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1}, \mathcal{L}_{n-1}) > \cosh^{-1}(\deg_L(f)) + 18\log 3 + \kappa$$

for every $n \geq 1$. Since $d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1}, \mathcal{L}_n) = \cosh^{-1}(\deg_L(f))$, we obtain

$$d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1}, \mathcal{L}_n) > \max(d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_{n+1}, \mathcal{L}_n), d_{\mathbb{H}(\mathfrak{X})}(\mathcal{L}_n, \mathcal{L}_{n-1})) + 18\log 3 + \kappa.$$

Let W be the subspace of $\mathbb{H}(\mathfrak{X})$ spanned by $\{\mathcal{L}_n\}$. Then W is separable and for any $x, y \in W$, the geodesic segment $[x, y]$ is included in W. It follows that

$(W, d_{\mathbb{H}(\mathfrak{X})}|_W)$ is a separated geodesic and hyperbolic metric space which satisfies the Rips condition of constant $\log 3$. By Theorem 2.14, we get for $n > 0$

$$\cosh^{-1}(\deg_L(f^n)) = d_{\mathbb{H}(\mathfrak{P})}(\mathcal{L}_n, \mathcal{L}) > \kappa n,$$

which is equivalent to

$$\deg_L(f^n) > (e^{\kappa n} + e^{-\kappa n})/2 > e^{\kappa n}/2.$$

We conclude that $\lambda_1(f) \geq e^{\kappa} > q$.

Let us prove $(*)$. For any $u \geq 1$, $q \geq 1$, we have

$$\cosh^{-1}(3^{18}\sqrt{2}u) = \log(3^{18}\sqrt{2}u + \sqrt{2\times 3^{36}u^2 - 1}) > \log(3^{18}u + 1 + \sqrt{2\times 3^{36}u^2 - 1})$$

$$> \log(3^{18}u + \sqrt{2\times 3^{36}u^2}) = 18\log 3 + \log(u + \sqrt{2u^2}) \geq 18\log 3 + \log(u + \sqrt{u^2+1})$$

$$= \cosh^{-1}(u) + 18\log 3,$$

and

$$\cosh^{-1}(3^{18}\sqrt{2}uq) - \cosh^{-1}(3^{18}\sqrt{2}u) = \log\Big(\frac{3^{18}\sqrt{2}uq + \sqrt{2\times 3^{36}u^2q^2 - 1}}{3^{18}\sqrt{2}u + \sqrt{2\times 3^{36}u^2 - 1}}\Big).$$

It follows that

$$\frac{3^{18}\sqrt{2}uq + \sqrt{2\times 3^{36}u^2q^2 - 1}}{3^{18}\sqrt{2}u + \sqrt{2\times 3^{36}u^2 - 1}} = q - \frac{\sqrt{2\times 3^{36}u^2q^2 - q^2} - \sqrt{2\times 3^{36}u^2q^2 - 1}}{3^{18}\sqrt{2}u + \sqrt{2\times 3^{36}u^2 - 1}} \geq q,$$

which concludes the proof. □

Our Key Lemma implies the following estimate on $\lambda_1(f)$ knowing $\deg_L(f^n)$ and $\deg_L(f^{2n})$ for some n sufficiently large.

Corollary 3.1. *Let f be a birational transformation of a smooth projective surface X over $\mathbf{k}$. For any integer $n > 0$, we set $q_n := \frac{\deg_L(f^{2n})}{3^{18}\sqrt{2}\deg_L(f^n)}$. If $q_n \geq 1$, we have*

$$q_n^{1/n} < \lambda_1(f) \text{ and } \lim_{n\to\infty} q_n^{1/n} = \lambda_1(f).$$

Proof. Assume that

$$q_n = \frac{\deg_L(f^{2n})}{3^{18}\sqrt{2}\deg_L(f^n)} \geq 1.$$

Our Key Lemma implies

$$\lambda_1(f)^n = \lambda_1(f^n) > q_n$$

and we conclude

$$\lambda_1(f) > q_n^{1/n}$$

as required. □

4. The behavior of λ_1 in family

4.1. **Lower semi-continuity of λ_1.** In this subsection we use our Key Lemma to study the behavior of the first dynamical degree in families. We aim at proving a version of Theorem 1.4 in the general context of integral schemes, see Theorem 4.3 below. We shall rely on the following

Lemma 4.1. *Let S be a smooth integral scheme, and $\pi : X \to S$ be a smooth projective and surjective morphism such that $\dim_S X = 2$. Let $L \to X$ be a line bundle which is nef over S, and $f : X \dashrightarrow X$ be a birational transformation over S such that for any point $p \in S$, f induces a birational transformation f_p of the special fiber X_p. Set $L_p := L|X_p$.*

Then $p \mapsto \deg_{L_p}(f_p)$ is a lower semi-continuous function on S.

Observe that $p \mapsto \deg_{L_p}(f_p)$ is not continuous in general as the following example shows.

Example 4.2. The map

$$f[x : y : z] = [xz : yz + 2xy : z^2]$$

is a birational transformation of $\mathbb{P}^2$ over $\operatorname{Spec} \mathbb{Z}$. Denote by L the hyperplane line bundle on $\mathbb{P}^2_{\mathbb{Z}}$. Then f_p is birational for any prime p, $\deg_{L_2}(f_2) = 1$ for $p = 2$ and $\deg_{L_p}(f_p) = 2$ for any odd prime.

Proof of Lemma 4.1. Denote by κ the generic point of S. We claim that on any integral scheme S we have $\deg_{L_p}(f_p) \leq \deg_{L_\kappa}(f_\kappa)$ on S with equality on a Zariski open subset of S.

The lower semicontinuity then follows. Indeed pick any $\lambda \in \mathbb{R}$ and define $R = \{x \in S, \deg_{L_x}(f_x) \leq \lambda\}$. Pick any irreducible component Z of the Zariski closure of R. Our claim applied to Z implies $\deg_{L_\kappa}(f_\kappa) = \deg_{L_p}(f_p)$ for some p hence $\deg_{L_\kappa}(f_\kappa) \leq \lambda$. And it follows that $\deg_{L_p}(f_p) \leq \deg_{L_\kappa}(f_\kappa)$ for all $p \in Z$ so that $R \bigcap Z = Z$ is Zariski closed.

We now prove the claim. Let $\Gamma \subseteq X \times_S X$ be the graph of f, and $\pi_1, \pi_2 : \Gamma \to X$ be the natural projections such that $\pi_2 \circ \pi_1^{-1} = f$. For any point $x \in S$, let Γ_x be the fiber of Γ above x, and π_{1x}, π_{2x} be the restrictions of π_1 and π_2 respectively to Γ_x. Denote by κ the generic point of S, then the function

$$\int_{\Gamma_x} \pi_{1x}^* L_x \cdot \pi_{2x}^* L_x = \int_{\Gamma_\kappa} \pi_{1\kappa}^* L_\kappa \cdot \pi_{2\kappa}^* L_\kappa = \deg_{L_\kappa}(f_\kappa)$$

is constant on S by [22, Proposition 10.2].

For $x \in S$, Γ_x may have several irreducible components, but there is only one component Γ'_x that satisfies $\pi_{1x}(\Gamma'_x) = X$, hence $\deg_{L_x}(f_x) = \int_{\Gamma'_x} \pi_{1x}^* L_x \cdot \pi_{2x}^* L_x \leq \int_{\Gamma_x} \pi_{1x}^* L_x \cdot \pi_{2x}^* L_x$. Since there is a nonempty open set U of S such that for any point $x \in U$, Γ_x is irreducible, it follows that $\deg_{L_x}(f_x) = \int_{\Gamma_x} \pi_{1x}^* L_x \cdot \pi_{2x}^* L_x = \deg_{L_\kappa}(f_\kappa)$ for $x \in U$. □

Theorem 4.3. *Let S be an integral scheme, $\pi : X \to S$ be a smooth projective and surjective morphism where the relative dimension $\dim_S X = 2$. Let $f : X \dashrightarrow X$ be a birational transformation over S such that for any $p \in S$, the reduction*

f_p is a birational transformation. Then the function $p \in S \mapsto \lambda_1(f_p)$ is lower semi-continuous.

Proof. As in the proof of Lemma 4.1, it is sufficient to check that for any integral scheme S then $\lambda_1(f_p) \leq \lambda_1(f_\kappa)$ for all $p \in S$ and that and for any $\lambda < \lambda_1(f_\kappa)$, there is a nonempty open set U of S, such that for every point $p \in U$, $\lambda_1(f_\kappa) \geq \lambda_1(f_p) > \lambda$.

For any $p \in S$, and any integer $n > 0$, we have $\deg_{L_p}(f_p^n) \leq \deg_{L_\kappa}(f_\kappa^n)$ hence

$$\lambda_1(f_p) \leq \lambda_1(f_\kappa).$$

The theorem trivially holds in the case $\lambda \leq 1$, so we may assume that $\lambda_{1,\kappa} > \lambda > 1$. For every $\lambda_{1,\kappa} > \lambda > 1$, there is an integer $n > 0$ such that

$$\Big(\frac{\deg_{L_\kappa}(f_\kappa^{2n})}{3^{18}\sqrt{2}\deg_{L_\kappa}(f_\kappa^n)}\Big)^{1/n} > \lambda > 1$$

by Corollary 3.1. By Lemma 4.1, there is an nonempty open set $U \subseteq S$ such that for any $q \in U$, $\deg_{L_q}(f_q^n) = \deg_{L_\kappa}(f_\kappa^n)$ and $\deg_{L_q}(f_q^{2n}) = \deg_{L_\kappa}(f_\kappa^{2n})$, hence

$$\Big(\frac{\deg_{L_q}(f_q^{2n})}{3^{18}\sqrt{2}\deg_{L_q}(f_q^n)}\Big)^{1/n} > \lambda.$$

By Corollary 3.1, we conclude that for any $q \in U$,

$$\lambda_1(f_q) \geq \Big(\frac{\deg_{L_q}(f_q^{2n})}{3^{18}\sqrt{2}\deg_{L_q}(f_q^n)}\Big)^{1/n} > \lambda,$$

as required. □

Corollary 4.4. *Let S be an integral scheme, $\pi : X \to S$ be a smooth projective and surjective morphism such that $\dim_S X = 2$. Let $f : X \dashrightarrow X$ be a birational transformation over S such that for any $p \in S$, f_p is birational. Then there is an integer $M > 0$ such that for every $p \in S$, $\lambda_1(f_p) = 1$ if and only if*

$$\frac{\deg_{L_p}(f_p^{2n})}{3^{18}\sqrt{2}\deg_{L_p}(f_p^n)} < 1$$

for $n = 1, 2, \cdots, M$.

Proof. Fix any integer $m > 0$, we set

$$Z_m := \{p \in S | \frac{\deg_{L_p}(f_p^{2n})}{3^{18}\sqrt{2}\deg_{L_p}(f_p^n)} < 1, \text{ for any } 0 < n \leq m\}$$

and $Z = \{p \in S | \lambda_1(f_p) = 1\}$. By Theorem 4.3 and Corollary 3.1, Z is closed and we have $Z = \bigcap_{m \geq 1} Z_m$. Since $\overline{Z_m}$ is a decreasing sequence of Zariski closed subsets then $\overline{Z_M} = \bigcap_{m \geq 1} \overline{Z_m}$ for some integer M and $Z \subset \overline{Z_M}$.

Suppose by contradiction $Z \neq \overline{Z_M}$, and pick a point $x \in \overline{Z_M} \setminus Z$. Let Y be an irreducible component of $\overline{Z_M}$ containing x and κ be the generic point of Y. Then

$Y = \overline{Y \bigcap Z_N}$ for every $N \geq M$. Since $\lambda_1(f_x) > 1$, we have $\lambda_1(f_\kappa) > 1$ by Lemma 4.1. There exists $N \geq M$ such that κ is not in Z_N, and we have

$$\frac{\deg_{L_\kappa}(f_\kappa^{2N})}{3^{18}\sqrt{2}\deg_{L_\kappa}(f_\kappa^N)} \geq 1.$$

By lemma 4.1, there is an open subset U of Y such that for any point $y \in U$ and $n = 1, 2, \cdots, N$, we have $\deg_{L_y}(f_y^n) = \deg_{L_\kappa}(f_\kappa^n)$ and $\deg_{L_y}(f_y^{2n}) = \deg_{L_\kappa}(f_\kappa^{2n})$. In particular $U \bigcap Z_N = \emptyset$, which contradicts the fact that $Y = \overline{Y \bigcap Z_N}$.

We get $Z = \overline{Z_M}$, and $\overline{Z_M} \supseteq Z_M \supseteq Z = \overline{Z_M}$, so that $Z = Z_M$ as required. $\square$

4.2. **Proof of Theorem 1.5.** If $f : \mathbb{P}^2_{\mathbf{k}} \to \mathbb{P}^2_{\mathbf{k}}$ is a birational transformation defined over an algebraically closed field $\mathbf{k}$, we set $\mathrm{alg.deg}(f) := \deg_{\mathcal{O}(1)}(f)$ and call it the degree of f. We denote by Bir_d the space of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ of degree d. It has a natural algebraic structure which makes it a quasi-projective space, see [5] for details.

By Theorem 4.3, it is easy to see that if a component of Bir_d contains a point with $\lambda_1 > 1$, then $\lambda_1 > 1$ for a general point in this component. However since there are few informations on the geometry of the components of Bir_d for $d \geq 3$ [10, 12], it is a priori non obvious to decide which component contains such a point.

The rest of the section is devoted to the proof of Theorem 1.5 stated in the introduction.

Theorem 1.5. *Let $\mathbf{k}$ be an algebraically closed field and $d \geq 2$ be an integer. Denote by Bir_d the space of birational transformations of $\mathbb{P}^2_{\mathbf{k}}$ of degree d. Then for any $\lambda < d$, the set $U_\lambda = \{f \in \mathrm{Bir}_d | \lambda_1(f) > \lambda\}$ is open and Zariski dense in Bir_d.*

In particular, for a general birational transformation f of degree $d > 1$, we have $\lambda_1(f) > 1$.

Remark 4.5. If the base field is uncountable, then the set $\{f \in \mathrm{Bir}_d | \lambda_1(f) = d\} = \bigcap_{n=1}^\infty U_{d-1/n}$ is dense in Bir_d. So for very general point $f \in \mathrm{Bir}_d$ we have $\lambda_1(f) = d$.

Remark 4.6. Our proof actually shows that for any $f \in \mathrm{Bir}_d$, the set $\{A \in \mathrm{PGL}_3(\mathbf{k}) | \lambda_1(A \circ f) > \lambda\}$ is dense in $\mathrm{PGL}_3(\mathbf{k})$.

Proof of Theorem 1.5. We claim that for any irreducible component S of Bir_d, there is a point $f \in S$ such that $\lambda_1(f) > \lambda$.

Since the function $f \mapsto \lambda_1(f)$ is lower semi-continuous by Theorem 4.3, the claim immediately implies that the set $\{f \in S | \lambda_1(f) > \lambda\}$ is Zariski open and dense which concludes the proof of the Theorem 1.5.

It thus remains to prove the claim. For that purpose, choose $f \in \mathrm{Bir}_d$, and consider the map

$$T_f : \mathrm{PGL}_3(\mathbf{k}) \to \mathrm{Bir}_d$$

sending A to $A \circ f$. Let $I(f) = \{z_1, z_2, \cdots, z_m\}$, and $I(f^{-1}) = \{x_1, x_2, \ldots, x_n\}$ be the indeterminacy sets of f and f^{-1} respectively. For any $i = 1, 2, \cdots, n$, there is a curve C_i such that $f(C_i) = x_i$. Let y_i be any point in $C_i \setminus (I(f) \bigcup I(f^{-1}))$, and pick a point $A_i \in \mathrm{PGL}_3(\mathbf{k})$ such that $A_i(x_i) = y_i$. Then for any $n \geq 0$, we have $(A_i \circ f)^n \circ A_i(x_i) = y_i$.

For any $i = 1, 2, \cdots, n$, we define the map

$$V_{1,i} : \mathrm{PGL}_3(\mathbf{k}) \to \mathbb{P}^2$$

by $V_{1,i}(A) := A(x_i)$. Let $U_{0,i} = \mathrm{PGL}_3(\mathbf{k})$ and set

$$U_{1,i} = V_{1,i}^{-1}(\mathbb{P}^2 \setminus I(f)).$$

Then $U_{1,i}$ is an open subset of $\mathrm{PGL}_3(\mathbf{k})$. Since $A_i(x_i) = y_i$ does not belong to $I(f)$, A_i lies in $U_{1,i}$, so $U_{1,i}$ is not empty. Now we define the map

$$V_{2,i} : U_{1,i} \to \mathbb{P}^2,$$

by

$$V_{2,i}(A) = A \circ f \circ A(x_i)$$

and set

$$U_{2,i} = V_{2,i}^{-1}(\mathbb{P}^2 \setminus I(f)).$$

Since $A \circ f \circ A_i(x_i) = y_i$, $U_{2,i}$ is as before an open set containing A_i.

By induction, for any i we build a sequence of non empty open subsets $U_{l,i} \subseteq U_{l-1,i}$ of $\mathrm{PGL}_3(\mathbf{k})$, and maps $V_{l+1,i} : U_{l,i} \to \mathbb{P}^2$ sending A to $(A \circ f)^l \circ A(x_i)$ such that for any $A \in U_{l,i}$, $(A \circ f)^t \circ A(x_i)$ is not in $I(f)$ for $t = 0, \cdots, l-1$. Let $U_l = \bigcap_{i=1}^n U_{l,i}$. Since the $U_{l,i}$'s are nonempty and $\mathrm{PGL}_3(\mathbf{k})$ is irreducible, then U_l is also non empty and Zariski dense. For any A in U_l, we have $\mathrm{alg.deg}((A \circ f)^s) = (\mathrm{alg.deg}(A \circ f))^s = d^s$ for $s = 0, 1, \cdots, l+1$.

Pick l sufficiently large such that $(d^l/3^{18}\sqrt{2})^2 > \lambda$. For any point $A \in U_{2l-1}$, we have $\mathrm{alg.deg}(A \circ f)^{2l} = d^{2l}$ and $\mathrm{alg.deg}(A \circ f)^l = d^l$, and by our Key Lemma, we conclude $\lambda_1(A \circ f) > \lambda$.

Pick any irreducible component S of Bir_d. There is a birational transformation

$$G : \mathbb{P}^2 \times S \dashrightarrow \mathbb{P}^2 \times S$$

over S by $G(x, f) = (f(x), f)$. Then the map G_f on the fiber at $f \in S$ induced by G is exactly f. For any $f \in S$ that is not lying in any other component of Bir_d, and for any $A \in \mathrm{PGL}_3(\mathbf{k})$, we have $A \circ f \in S$ since $\mathrm{PGL}_3(\mathbf{k})$ is irreducible. By the discussion of the previous paragraph, there is a point $A \in \mathrm{PGL}_3(\mathbf{k})$ such that $\lambda_1(A \circ f) > \lambda$. Let κ be the generic point of S. Then by Theorem 4.3, we get

$$\lambda_1(G_\kappa) \geq \lambda_1(G_{A \circ f}) = \lambda_1(A \circ f) > \lambda.$$

Applying again Theorem 4.3, we conclude that $\lambda_1(f) > \lambda$ for a general point f in S. □

4.3. **An example.** In this section we provide an example of a birational transformation f on $\mathbb{P}^2$ over $\mathbb{Z}$ such that $\lambda_1(f_p) < \lambda_1(f)$ for any prime number $p > 0$. Let us introduce the following two birational transformations $g = [xy : xy + yz : z^2]$ and $h = [x : x - 2z : -x + y + 3z]$.

Proposition 4.7. *The map $f = h \circ g = [xy : xy - 2z^2 : yz + 3z^2]$ is algebraically stable.*

Proof. A direct computation shows that

$$f^{-1} = [2x^2 - 2xy : (-3x + 3y + 2z)^2 : (x - y)(-3x + 3y + 2z)],$$

$$I(f) = \{[1 : 0 : 0], [0 : 1 : 0]\} \text{ and } I(f^{-1}) = \{[1 : 1 : 0], [0 : -2 : 3]\}.$$

Observe that the line $C := \{x = 0\}$ is f-invariant, and that $f([0 : y : z]) = [0 : -2z : y + 3z]$. Let us compute the orbits of the points in $I(f^{-1})$. Since $[1 : 1 : 0]$ is a fixed point of f, its orbit does not meet $I(f)$. Let i be the automorphism of C sending $[0 : y : z]$ to $[0 : y - 2z : -y + z]$, then

$$l := i^{-1} \circ f_{|C} \circ i([0 : y : z]) = [0 : 2y : z],$$

$$i^{-1}([0 : 1 : 0]) = [0 : 1 : 1] \text{ and } i^{-1}([0 : -2 : 3]) = [0 : 4 : 1].$$

In particular the orbit of $[0 : -2 : 3]$ is equal to $i(\{[0 : 2^{l+2} : 1] | l = 0, 1, 2 \cdots \})$ which does not meet $I(f)$. We conclude that f is algebraically stable. □

Since f is algebraically stable, we have $\lambda_1(f) = \text{alg.deg}(f) = 2$.

Proposition 4.8. *For any prime $p > 2$, f_p is a birational transformation of $\mathbb{P}^2_{\overline{\mathbb{F}_p}}$ and $\lambda_1(f) < 2$.*

Proof. Observe that $f^{-1} \circ f = [4xyz^2 : 4y^2z^2 : 4yz^3]$ so that f_p is birational as soon as $p \geq 3$. Let $n_p \geq 2$ be the order of 2 in the multiplicative group $\mathbb{F}_p^{\times}$. We have

$$l_p^{n_p-2}([0 : 4 : 1]) = [0 : 1 : 1]$$

over $\overline{\mathbb{F}_p}$, which implies

$$f^{n_p-2}([0 : -2 : 3]) = [0 : 1 : 0] \in I(f_p).$$

In particular f_p is not algebraically stable. It follows that there is a number n such that $\text{alg.deg}(f_p^n) < \text{alg.deg}(f_p)^n = 2^n$ (in fact the least number having this property is $n_p - 1$), and $\lambda_1(f_p) \leq \text{alg.deg}(f_p^n)^{1/n} < 2$ by Corollary 3.1. □

Remark 4.9. We can compute $\lambda_1(f_p)$ by explicitly constructing an algebraically stable model dominating $\mathbb{P}^2_{\overline{\mathbb{F}_p}}$. For $p > 2$, we find that $\lambda_1(f)$ is the greatest real root of the polynomial

$$x_p^{n_p} - 2x_p^{n_p-1} + 1 = 0.$$

Define $F_n(x) = (x - 2)x^{n-1} + 1$. When $n > 2$, observe that $F_n(3/2) < 0$, and $F_n(2) = 1 > 0$, so that the largest root x of $F_n(x) = 0$ satisfies $2 > x > 3/2$. Since $2^{n_p-1}(x_p - 2) + 1 < 0 = (x_p - 2)x_p^{n_p-1} + 1 < (x_p - 2)(3/2)^{n_p-1} + 1$, we get

$$(1/2)^{n_p-1} < 2 - \lambda_1(f) < (2/3)^{n_p-1}.$$

5. The case $\lambda_1 > 1$

The purpose of this section is to prove Theorem 1.2.

5.1. **The case of finite fields.** First we recall the following theorem of Hrushovski.

Theorem 5.1 ([26]). *Let $g : X \to \operatorname{Spec} k$ be an irreducible affine variety of dimension r over an algebraically closed field k of characteristic p, and let q be a power of p. We denote by ϕ_q the q-Frobenius map of k, and by X^{ϕ_q} the same scheme as X with g replaced by $g \circ \phi_q^{-1}$. Let $V \subseteq X \times X^{\phi_q}$ be an irreducible subvariety of dimension r such that both projections*

$$\pi_1 : V \to X \text{ and } \pi_2 : V \to X^{\phi_q}$$

are dominant and the second one is quasi-finite. Let $\Phi_q \subseteq X \times X^{\phi_q}$ be the graph of the q-Frobenius map ϕ_q. Set

$$u = \frac{\deg \pi_1}{\deg_{insep} \pi_2},$$

where $\deg \pi_1$ denotes the degree of field extension $K(V)/K(X)$ and $\deg_{insep} \pi_2$ is the purely inseparable degree of the field extension $K(V)/K(X)$.

Then there is a constant C that does not depend on q, such that

$$|\#(V \bigcap \Phi_q) - uq^r| \leq Cq^{r-1/2}.$$

Building on [19, Proposition 5.5], we show that the set of periodic points of a birational transformation is Zariski dense over the algebraic closure a finite field.

Proposition 5.2. *Pick any prime $p > 0$, and let X be an algebraic variety over $\overline{\mathbb{F}_p}$. Let $f : X \dashrightarrow X$ be a birational transformation. Then the subset of $X(\overline{\mathbb{F}_p})$ consisting of non-critical periodic points of f is Zariski dense in X.*

Remark 5.3. In our paper, we only talk about birational transformations, but in fact our proof of Proposition 5.2 holds for arbitrary dominant rational endomorphisms with minor modifications.

Recall that we denote by $\mathcal{E}(f)$ the critical set of f and a non critical periodic point is a point whose orbit under f meets neither the indeterminacy set of f nor its critical set and is finite.

Proof of Proposition 5.2. Let Z be the Zariski closure of the set of non-critical periodic points of f in $X(\overline{\mathbb{F}_p})$ and suppose by contradiction that $Z \neq X$. Set $Y := Z \bigcup I(f) \bigcup \mathcal{E}(f)$ and then Y is a proper closed subset of X. Let $q = p^n$ be such that X and f are defined over the subfield $\mathbb{F}_q$ of $\overline{\mathbb{F}_p}$ having exactly q elements. Let ϕ_q denote the Frobenius morphism acting on X and let Γ_f(resp. Γ_m) denote the graph of f (resp. ϕ_q^m) in $X \times X$. Let U be an irreducible affine open subset of $X \setminus Y$ that is also defined over $\mathbb{F}_q$ and such that f is an open embedding from U to X. Set $V = \Gamma_f \bigcap (U \times U)$. By Theorem 5.1 there exists an integer $m > 0$ such that $(V \bigcap \Gamma_m)(\overline{\mathbb{F}_p}) \neq \emptyset$ i.e. there exists $u \in U(\overline{\mathbb{F}_p})$ such that $f(u) = \phi_q^m(u) \in U$. Since f is defined over $\mathbb{F}_q$, it follows that $f^l(u) = \phi_q^{lm}(u) \in U$ for all $l \geq 0$. In particular u is a non-critical periodic point of f. This contradicts the definition of Y and U, and the proof is complete. $\square$

For the convenience of the reader, we repeat the arguments of [19, Theorem 5.1] which allows us to lift any isolated periodic point from the special fiber to the generic fiber.

Lemma 5.4. *Let* $\mathbf{X}$ *be a projective scheme, flat over a discrete valuation ring* R *with fraction field* K *and residue field* k*. Let* F *be a birational map* $\mathbf{X} \dashrightarrow \mathbf{X}$ *over* R *which is well defined at least at one point on the special fiber. Let* X *be the special fiber of* $\mathbf{X}$ *and* X' *be the generic fiber of* $\mathbf{X}$ *,* f *be the restriction of* F *to* X*, and* f' *be the restriction of* F *to* X'*.*

If the set of periodic $\overline{k}$*-points of* f *is Zariski dense in* X*, and moreover there are only finitely many curves of periodic points in* X*, then the set consisting of periodic* $\overline{K}$*-points of* f' *is Zariski dense in the generic fiber of* X'*.*

Proof. The set of periodic $\overline{k}$-points of f of period dividing n can be viewed as the set of $\overline{k}$-points in $\Delta_X \bigcap \Gamma_{f^n}$, where Δ_X is the diagonal and Γ_{f^n} is the graph of f^n in $X \times X$.

For any positive integer n, consider the subscheme $\Delta_{\mathbf{X}} \bigcap \Gamma_{F^n}$ of $\mathbf{X} \times_R \mathbf{X}$, where $\Delta_{\mathbf{X}}$ is the diagonal and Γ_{F^n} is the graph of F^n in $\mathbf{X} \times_R \mathbf{X}$. If $x \in X \setminus \mathrm{Sing}X$ is a periodic point of f that does not lie in any curve of periodic points, then (x, x) is contained in a closed subscheme of $\Delta_{\mathbf{X}} \bigcap \Gamma_{F^n}$ of dimension one. Since x is not in any curve of periodic points, the generic point x' of this subscheme is in $\Delta_{X'} \bigcap \Gamma_{f'^n}$ the generic fiber of $\Delta_{\mathbf{X}} \bigcap \Gamma_{F^n}$, so that x' is a periodic point. Since x is non critical, F^k is a local isomorphism on a neighborhood $\mathbf{U}^k$ of x in $\mathbf{X}$. Since x' is the generic point of a curve containing x, we get $x' \in \mathbf{U}^k$ hence it is non-critical.

We identify $\mathbf{X}$ with $\Delta_{\mathbf{X}}$. For any open subset U' of X', let Z' be a Weil-divisor of X' containing $X' \setminus U'$. Let $\mathbf{Z}$ be the closure of Z' in $\mathbf{X}$, then $\mathrm{codim}(\mathbf{Z}) = 1$ and each component of $\mathbf{Z}$ meets Z'. Each component of X is of codimension 1. If $X \subseteq \mathbf{Z}$, each component of X is a component of $\mathbf{Z}$. Since $X' \bigcap X = \emptyset$, we get $X \nsubseteq \mathbf{Z}$. Let $\mathbf{U} = \mathbf{X} \setminus \mathbf{Z}$ and $U = \mathbf{U} \bigcap X$, then $\mathbf{U} \bigcap X' = U'$ and $U \neq \emptyset$. There is a periodic point x of f not in any curve of periodic points which is a smooth point in U, then x' is in U' and is a non-critical periodic point of f'. □

5.2. **Invariant curves.** From the previous subsection, we see that curves of periodic points are the main obstructions to lift periodic points from finite fields. The following theorem which essentially is [7, Corollary 3.3] of Cantat tells us that if $\lambda_1 > 1$ on the special fiber, then this obstruction can be removed.

Theorem 5.5 ([7, 16])**.** *Let* X *be a smooth projective surface defined over* $\mathbf{k}$*. Then a birational transformation* $f : X \dashrightarrow X$ *with* $\lambda_1(f) > 1$ *admits only finitely many periodic curves.*

In particular, there are only finitely many curves of periodic points.

Observe that [7, Corollary 3.3] is stated over the field of complex numbers. However the proof relies mainly on the fact that $\dim H^i(X, \Omega^1_X)$, $i = 0, 1$ is finite and thus works over any projective varieties defined over an algebraically closed field of any characteristic endowed with its Zariski topology.

Proof. Assume by contradiction that there exists infinitely many $f-$invariant curves. By [7, Corollary 3.3], there is a rational function $\Phi : X \dashrightarrow \mathbb{P}^1$ and a non-zero constant α such that $\Phi \circ f = \alpha\Phi$. This implies $\lambda_1(f) = 1$. For the convenience of the reader, we give a proof of this fact. By Theorem 2.6, we may assume that f is algebraically stable on X. Then there is a nef class $\omega \in N^1(X)_{\mathbb{R}} \setminus \{0\}$, such that $f^*(\omega) = \lambda_1(f)\omega$. Let $[F] \in N^1(X)_{\mathbb{R}}$ be the class of a fiber of the invariant fibration. Since f is birational, we have $f^*[F] = f_*[F] = [F]$, and

$$(\omega \cdot [F]) = (\omega \cdot f_*[F]) = (f^*\omega \cdot [F]) = \lambda_1(f)(\omega \cdot [F]).$$

If $(\omega \cdot [F]) \neq 0$, we are done. Otherwise, since $(F^2) = 0$ and L is nef, then $L = lF$ for some $l \in \mathbb{R}$ by Theorem 2.2. In this case we also have that $\lambda_1(f) = 1$, because $f^*[F] = [F]$. □

5.3. **Proof of Theorem 1.2.** Let L be any very ample line bundle on X. We may assume that the transcendence degree of $\mathbf{k}$ over its prime field F is finite, since we can find a subfield of $\mathbf{k}$ which is finitely generated over F such that X, f and L are all defined over this subfield. We complete the proof by induction on the transcendence degree of $\mathbf{k}$ over F.

If $\mathbf{k}$ is the closure of a finite field, then the theorem holds by Proposition 5.2.

If $\mathbf{k} = \overline{\mathbb{Q}}$, there is a regular subring R of $\overline{\mathbb{Q}}$ which is finitely generated over $\mathbb{Z}$, such that X, L, f are defined over R. By Theorem 4.3, there is a maximal ideal $\mathfrak{m}$ of R such that the fiber $X_{\mathfrak{m}}$ is smooth and the restriction $f_{\mathfrak{m}}$ of f on this fiber is a birational transformation with $\lambda_1(f_{\mathfrak{m}}) > 1$. Since R is regular and finitely generated over $\mathbb{Z}$, the localization $R_{\mathfrak{m}}$ of R at $\mathfrak{m}$ is a discrete valuation ring such that $\overline{\mathrm{Frac}\,(R_{\mathfrak{m}})} = \overline{\mathbb{Q}}$ and $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} = R/\mathfrak{m}$. Then, by Proposition 5.2 the set of non-critical periodic points of $f_{\mathfrak{m}}$ is Zariski dense in the special fiber. Since $\lambda_1(f_{\mathfrak{m}}) > 1$, Theorem 5.5 shows that the number of curves of periodic points is finite. We are thus in position to apply Lemma 5.4: the set of non-critical periodic points of f forms a Zariski dense subset of X, and the theorem holds in this case.

If the transcendence degree of $\mathbf{k}$ over F is greater than 1, we pick an algebraically closed subfield K of $\mathbf{k}$ such that the transcendence degree of K over F equals the transcendence degree of $\mathbf{k}$ over F minus 1. Then we pick a subring R of $\mathbf{k}$ which is finitely generated over K, such that X, L and f are all defined over R. Since $\mathrm{Spec}\, R$ is regular on an open set, we may assume that R is regular by adding finitely many inverses of elements in R. We may repeat the same arguments as in the case $\mathbf{k} = \overline{\mathbb{Q}}$.

5.4. **Existence of Zariski dense orbits.** In this subsection, we denote by $\mathbf{k}$ an algebraically closed field of characteristic 0. Our aim is to show the following result from the introduction:

Theorem 1.4. *Let X be a projective surface over an algebraically closed field $\mathbf{k}$ of characteristic 0. Let $f : X \dashrightarrow X$ be a birational transformation with $\lambda_1(f) > 1$. Then there is a point $x \in X$ such that $f^n(x) \in X \setminus I(f)$ for any $n \in \mathbb{Z}$ and $\{f^n(x) | n \in \mathbb{Z}\}$ is Zariski dense.*

As a first intermediate step, we extend E. Amerik's results [2] to arbitrary algebraically closed field of characteristic 0. Namely we prove

Theorem 5.6. *Let X be a variety over an algebraically closed field $\mathbf{k}$ of characteristic 0, and $f : X \dashrightarrow X$ be a birational transformation, then there is a point $x \in X$ such that $f^n(x) \in X \setminus I(f)$ for all $n \in \mathbb{Z}$ and $\{f^n(x)|n \in \mathbb{Z}\}$ is infinite.*

Remark 5.7. When $\mathbf{k}$ is of positive characteristic, we can still prove the existence of a point $x \in X$ such that $f^n(x) \in X \setminus I(f)$ for all $n \in \mathbb{Z}$ by a similar method. But the existence of a non-preperiodic point may not be true over $\overline{\mathbb{F}_p}$, since for any $q = p^n$, $\#X(\mathbb{F}_q)$ is finite.

To do so we shall rely on the following Lemma which is completely standard and whose proof is left to the reader.

Lemma 5.8. *Let $\pi : X \to Y$ be a dominant morphism between two irreducible varieties defined over an algebraically closed field. For every point $x \in X$, there is an irreducible subvariety S through x of X, such that $\dim S = \dim Y$, and the restriction of π on S is dominant to Y.*

We are now in position to prove Theorem 5.6. This proof is standard and it is included here for the sake of completeness.

Proof of Theorem 5.6. In the case $\mathbf{k} = \overline{\mathbb{Q}}$, the theorem is due to E. Amerik in [2].

In the general case, there is a subring R of $\mathbf{k}$ which is finitely generated over $\overline{\mathbb{Q}}$ on which X and f are defined, and we may assume that $\mathbf{k}$ is the algebraic closure of the fraction field of R. We then pick a scheme $\pi : X_R \to \operatorname{Spec} R$ and a birational transformation f_R of X_R over R such that the geometric generic fiber of X_R is X and the restriction of f_R on X is f. Pick any closed point $\mathfrak{m} \in \operatorname{Spec} R$ such that the restriction $f_{\mathfrak{m}}$ of f on the special fiber $X_{\mathfrak{m}}$ at $\mathfrak{m}$ is birational. Since $R/\mathfrak{m} = \overline{\mathbb{Q}}$, there is a point $y \in X_{\mathfrak{m}}$ such that $f_{\mathfrak{m}}^n(y) \in X_{\mathfrak{m}} \setminus I(f_{\mathfrak{m}})$ for any $n \in \mathbb{Z}$ and $\{f_{\mathfrak{m}}^n(y)|n \in \mathbb{Z}\}$ is infinite. By Lemma 5.8, there is an irreducible subvariety S of X containing y and such that $\dim S = \dim R$, and the restriction of π to S is dominant to $\operatorname{Spec} R$. Let x be the generic point of S, then we have $x \in X_R(\mathbf{k}) = X$. We thus get $f^n(x) \in X \setminus I(f)$ for any $n \in \mathbb{Z}$ and $\{f^n(x)|\ n \in \mathbb{Z}\}$ is infinite. □

Proof of Theorem 1.3. By Theorem 5.5, there are only finitely many invariant curves for f. Let C be the union of these curves and set $U = X \setminus C$. By Theorem 5.6, there is a non-preperiodic point $x \in U$ such that $f^n(x) \in U \setminus I(f)$ for any $n \in \mathbb{Z}$. Let O be the Zariski closure of $\{f^n(x)|n \in \mathbb{Z}\}$ in X. If $O \neq X$, then $\dim O = 1$ since it is infinite. The union of all one-dimensional irreducible components of O is then invariant and intersects U which is a contradiction. □

6. The case $\lambda_1 = 1$

In this section, we denote by $\mathbf{k}$ an algebraically closed field of characteristic different from 2 and 3. Our aim is to prove Theorem 1.1 in the remaining case $\lambda_1 = 1$.

Recall the following structure theorem for this class of maps.

Theorem 6.1. *([15, 24]) Let X be a smooth projective surface over* $\mathbf{k}$*, let $L \to X$ be an ample line bundle, and let f be a birational transformation of X. Assume $\lambda_1(f) = 1$. Then up to a birational conjugacy, we are in one of the following three cases:*

(i) *the sequence* $\deg_L(f^n)$ *is bounded, f is an automorphism and some positive iterate of f acts on $N^1(X)$ as the identity;*
(ii) *the sequence* $\deg_L(f^n)$ *is equivalent to cn for some $c > 0$, and f preserves a rational fibration;*
(iii) *the sequence* $\deg_L(f^n)$ *is equivalent to cn^2, for some $c > 0$ and f is an automorphism preserving an elliptic fibration.*

We shall argue case by case.

6.1. **The case when** $\deg_L(f^n)$ **is bounded.**

Proposition 6.2. *Let X be a projective variety, f be an automorphism of X which acts on $N^1(X)$ as the identity. If the periodic points of f are Zariski dense, then there is an integer $n > 0$ such that $f^n = \mathrm{id}$.*

Remark 6.3. When k is the algebraic closure of a finite field, any automorphism f acting trivially on $N^1(X)$ satisfies $f^n = \mathrm{id}$ for some $n \geq 1$ since all points in X are periodic. However in any other field there are some automorphisms acting trivially on $N^1(X)$ of infinite order such as $f_t : [x : y : z] \mapsto [x : ty : z]$ on $X = \mathbb{P}^2$ where $t \in k$ is not a root of unity. Moreover, by conjugating such f_t by a birational transformation $g : [x : y : z] \mapsto [yz : xz + yz : xy + y^2]$ on $\mathbb{P}^2_k$, we get a birational transformation $h_t := g^{-1} \circ f_t \circ g : [x : y : z] \mapsto [xy : (1-t)xy + y^2 : (1-t)xz + zy]$ satisfying $\deg(h_t^n) = O(1)$ which is of infinite order and not an automorphism on $\mathbb{P}^2_k$.

Remark 6.4. When the action of f on the Picard group is the identity, then the arguments of [19, Proposition 2.1] apply directly. We can thus find an embedding of X in $\mathbb{P}^N$ and an automorphism $A \in \mathrm{PGL}_{N+1}$ such that $f = A|_X$. The result then follows easily.

Proof of Proposition 6.2. We denote by $\mathrm{Aut}(X)$ the automorphism group of X. Pick a line bundle $L \to X$ and let $\mathrm{Aut}_{[L]}$ be the subgroup of all automorphisms fixing the class $[L] \in N^1(X)$. We denote by $\mathrm{Aut}_0(X)$ the irreducible component of the identity. For any $g \in \mathrm{Aut}_{[L]}(X)$, let $\Gamma_g \subseteq X \times X$ be the graph of g. We denote by π_1, π_2 the projections onto the first and second factors. Since $\pi_1^* L \otimes \pi_2^* L$ is ample on $X \times X$, we may consider the Hilbert polynomial $P_g(m)$ of Γ_g:

$$P_g(m) = \chi(\Gamma_g, (\pi_1^* L \otimes \pi_2^* L)^{\otimes m}) = \chi(X, (L \otimes g^* L)^{\otimes m}).$$

By the Hirzebruch-Riemann-Roch theorem, we see that $\chi(X, (L \otimes g^* L)^{\otimes m})$ is a polynomial function of m whose coefficients only depend on the numerical class $[L \otimes g^* L] = 2[L] \in N^1(X)$, it follows that $P := P_g$ is independent of g. Let Y be the Hilbert scheme parameterizing closed subschemes of $X \times X$ with Hilbert polynomial P: it is a scheme that admits finitely many irreducible components and $\mathrm{Aut}_{[L]}(X)$ is an open subvariety of Y.

Since f acts on $N^1(X)$ as the identity, it follows that $f \in \mathrm{Aut}_{[L]}(X)$, hence $f^M \in \mathrm{Aut}_0(X)$ for some $M \geq 1$. We may thus assume that $f \in \mathrm{Aut}_0(X)$. Let S_m be the set of fixed points of $f^{m!}$ for $m \geq 1$, so that $S_m \subseteq S_{m+1}$ for any $m \geq 1$. Let $F_m = \{g \in \mathrm{Aut}_0(X)|\ g|_{S_m} = \mathrm{id}\}$. Then F_m is a closed set and $F_{m+1} \subseteq F_m$ for any $m \geq 1$. By noetherianity there is an integer l such that $F_l = \bigcap_{m\geq 1} F_m$, and it follows that then $f^n \in F_l = \bigcap_{m\geq 1} F_m$ for $n = l!$. In particular, we have $f^n|S_m = \mathrm{id}$ for any $m \geq 1$. Since the Zariski closure of $\bigcup_{m\geq 1} S_m$ is X by assumption, we conclude that $f^n = \mathrm{id}$. □

Proposition 6.5. *Let X be a smooth projective surface over $\mathbf{k}$, and $L \to X$ be an ample line bundle. Let f be a birational transformation of X, such that the sequence $\deg_L(f^n)$ is bounded. If the set of non critical periodic points of f is Zariski dense, then there is an integer $n > 0$ such that $f^n = \mathrm{id}$.*

Proof. By Theorem 6.1, we may assume that f is an automorphism and acts on $N^1(X)$ as the identity and we conclude by Proposition 6.2. □

6.2. **The linear growth case.**

Proposition 6.6. *Let X be a projective smooth surface over $\mathbf{k}$, and $L \to X$ be an ample line bundle. Let f be a birational transformation of X, such that $\deg_L(f^n) \sim cn$ for some $c > 0$. Then the set of non-critical periodic points of f is Zariski dense if and only if its action on the base of its invariant rational fibration is periodic.*

Proof. Suppose first that the set of non-critical periodic points is Zariski dense. By Theorem 6.1, we may assume that f is written under the form

$$f(x,y) = \Big(g(x), \frac{A_1(x)y + B_1(x)}{A_2(x)y + B_2(x)}\Big)$$

where g is an automorphism of C and $A_1(x), B_1(x), A_2(x), B_2(x)$ are rational functions on C such that $A_1(x)B_2(x) - A_2(x)B_1(x) \neq 0$. Since the set non-critical periodic points of f is Zariski dense, the set of all periodic points of g is also Zariski dense, hence $g^n = \mathrm{id}$ for some $n \geq 0$. Replacing f by a suitable iterate, we may thus assume that $g = \mathrm{id}$, and

$$f = \Big(x, \frac{A_1(x)y + B_1(x)}{A_2(x)y + B_2(x)}\Big). \tag{$**$}$$

Conversely suppose that $\deg_L(f^n) \to \infty$ and that f can be written under the form $(**)$. We denote the function field of C by K. Let

$$T(x) = (A_1(x) + B_2(x))^2/(A_1(x)B_2(x) - A_2(x)B_1(x))$$

and let $t_1, t_2 \in \overline{K}$ be the two eigenvalues of the matrix

$$\begin{pmatrix} A_1(x) & B_1(x) \\ A_2(x) & B_2(x) \end{pmatrix}.$$

If $(A_1(x) + B_2(x))^2/(A_1(x)B_2(x) - A_2(x)B_1(x)) \in \mathbf{k}$, then

$$t_1/t_2 + t_2/t_1 + 2 = (A_1(x) + B_2(x))^2/(A_1(x)B_2(x) - A_2(x)B_1(x)) \in \mathbf{k},$$

which implies $t_1/t_2 \in \mathbf{k}$, since $\mathbf{k}$ is algebraically closed.

If $t_1 = t_2$, then $t_1 = t_2 = (A_1(x) + B_2(x))/2 \in K$. We may replace $A_i(x)$ (resp. $B_i(x)$) by $2A_i(x)/(A_1(x) + B_2(x))$ (resp. $2B_i(x)/(A_1(x) + B_2(x))$), so that we may assume that $t_1 = t_2 = 1$. Changing coordinates if necessary, f can be the written as $(x, y + B(x))$ where $B(x) \in K$. It follows that $\deg_L f^n$ is bounded, which is a contradiction.

If $t_1 \neq t_2$, then $K(t_1)$ is a finite extension over K. There is a curve $\pi : B \to C$ corresponding to this field extension. Since f acts on C trivially, it induces a map $\widetilde{f}$ on $\mathbb{P}^1 \times_C B$. We set $\widetilde{L} = (\mathrm{id} \times_C \pi)^* L$. Since t_1, t_2 are rational functions on B, $\widetilde{f}$ is under the form $(x, (t_1/t_2)y)$, this implies that $\deg_{\widetilde{L}} \widetilde{f}^n$ is bounded. Since $\deg_{\widetilde{L}} \widetilde{f}^n = \deg \pi \times \deg_L f^n$, we get a contradiction.

We have shown that $T(x)$ is a non-constant rational function on B. For any $n > 0$, pick a primitive $n-$th root r_n of unity, there is at least one point $x \in C$ such that $T(x) = 2 + r_n + 1/r_n$. Changing coordinates if necessary, f acts on this fiber as $y \mapsto r_n y$ has finite order. It follows that periodic points of f are Zariski dense. $\square$

6.3. **The quadratic growth case.** The proof of the following theorem is similar to the proof of [6, Proposition 7.4].

Proposition 6.7. *Let X be a smooth projective surface over $\mathbf{k}$, and $L \to X$ be an ample line bundle. Let $f : X \to X$ be an automorphism of X such that $\deg_L(f^n) \to \infty$. Then the set of periodic points of f is Zariski dense if and only if its action on the base of its invariant elliptic fibration is periodic.*

Proof. Let $\pi : X \to C$ be the invariant elliptic fibration. Denote by g the automorphism on the base curve C induced by f.

Suppose first that the set of periodic points of f is Zariski dense. Then the set of periodic points of g is Zariski dense too. So there is an integer $N > 0$, such that $g^N = \mathrm{id}$.

Conversely suppose that $\deg_L(f^n) \to \infty$ and $g = \mathrm{id}$. Since f is an elliptic fibration, then all but finitely many fibers are elliptic curves.

By [30, Theorem 10.1 III], the order of the automorphism group of an elliptic curve (as an algebraic group) is a divisor of 24. We may replace f by f^{24}, so that the restriction of f to each smooth fiber is a translation. Observe that $f|_{E_x}$ admits a fixed point on a smooth fiber E_x if and only if $f|_{E_x}$ is the identity. Assume by contradiction that the set of periodic points of f is not Zariski dense. Then there is a set $T = \{x_1, \cdots, x_m\} \in C$, such that for any $x \in C \setminus T$, E_x is a smooth elliptic curve and f has no periodic points in E_x.

By replacing L by a sufficiently large power, we may assume it is very ample. By Bertini's Theorem (see [25]), we can find a general section S of L such that for any $x \in T$, the intersection of S and F_x is transverse, and these intersection points are smooth in S and in F_x.

Let H be the set of periodic points which lies in S. Then $H \subseteq S \bigcap (\bigcup_{x \in T} E_x)$ is a finite set. By replacing f by f^l for some $l > 0$, we may assume that all points in H are fixed. Observe that for any $i \neq j$, we have $f^i(S) \bigcap f^j(S) = H$. Let x be any point in H and write $\pi(x) = y$. In some local coordinates (z_1, z_2) at x, since both S and F_y are smooth at x and the intersection of S and E_y at x is transverse, we may assume that $S = (z_2 = 0)$, $E_y = (z_1 = 0)$ and π depends only on z_1. Then f^{-1} can be written as

$$f^{-1} = (z_1, z_2 + h(z_1, z_2))$$

where $h(0,0) = 0$. Since the fixed point locus is given by $h = 0$ and this set lie in $z_1 = 0$ in this chart, it follows that we can write $h = z_1^l(a + b(z_1, z_2))$ where $l \geq 1$, $a \neq 0$ and $b(0,0) = 0$.

Lemma 6.8. *One can find local coordinates (z_1, z_2) such that for any integer n, one has*

$$f^{-n} = (z_1, z_2 + z_1^l(na + b_n(z_1, z_2)))$$

where $b_1 = b$ and $b_n(0,0) = 0$.

It follows that $f^n(S)$ is defined by the equation $z_2 + z_1^l(na + b_n(z_1, z_2)) = 0$. When n is not divisible by the characteristic of $\mathbf{k}$, it follows that the intersection product $(S \cdot f^n(S))_x$ is equal to l independently on n. It follows that $(f^*L \cdot L) = \Sigma_{x \in H}(S \cdot f^n(S))_x$ is bounded which is gives a contradiction. □

Proof of Lemma 6.8. We proceed by induction on n. If it is true for n, then we have

$$f^{-(n+1)} = f^{-1} \circ f^{-n},$$

so that

$$\begin{aligned} f^{-n-1} &= (z_1, z_2 + z_1^l(na + b_n(z_1, z_2)) + z_1^l(a + b(z_1, z_2 + z_1^l(na + b_n(z_1, z_2)))))) \\ &= (z_1, z_2 + z_1^l((n+1)a + b_{n+1}(z_1, z_2))) \end{aligned}$$

where $b_{n+1} = b_n(z_1, z_2) + b(z_1, z_2 + z_1^l(na + b_n(z_1, z_2)))$. In particular, we have $b_{n+1}(0,0) = 0$. □

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