On the dynamical Mordell–Lang conjecture in positive characteristic

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Abstract

We prove that the dynamical Mordell–Lang conjecture in positive characteristic holds for bounded-degree self-maps of projective varieties. The key ingredient of the proof is a Mordell–Lang-type result for arbitrary algebraic groups over algebraically closed fields of positive characteristic, which is also interesting on its own. Moreover, we propose a geometric version of dynamical Mordell–Lang conjecture in positive characteristic.

1 Introduction

In this paper, as a matter of convention, every variety is assumed to be integral but the closed subvarieties can be reducible. For a rational map $f : X \dashrightarrow Y$ between two varieties, we denote $\mathrm{Dom}(f) \subseteq X$ as the domain of definition of f. Let X be a variety over an algebraically closed field K and let f be a rational self-map of X. For a point $x \in X(K)$, we say the orbit $\mathcal{O}_f(x) := \{f^n(x) |\ n \in \mathbb{N}\}$ is well-defined if every iterate $f^n(x)$ lies in $\mathrm{Dom}(f)$. We denote $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$. An arithmetic progression is a set of the form $\{mk + l |\ k \in \mathbb{Z}\}$ for some $m, l \in \mathbb{Z}$ and an arithmetic progression in $\mathbb{N}$ is a set of the form $\{mk + l |\ k \in \mathbb{N}\}$ for some $m, l \in \mathbb{N}$.

The dynamical Mordell–Lang conjecture, which is one of the core problems in the field of arithmetic dynamics, asserts that for any rational self-map f of a variety X over $\mathbb{C}$, the return set $\{n \in \mathbb{N} |\ f^n(x) \in V(\mathbb{C})\}$ is a finite union of arithmetic progressions in $\mathbb{N}$ where $x \in X(\mathbb{C})$ is a point such that the orbit $\mathcal{O}_f(x)$ is well-defined and $V \subseteq X$ is a closed subvariety. There is an extensive literature on various cases of this 0-DML conjecture ("0" stands for the characteristic of the base field). Two significant cases are as follows:

(i) If X is a quasi-projective variety over $\mathbb{C}$ and f is an étale endomorphism of X, then the 0-DML conjecture holds for (X, f). See [BGT10, Theorem 1.3].

(ii) If $X = \mathbb{A}^2_{\mathbb{C}}$ and f is an endomorphism of X, then the 0-DML conjecture holds for (X, f). See [Xie17] and [Xie, Theorem 3.2].

One can consult [BGT16, Xie] and the references therein for more known results. However, we remark that not much is known about the 0-DML conjecture when f is just a *rational* self-map of the variety X. The following problem might reflect the issue in some sense. It seems that the dynamical Mordell–Lang problem is not quite compatible with birational transformations. More precisely, let X, Y be varieties over $\mathbb{C}$, f, g be dominant rational self-maps of X, Y respectively and $\pi : Y \dashrightarrow X$ be a birational map such that $f \circ \pi = \pi \circ g$. Even if the 0-DML conjecture holds for (Y, g), generally we do not know how to deduce that the 0-DML conjecture holds for (X, f).

The statement of the 0-DML conjecture fails when the base field has positive characteristic. See [BGT16, Example 3.4.5.1] for an example. Consequently, Ghioca and Scanlon proposed a dynamical Mordell–Lang conjecture in positive characteristic. See [BGT16, Conjecture 13.2.0.1]. Here, we will introduce a slightly modified version of this conjecture. Firstly, we review the concept of "p-normal set" introduced in [Der07, Definition 1.7].

Definition 1.1. *Let p be a prime number and let $q = p^e$ for some positive integer e. Suppose that $d \in \mathbb{Z}_+$, $c_0, c_1, \ldots, c_d \in \mathbb{Q}$ with $(q-1)c_i \in \mathbb{Z}$ for all i, $c_0 + c_1 + \cdots + c_d \in \mathbb{Z}$ and $c_i \neq 0$ for $i = 1, 2, \ldots, d$. Then we define*

$$S_q(c_0; c_1, \ldots, c_d) = \{c_0 + c_1 q^{k_1} + \cdots + c_d q^{k_d} \mid k_1, \ldots, k_d \in \mathbb{N}\} \subseteq \mathbb{Z}.$$

We define a p-normal set in $\mathbb{Z}$ *as a union of finitely many arithmetic progressions (possibly singleton) and finitely many sets of form $S_q(c_0; c_1, \ldots, c_d)$ described as above. A* p-normal set in $\mathbb{N}$ *is a subset of $\mathbb{N}$ which is, up to a finite set, equal to the intersection of a p-normal set in $\mathbb{Z}$ and $\mathbb{N}$.*

Here, we say two sets S, T are equal up to a finite set if the symmetric difference $(S \backslash T) \cup (T \backslash S)$ is finite, as in [Der07].

In the rest of this article, "p-normal set" will be the abbreviation of "p-normal set in $\mathbb{Z}$".

Remark 1.2. *A p-normal set intersects $\mathbb{N}$ is a p-normal set in $\mathbb{N}$. A finite union of p-normal sets (resp. p-normal sets in $\mathbb{N}$) is still a p-normal set (resp. p-normal set in $\mathbb{N}$). Moreover, $aS + b$ is a p-normal set (resp. p-normal set in $\mathbb{N}$) if S is a p-normal set (resp. p-normal set in $\mathbb{N}$) and $a, b \in \mathbb{Z}$ (resp. $\mathbb{N}$).*

Now we can state the dynamical Mordell–Lang conjecture in positive characteristic.

Conjecture 1.3. *(pDML, arithmetic version) Let X be a variety over an algebraically closed field K of characteristic $p > 0$ and let f be a rational self-map of X. Let $x \in X(K)$ be a closed point such that the orbit $\mathcal{O}_f(x)$ is well-defined and let $V \subseteq X$ be a closed subvariety. Then $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\}$ is a p-normal set in $\mathbb{N}$.*

This pDML conjecture is expected to be very difficult. As an evidence, the argument in [Der07, p.189–p.190] shows that in the positive characteristic case, the return set can be arbitrarily complicated even for the linear recurrence sequences (or in other words, automorphisms of the projective space). Moreover, combining this argument with the statements in [Der07, Section 9], one can construct examples which show that even for surjective endomorphisms of projective varieties,

(i) one need to allow $c_i < 0$ for some $1 \leq i \leq d$ in the definition of the p-set $S_q(c_0; c_1, \ldots, c_d)$, and

(ii) the phrase "up to a finite set" in the definition of "p-normal set in $\mathbb{N}$" cannot be dropped.

But we believe that the return set should be a p-normal set in $\mathbb{Z}$ for automorphisms, i.e. there is no need to delete finitely many elements in such case. For the latest progress on this pDML conjecture, one may refer to [CGSZ21], [Xie23, Theorem 1.4, Theorem 1.5] and [Yan23].

People (for instance, [Xie23] and also Ghioca and Scanlon) guessed that the p-part in p-normal sets appears in the return set only when the endomorphism f involves or "comes from" some group actions. Otherwise, it is conjectured that the initial statement of the 0-DML conjecture should still valid for f. We will form a rigorous conjecture towards this perspective in this paper (see Conjecture 5.2). We think that p-sets come from *bounded-degree self-maps*, which is a synonym of "come from group actions" in some sense but has a more dynamical flavor.

We shall briefly introduce how to measure the complexity of a dominant rational self-map $f : X \dashrightarrow X$ in which X is a projective variety over an algebraically closed field. We use the concept of the *degree sequence* of f. Since there are many references of this concept in the literature (see for example [Dan20], [Tru20], [Xie23, Section 2.1] and [Yan, Section 5]), we will just state the definition and some basic properties here.

Let $L \in \mathrm{Pic}(X)$ be a big and nef line bundle. We consider the graph $\Gamma_f \subseteq X \times X$ which is an irreducible closed subvariety. Let $\pi_1, \pi_2 : \Gamma_f \to X$ be the two projections. Then π_1 is a birational proper morphism and π_2 is surjective proper. We define the *first degree* $\deg_{1,L}(f)$ of f with respect to L as the intersection number $(\pi_2^*(L) \cdot \pi_1^*(L)^{\dim(X)-1})$ on Γ_f. Then we get a sequence $\{\deg_{1,L}(f^n) |\ n \in \mathbb{N}\}$ of positive integers.

For two sequences $\{a_n\}, \{b_n\} \in (\mathbb{R}_{\geq 1})^{\mathbb{N}}$, we say that they have the same speed of growth if $\{\frac{a_n}{b_n} |\ n \in \mathbb{N}\}$ has an upper bound and a positive lower bound. Let $\deg_1(f)$ be the class of the speed of growth of the sequence $\{\deg_{1,L}(f^n) |\ n \in \mathbb{N}\}$, which by [Dan20, Theorem 1(ii)] is irrelevant with the choice of the big and nef line bundle L. Notice that although [Dan20, Theorem 1(ii)] was stated for normal projective variety X, the result also holds for arbitrary irreducible projective variety X because one can pass to the normalization of X. Then we can abuse notation and say that $\deg_1(f)$ is the *degree sequence* of f. We also remark that in fact $\deg_1(f)$ remains the same on different birational models. See [Dan20, top of p. 1269].

Definition 1.4. *Let X be a projective variety over an algebraically closed field. We say a dominant rational self-map $f : X \dashrightarrow X$ is of bounded-degree if* $\deg_1(f)$ *is a bounded sequence.*

Notice that according to our definition, a bounded-degree self-map is a priori dominant.

Remark 1.5. *If f is a surjective endomorphism of a projective variety X in the definition above, then f is of bounded-degree if and only if the sequence $\{(f^n)^*(L) \cdot L^{\dim(X)-1} |\ n \in \mathbb{N}\}$ is bounded for some (and hence every) ample line bundle $L \in \mathrm{Pic}(X)$.*

Now we can state the main theorem of this paper.

Theorem 1.6. *Let X be a projective variety over an algebraically closed field K of characteristic $p > 0$ and let f be a bounded-degree self-map of X. Let $x \in X(K)$ be a closed point such that the orbit $\mathcal{O}_f(x)$ is well-defined and let $V \subseteq X$ be a closed subvariety. Then $\{n \in \mathbb{N}|\ f^n(x) \in V(K)\}$ is a p-normal set in $\mathbb{N}$.*

Remark 1.7. *(i) We only consider bounded-degree self-maps of projective varieties in this article, but one can also define this notion on quasi-projective varieties since the degree sequence is independent of the choice of birational models as we have mentioned before. Then our result automatically extends to bounded-degree self-maps of quasi-projective varieties over an algebraically closed field of positive characteristic. Using [CGSZ21, Proposition 4.3] and imitating the reduction steps in [LN], one can realize sets of the form "a finite union of arithmetic progressions in $\mathbb{Z}$ along with finitely many sets of the form '$S_q(c_0; c_1, \ldots, c_d)$ where $c_1, \ldots, c_d$ have the same sign'" as the return sets of bounded-degree automorphisms of tori over $\overline{\mathbb{F}_p(t)}$.*

(ii) We focus on the positive characteristic case in this article, but our method is also valid for the 0-characteristic case. Notice [CS93, Theorem 7] implies that $\{n \in \mathbb{Z}|\ g^n \in X(K)\}$ is a finite union of arithmetic progressions in which G is an algebraic group over an algebraically closed field K of characteristic 0, $g \in G(K)$ is a closed point and $X \subseteq G$ is a closed subvariety. By using this result instead of Proposition 3.1 in the proof of Theorem 1.6, one can show that the set $\{n \in \mathbb{N}|\ f^n(x) \in V(K)\}$ is a finite union of arithmetic progressions in $\mathbb{N}$ if K is an algebraically closed field of characteristic 0 (and the other conditions remain the same).

At the end of the Introduction, we describe the structure of this paper. In Section 2, we will propose a Mordell–Lang-type result for arbitrary algebraic groups over algebraically closed fields of positive characteristic. This result is a preliminary of our work on the dynamical Mordell–Lang conjecture in positive characteristic and it also has its own interest. Then, in Section 3, we will apply this result to prove Conjecture 1.3 in the case of translation of algebraic groups. After that, we will finish the proof of Theorem 1.6 in Section 4, using the philosophy that bounded-degree self-maps come from group actions. Finally, we will make further discussions and propose a geometric version of the pDML conjecture in Section 5.

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2 Preparation: Mordell–Lang in positive characteristic

The Mordell–Lang conjecture for a semiabelian variety G over an algebraically closed field K of characteristic 0 states that $X(K) \cap \Gamma$ is a finite union of cosets of subgroups of Γ where $X \subseteq G$ is a closed subvariety and $\Gamma \subseteq G(K)$ is a finitely generated subgroup. This conjecture was proved by Faltings [Fal94] in the case of abelian varieties and by Vojta [Voj96] in the general case of semiabelian varieties. But this statement may not hold for other algebraic groups (see [GHSZ19] for some further discussions). However, when the ground field K has positive characteristic, we are able to describe the intersection set $X(K) \cap \Gamma$ where the ambient algebraic group is arbitrary. Our argument is by combining the known results for semiabelian varieties in positive charcteristic and certain facts for algebraic groups. This kind of observation firstly appeared in [BGM, Section 2].

We would like to name our result as the arithmetic version of the p-Mordell–Lang problem. On the contrary, the groundbreaking work [Hru96, Theorem 1.1] is regarded as the geometric version of the p-Mordell–Lang problem. We will make comparisons between the arithmetic version and the geometric version, and between the p-Mordell–Lang problem and the dynamical p-Mordell–Lang problem in Section 5. Our result is not quite concise because it is of full generality. One may consult [MS04, Ghi] for results in the case that the ambient algebraic group is an isotrivial semiabelian variety in positive characteristic.

Theorem 2.1. *(pML, arithmetic version) Let G be an algebraic group over an algebraically closed field K of characteristic $p > 0$. Let $X \subseteq G$ be a closed subvariety and let $\Gamma \subseteq G(K)$ be a finitely generated commutative subgroup. Then $X(K) \cap \Gamma$ is a finite union of sets of the form*

$$x_0 + (\pi|_{\Gamma_0})^{-1}(S)$$

where

- *$x_0 \in \Gamma$,*
- *$G_0 \subseteq G$ is an algebraic subgroup which is a semiabelian variety over K, and $\Gamma_0 = G_0(K) \cap \Gamma$,*
- *H_0 is a semiabelian variety over a finite subfield $\mathbb{F}_q \subseteq K$, and F_0 is the absolute Frobenius endomorphism of H_0 corresponding to $\mathbb{F}_q$,*
- *$H = H_0 \times_{\mathbb{F}_q} K$, and $F = F_0 \times_{\mathbb{F}_q} K$ is the Frobenius endomorphism of H,*
- *$\pi : G_0 \to H$ is a surjective algebraic group homomorphism, and*
- *S is a subset of $\pi(\Gamma_0)$ of the form $\{\alpha_0 + \sum_{j=1}^{r} F^{n_j}(\alpha_j) |\ n_1, \ldots, n_r \in \mathbb{N}\}$ where $r \in \mathbb{N}$ and $\alpha_0, \alpha_1, \ldots, \alpha_r \in H(K)$.*

Note that since Γ is assumed to be commutative, we use " + " for the multiply operation of elements in Γ.

Remark 2.2. *When G is itself a semiabelian variety, Theorem 2.1 is just [GY24, Theorem 1.10]. According to the notions in [GY24, Definition 1.2, Definition 1.8], S is called a groupless F-set in*

$\pi(\Gamma_0)$ *and the set* $x_0 + (\pi|_{\Gamma_0})^{-1}(S)$ *is called a pseudo-generalized* F*-set in* Γ*. Notice that although the set* S *in the original definition of "pseudo-generalized* F*-set in* Γ*" is allowed to have the slightly broader form* $\{\alpha_0 + \sum_{j=1}^{r} F^{kn_j}(\alpha_j)|\ n_1, \ldots, n_r \in \mathbb{N}\}$ *in which* k *is some positive integer, we may assume* $k = 1$ *by viewing* H *as a semiabelian variety defined over* $\mathbb{F}_{q^k}$*.*

Remark 2.3. *One may consider the case when* G *has no nontrivial subgroup which is a semiabelian variety. In this case, Theorem 2.1 says that* $G(K)$ *is a torsion group, which can also be deduced by statements in [Bri17]. In fact, we will use those statements in [Bri17] to deduce Theorem 2.1 from [GY24,* Theorem 1.10]*.*

In order to prove Theorem 2.1, we fix an algebraically closed field K of characteristic $p > 0$ in this Section. Every algebraic group is over K unless otherwise stated.

For convenience, we will say an algebraic group satisfies the pML property if it satisfies the result of Theorem 2.1. Our goal is to prove that every algebraic group satisfies the pML property. As we have mentioned in Remark 2.2, [GY24, Theorem 1.10] says that semiabelian varieties satisfy the pML property. We will firstly extend this result to commutative algebraic groups. The next lemma is an easy but important observation.

Lemma 2.4. *Let* $0 \to N \to G \to Q$ *be an exact sequence of algebraic groups in which* G *(and hence* N*) is commutative. If* N *satisfies the pML property and* $Q(K)$ *is a torsion group, then* G *satisfies the pML property.*

Proof. Let $X \subseteq G$ be a closed subvariety and let $\Gamma \subseteq G(K)$ be a finitely generated subgroup. Let $\Gamma' = N(K) \cap \Gamma$. Since Γ is finitely generated and $Q(K)$ is torsion, there exists a positive integer n such that $n\Gamma \subseteq \Gamma'$. Therefore, Γ' is a finite index subgroup of Γ. Write $\Gamma = \bigsqcup_{i=1}^{M}(x_i + \Gamma')$ in which $x_1, \ldots, x_M \in \Gamma$.

Now since $X(K) \cap \Gamma = \bigsqcup_{i=1}^{M}(X(K) \cap (x_i + \Gamma'))$, we have to show that each $X(K) \cap (x_i + \Gamma')$ is a finite union of pseudo-generalized F-sets in Γ (for the notion, see Remark 2.2). For each $i = 1, 2, \ldots, M$, denote $X_i = (-x_i + X) \cap N$ which is a closed subvariety of N. Then we have $X(K) \cap (x_i + \Gamma') = x_i + ((-x_i + X)(K) \cap \Gamma') = x_i + (X_i(K) \cap \Gamma')$ because $\Gamma' \subseteq N(K)$.

Since N satisfies the pML property, each $X_i(K) \cap \Gamma'$ is a finite union of pseudo-generalized F-sets in Γ'. Notice that for any algebraic subgroup $G_0 \subseteq N$, $\Gamma_0 = G_0(K) \cap \Gamma'$ is just the same as $G_0(K) \cap \Gamma$. Then we may conclude that $x_i + (X_i(K) \cap \Gamma')$ is a finite union of pseudo-generalized F-sets in Γ for each $i = 1, 2, \ldots, M$. As a result, G satisfies the pML property. $\square$

Now we can prove Theorem 2.1 for commutative algebraic groups. We will use some statements in [Bri17], as mentioned in Remark 2.3.

Proposition 2.5. *Let* G *be a commutative algebraic group. Then* G *satisfies the pML property.*

Proof. The first sentence of [Bri17, Theorem 2.11(i)] states that G has a smallest algebraic subgroup H such that G/H is unipotent. Moreover, the proof in there indicates that there exists an algebraic subgroup $S \subseteq H$ which is a semiabelian variety such that H/S is a finite group (see the exact sequence at the bottom of [Bri17, p. 689]). Now taking $\operatorname{char} K > 0$ into account, we know that both $(G/H)(K)$ and $(H/S)(K)$ are torsion groups. So in view of Lemma 2.4, we finish the proof by [GY24, Theorem 1.10] (see also Remark 2.2). $\square$

Now we prove Theorem 2.1.

Proof of Theorem 2.1. Firstly, one can see that there is a smooth commutative algebraic subgroup $H \subseteq G$ whose underlying set is $\overline{\Gamma}$. In particular, H is a commutative algebraic subgroup of G such that $\Gamma \subseteq H(K)$. Denote $X_0 = X \cap H$ which is a closed subvariety of H. Then $X(K) \cap \Gamma = X_0(K) \cap \Gamma$. But Proposition 2.5 asserts that $X_0(K) \cap \Gamma$ is a finite union of pseudo-generalized F-sets in Γ, hence so does $X(K) \cap \Gamma$. Thus we are done. $\square$

3 An application of p-Mordell–Lang

In this Section, we prove that Conjecture 1.3 holds for translation of algebraic groups. This case is interesting as it connects the p-Mordell–Lang problem and the dynamical p-Mordell–Lang problem.

Proposition 3.1. *Let G be an algebraic group over an algebraically closed field K of characteristic $p > 0$. Let $g \in G(K)$ be a closed point and let $X \subseteq G$ be a closed subvariety. Then $\{n \in \mathbb{Z} | \ g^n \in X(K)\}$ is a p-normal set.*

We need the following lemma.

Lemma 3.2. *Let K, H and F be as in the statement of Theorem 2.1. Let $\Gamma \subseteq H(K)$ be an infinite cyclic subgroup. Choose a generator g of Γ and identify Γ with $\mathbb{Z}$. Let S be a subset of Γ of the form $\{\alpha_0 + \sum_{j=1}^{r} F^{n_j}(\alpha_j) | \ n_1, \ldots, n_r \in \mathbb{N}\}$ where $r \in \mathbb{N}$ and $\alpha_0, \alpha_1, \ldots, \alpha_r \in H(K)$. Then S is a p-normal set.*

Proof. Firstly, we know that F admits an equation $P(x) = 0$ in which $P \in \mathbb{Z}[x]$ is a polynomial with leading coefficient 1 such that every integer root of P has the form $\pm p^e$ for some positive integer e. Let us analyze what the condition $S = \{\alpha_0 + \sum_{j=1}^{r} F^{n_j}(\alpha_j) | \ n_j \in \mathbb{N}\} \subseteq \Gamma$ means. One can see that $\alpha_0 + \alpha_1 + \cdots + \alpha_r \in \Gamma$ and $F(\alpha_j) - \alpha_j \in \Gamma$ for each $j = 1, 2, \ldots, r$. Write $\alpha_0 + \alpha_1 + \cdots + \alpha_r = cg$ and $F(\alpha_j) - \alpha_j = l_j g$ for some $c, l_1, \ldots, l_r \in \mathbb{Z}$. We may assume $l_j \neq 0$ for each j since otherwise $F^{n_j}(\alpha_j) = \alpha_j$ for all $n_j \in \mathbb{N}$ and so that this term can be absorbed into the constant term α_0. Now the condition $S \subseteq \Gamma$ can be read as $l_j F^n(g) \in \Gamma$ for any $j \in \{1, \ldots, r\}$ and $n \in \mathbb{N}$.

Write $l_j F^n(g) = l_j^{(n)} g$ for some $l_j^{(n)} \in \mathbb{Z}$ (so $l_j^{(0)} = l_j$). Then $l_j^{(n)} g = l_j F^n(g) = F(l_j F^{n-1}(g)) = F(l_j^{(n-1)} g) = l_j^{(n-1)} F(g)$ for any $j \in \{1, \ldots, r\}$ and $n \in \mathbb{Z}_+$. Since g is not a torsion point, we deduce $(l_j^{(n-1)})^2 = l_j^{(n-2)} l_j^{(n)}$ for any $n \geq 2$. Thus $l_j | l_j^{(1)}$ must hold because $l_j^{(0)} = l_j \neq 0$ and each $l_j^{(n)}$ is an integer. So we can write $l_j^{(1)} = l_j t_j$ for some integer t_j and hence $l_j(F(g) - t_j g) = 0$ for each j.

Since each $F(g) - t_j g$ is a torsion point but g is not torsion, we can see that $t_1 = \cdots = t_m := t \in \mathbb{Z}$. Write $F(g) = tg + h$ for some torsion point $h \in H(K)$. Then $F^n(g) = t^n g + t^{n-1} h + t^{n-2} F(h) + \cdots + F^{n-1}(h)$ so that $F^n(g) - t^n g$ is a torsion point for each $n \in \mathbb{Z}_+$. But $P(F) = 0$, so $P(t) \cdot g$ is a torsion point and hence $P(t) = 0$ because g is non-torsion. As a result, t has the form $\pm p^e$ for some positive integer e.

Now, we calculate $F^{n_j}(\alpha_j)$. We have $F^{n_j}(\alpha_j) = \alpha_j + (F(\alpha_j) - \alpha_j) + \cdots + (F^{n_j}(\alpha_j) - F^{n_j - 1}(\alpha_j)) = \alpha_j + l_j g + \cdots + l_j F^{n_j - 1}(g) = \alpha_j + l_j^{(0)} g + \cdots + l_j^{(n_j - 1)} g$. Since $l_j^{(0)} = l_j, l_j^{(1)} = l_j t$ and $(l_j^{(n-1)})^2 = l_j^{(n-2)} l_j^{(n)}$ for any $n \geq 2$, we deduce $l_j^{(n)} = l_j t^n$ for each $n \in \mathbb{N}$ by $l_j t \neq 0$. Thus $F^{n_j}(\alpha_j) = \alpha_j + l_j(1 + t + \cdots + t^{n_j - 1}) g = \alpha_j + l_j \frac{t^{n_j} - 1}{t - 1} \cdot g$.

So we know $\alpha_0 + \sum\limits_{j=1}^{r} F^{n_j}(\alpha_j) = (\alpha_0 + \alpha_1 + \cdots + \alpha_r) + \sum\limits_{j=1}^{r} l_j \frac{t^{n_j} - 1}{t - 1} \cdot g = (c + \sum\limits_{j=1}^{r} l_j \frac{t^{n_j} - 1}{t - 1}) \cdot g$ for nonnegative integers $n_1, \ldots, n_r$ in which $c, l_1, \ldots, l_r \in \mathbb{Z}$ and $t = \pm p^e$ with a positive integer e. If t is positive, we immediately see that $S = \{\alpha_0 + \sum\limits_{j=1}^{r} F^{n_j}(\alpha_j) | \ n_j \in \mathbb{N}\} \subseteq \Gamma$ is a p-normal set. If t is negative, we write $S = \bigcup\limits_{\epsilon_1, \cdots, \epsilon_r \in \{0,1\}} \{(c + \sum\limits_{j=1}^{r} l_j \frac{(t+1)(t^{\epsilon_j}(t^2)^{n_j} - 1)}{t^2 - 1}) \cdot g | \ n_j \in \mathbb{N}\}$ and then also conclude that S is a p-normal set. Thus we are done. □

Now we can prove Proposition 3.1.

Proof of Proposition 3.1. Without loss of generality, we may assume g is non-torsion. Let $\Gamma = \{g^n | \ n \in \mathbb{Z}\}$ which is an infinite cyclic subgroup of $G(K)$. Using Theorem 2.1, we only need to show that every pseudo-generalized F-set in Γ is a p-normal set. But taking Lemma 3.2 into account, the process of proving this statement is just routine. □

At the end of this Section, we would like to give an example. It is well-known that the set $\{n \in \mathbb{Z} | \ g^n \in X(K)\}$ in Proposition 3.1 can be a "p-set" when the ambient algebraic group is an algebraic torus. We shall give an explicit example to show that when the ambient algebraic group is an abelian variety, this set can also be something beyond a finite union of arithmetic progressions. This can also serve as an example of our main result Theorem 1.6, which is beyond automorphisms of projective spaces and also involves "p-sets".

Example 3.3. *Let E be a supersingular elliptic curve over $K = \overline{\mathbb{F}_p(t)}$ which is defined over $\mathbb{F}_p$. For simplicity, we just let $p = 5$ and let E be the elliptic curve $x_1^2 x_2 = x_0^3 + x_2^3$ in $\mathbb{P}_K^2$ with zero point $O = [0, 1, 0] \in E(K)$. Let $A = E \times E$ which is an abelian variety. We embed A into $\mathbb{P}_K^8$ by Segre embedding, i.e. $[x_0, x_1, x_2] \times [y_0, y_1, y_2] \mapsto [x_0 y_0, x_0 y_1, x_0 y_2, x_1 y_0, x_1 y_1, x_1 y_2, x_2 y_0, x_2 y_1, x_2 y_2]$. Let z_{ij} be the coordinate of $\mathbb{P}^8$ corresponding to $x_i y_j$ for any $0 \leq i, j \leq 2$. Let $X \subseteq A$ be the closed subvariety $\{z_{02} = z_{20} + z_{22}\} \cap A$. Let $g = (Q_1, Q_2) \in A(K)$ in which $Q_1 = (t+1, \sqrt{(t+1)^3 + 1}), Q_2 =$*

$(t, \sqrt{t^3+1})$ *are points lie in the affine chart of* $E(K)$*. Denote* $S = \{n \in \mathbb{N} | \ n \cdot g \in X(K)\}$*. Then we have*

(i) $\{p^{2n} | \ n \in \mathbb{N}\} \subseteq S$

(ii) $S \subseteq \{0\} \cup \{p^k m | \ k \in \mathbb{N}, m \in \mathbb{Z}_+, m \equiv \pm 1 \pmod{2p}\}$.

Thus S *cannot be a finite union of arithmetic progressions in* $\mathbb{N}$.

Proof. Let F be the Frobenius endomorphism Frob_p of E. Since E is supersingular, we have $F^2 = [-p] \in \mathrm{End}(E)$. As a result, $p^{2n} \cdot P = F^{4n}(P)$ for any $n \in \mathbb{N}$ and $P \in E(K)$. Thus we can see that (i) holds.

To prove (ii), we need the explicit formula of the multiplication-by-m map of an elliptic curve described in [Sil09, (III, Ex. 3.7)]. We apply this result to our E.

For any positive integer m, there exist $f_m(x) = x^{m^2}+$ (lower order terms) and $g_m(x) = m^2 x^{m^2-1}+$ (lower order terms) which are coprime polynomials in $\mathbb{F}_p[x]$, such that

$$m \cdot P = \begin{cases} \left(\frac{f_m(x)}{g_m(x)}, y'\right), & g_m(x) \neq 0 \\ O, & g_m(x) = 0 \end{cases}$$

where $P = (x, y)$ lies in the affine chart of $E(K)$. In particular, the points Q_1 and Q_2 are non-torsion.

Now $S = \{0\} \cup \{m \in \mathbb{Z}_+ | \ \frac{f_m(t+1)}{g_m(t+1)} = \frac{f_m(t)}{g_m(t)} + 1\}$. But since $f_m(x)$ and $g_m(x)$ are coprime, this condition on m yields $g_m(t+1) = g_m(t)$. So $g_m(x)$ must be a polynomial of $x^p - x$ and as a result, the number of different roots of $g_m(x)$ in K is a multiple of p.

Denote this number by pd_m and write $m = p^k m'$ in which $p \nmid m'$. Then by the supersingularity of E, we deduce

$$m'^2 = |E[m]| = \begin{cases} 1 + 2pd_m, & 2 \nmid m \\ 4 + 2(pd_m - 3), & 2 | m \end{cases}$$

But $p = 5$ cannot be a factor of $m'^2 + 2$, so we have $2 \nmid m$ and $m' \equiv \pm 1 \pmod{p}$. Thus we are done. $\square$

4 Bounded-degree self-maps

We will prove Theorem 1.6 in this Section. We shall use some knowledge in [Bri] as well as the Weil's regularization theorem (Theorem 4.5) to deduce Theorem 1.6 from Proposition 3.1. More precisely, we will deal with the case of bounded-degree automorphisms in subsection 4.1, and prove Theorem 1.6 in subsection 4.2 by reducing to the case of bounded-degree automorphisms.

Through this Section, we fix an algebraically closed field K of characteristic $p > 0$ and let everything be over this field. We require K to be of positive characteristic only because we use Proposition 3.1 in the proof of Proposition 4.4.

4.1 Bounded-degree automorphisms

In this subsection, let X be a projective variety and let f be a bounded-degree automorphism of X. We denote $\mathrm{N}^1(X)$ as the group of line bundles on X up to numerical equivalence, which is a finite free $\mathbb{Z}$-module. For $L \in \mathrm{Pic}(X)$, we denote by $[L]_{\mathrm{num}}$ the class of L in $\mathrm{N}^1(X)$. We will prove that Conjecture 1.3 holds for the bounded-degree system (X, f) at the end of this subsection. Now we start with the following proposition which says that bounded-degree automorphisms come from group actions.

Proposition 4.1. *Let X, f be as above. Then there exists a (not necessarily connected) group variety G, a group action $F : G \times X \to X$ and a point $g_0 \in G(K)$ such that $f = F_{g_0}$ in which F_{g_0} is the automorphism of X induced by the group action.*

Firstly, we shall show that the action of the bounded-degree automorphism f on $\mathrm{N}^1(X)$ is unipotent. We need the following lemma on intersection theory.

Lemma 4.2. *Let X be as above and let $C \subseteq X$ be an integral closed subcurve. Then there exists an ample line bundle $L_0 \in \mathrm{Pic}(X)$ such that for every ample line bundle $L \in \mathrm{Pic}(X)$, we have $L \cdot L_0^{\dim X - 1} \geq L \cdot C$.*

Proof. We prove the assertion by induction on $\dim X$. If $\dim X = 1$, then there is nothing to prove. So we may assume that $\dim X \geq 2$.

Denote $I_C \subseteq \mathcal{O}_X$ as the ideal sheaf of $C \subseteq X$. Pick an ample line bundle $L_1 \in \mathrm{Pic}(X)$ such that $I_C \otimes L_1$ is globally generated. Pick a nonzero global section $s \in \Gamma(X, I_C \otimes L_1) \subseteq \Gamma(X, L_1)$ (notice that $I_C \otimes L_1$ is a subsheaf of L_1). Let $D = (s)_0$ be the divisor of zeros of s, which is an effective Cartier divisor on X such that $L_1 \cong \mathscr{L}(D)$. Let $Y \subseteq X$ be the closed subscheme associated with D. Then $C \subseteq Y$ and Y is of pure codimension 1 in X.

Let $Y_0 \subseteq Y$ be an irreducible component of Y containing C and equip Y_0 with the reduced induced closed subscheme structure. Then Y_0 is a projective variety of dimension $\dim X - 1$. Let $i : Y_0 \hookrightarrow X$ be the closed immersion. By induction hypothesis, there is an ample line bundle $L_2 \in \mathrm{Pic}(Y_0)$ such that $(L' \cdot L_2^{\dim Y_0 - 1})_{Y_0} \geq (L' \cdot C)_{Y_0}$ for every ample line bundle $L' \in \mathrm{Pic}(Y_0)$. Now we choose an ample line bundle $L_0 \in \mathrm{Pic}(X)$ such that both $L_0 - L_1$ and $i^*L_0 - L_2$ are ample. We claim that L_0 has the desired property.

Indeed, for every ample line bundle $L \in \mathrm{Pic}(X)$, we have $(L \cdot L_0^{\dim X - 1}) \geq (L \cdot L_0^{\dim X - 2} \cdot L_1) = (L \cdot L_0^{\dim X - 2} \cdot Y) \geq (L \cdot L_0^{\dim X - 2} \cdot Y_0) = (i^*L \cdot (i^*L_0)^{\dim X - 2})_{Y_0} \geq (i^*L \cdot (L_2)^{\dim X - 2})_{Y_0} \geq (i^*L \cdot C)_{Y_0} = (L \cdot C)$. Thus we finish the proof by induction. $\square$

Lemma 4.3. *Let X, f be as above. Then there exists a positive integer n_0 such that $(f^{n_0})^* : \mathrm{N}^1(X) \to \mathrm{N}^1(X)$ is the identity map.*

Proof. Pick a $\mathbb{Z}$-basis $\{[L_1]_{\mathrm{num}}, \ldots, [L_d]_{\mathrm{num}}\}$ of $\mathrm{N}^1(X)$. Then there exists $\{C_1, \ldots, C_d\}$ which are $\mathbb{Q}$-coefficient 1-cycles in X, such that $L_i \cdot C_j = \delta_{ij}$ for all $1 \leq i, j \leq d$ where δ_{ij} is the Kronecker symbol. Let $A \in GL_d(\mathbb{Z})$ be the matrix corresponds to $f^* : \mathrm{N}^1(X) \to \mathrm{N}^1(X)$ under this basis. We have to show that there is a poistive integer n_0 such that $A^{n_0} = I_d$ which is the identity matrix.

For each nonnegative integer n, we have $A^n = ((f^n)^*(L_1), \dots, (f^n)^*(L_d))^\top \cdot (C_1, \dots, C_d)$. But by Remark 1.5 and Lemma 4.2, we can see that each sequence $\{(f^n)^*(L_i) \cdot C_j | \; n \in \mathbb{N}\}$ is bounded. So $\{A^n | \; n \in \mathbb{N}\}$ is a sequence in $GL_d(\mathbb{Z})$ in which each element is bounded. As a result, there are only finitely many different matrices in that sequence. Thus we are done because A is invertible. $\square$

Next, we need to recall some knowledge of the automorphism groups of projective varieties. For a reference, see [Bri, Section 2].

Let X be our projective variety. Let $\mathbf{Aut}_X$ be the contravariant functor from the category of (locally noetherian) K-schemes to the category of groups, which sends the K-scheme S to the group $\mathrm{Aut}(X \times S/S)$ (the products will always be taken over K). This functor is represented by a locally algebraic group Aut_X over K. Let $\mathrm{Aut}(X) = \mathrm{Aut}_{X,\mathrm{red}}$ be the reduced closed (locally algebraic) subgroup of Aut_X, then there are canonical bijections $\mathrm{Aut}(X/K) = \mathrm{Aut}_X(K) = \mathrm{Aut}(X)(K)$. Let $\mathrm{Aut}^0(X)$ be the identity component of $\mathrm{Aut}(X)$, which is a connected group variety. Then $\mathrm{Aut}(X)$ acts on $\mathrm{N}^1(X)$ and in fact $\mathrm{Aut}^0(X)$ acts trivially on it (see [Bri, Lemma 2.8] and the discussion above it).

Let $L \in \mathrm{Pic}(X)$ be an ample line bundle and let $\mathrm{Aut}(X, [L]_{\mathrm{num}})$ be the stabilizer of $[L]_{\mathrm{num}}$ under the action of $\mathrm{Aut}(X)$. Then [Bri, Theorem 2.10] says that $\mathrm{Aut}(X, [L]_{\mathrm{num}})$ is a closed algebraic subgroup of $\mathrm{Aut}(X)$. Notice that $\mathrm{Aut}(X, [L]_{\mathrm{num}})(K) \subseteq \mathrm{Aut}(X)(K)$ is canonically identified with $\{f \in \mathrm{Aut}(X/K) | \; f^*(L) \equiv L\} \subseteq \mathrm{Aut}(X/K)$ where "$\equiv$" stands for numerically equivalent.

Now we can prove Proposition 4.1.

Proof of Proposition 4.1. Let $g_0 \in \mathrm{Aut}_X(K) = \mathrm{Aut}(X)(K)$ be the closed point which corresponds to the bounded-degree automorphism f. Combining Lemma 4.3 and the discussion above, we can see that there is a positive integer n_0 such that $g_0^{n_0}$ lies in a closed algebraic subgroup of $\mathrm{Aut}(X)$ (as it lies in any $\mathrm{Aut}(X, [L]_{\mathrm{num}})$ where $L \in \mathrm{Pic}(X)$ is an ample line bundle). As a result, if we let $G \subseteq \mathrm{Aut}_X$ be the closed smooth (locally algebraic) subgroup whose underlying space is $\overline{\{g_0^n | \; n \in \mathbb{Z}\}}$, then G is in fact an (algebraic) group variety.

Now notice that there is a natural group action $\sigma : \mathrm{Aut}_X \times X \to X$ such that $f = \sigma_{g_0}$ (in which σ_{g_0} is the automorphism of X induced by the group action), we may just let $F : G \times X \to X$ be the group action induced by σ and then one can verify that the conclusion of Proposition 4.1 holds for G, F and $g_0 \in G(K)$. $\square$

Now we can prove Theorem 1.6 for bounded-degree automorphisms of projective varieties.

Proposition 4.4. *Let X, f be as in the beginning of this subsection. Then f satisfies the arithmetic pDML property, i.e. the conclusion of Theorem 1.6 holds.*

Proof. Let $x \in X(K)$ be a closed point and let $V \subseteq X$ be a closed subvariety. We will prove that $\{n \in \mathbb{Z}|\ f^n(x) \in V(K)\}$ is a p-normal set.

Let G, F and $g_0 \in G(K)$ be as in Proposition 4.1. Since $f = F_{g_0}$, we know that $f^n = F_{g_0^n}$ for every integer n where $F_{g_0^n}$ is the automorphism of X induced by F and $g_0^n \in G(K)$. So for each integer n, we have $f^n(x) = F(g_0^n, x)$. Now let $i_x : G \hookrightarrow G \times X$ be the closed immersion given by $g \mapsto (g, x)$ and let $j : G \to X$ be the composition $F \circ i_x$. Then we have $\{n \in \mathbb{Z}|\ f^n(x) \in V(K)\} = \{n \in \mathbb{Z}|\ g_0^n \in j^{-1}(V)\}$. Thus the result follows from Proposition 3.1. $\square$

4.2 Proof of Theorem 1.6

We will finish the proof of Theorem 1.6 in this subsection. Firstly, we reduce to the case in which the orbit $\mathcal{O}_f(x)$ is dense in X.

Lemma 4.5. *In order to prove Theorem 1.6, we may assume that the orbit $\mathcal{O}_f(x)$ is dense in X without loss of generality.*

Proof. Assume that we have proved Theorem 1.6 with the additional assumption that $\mathcal{O}_f(x)$ is dense in X, we want to prove Theorem 1.6. We will do induction on $\dim(X)$. The case in which $\dim(X) = 1$ is easy.

Now assume $f : X \dashrightarrow X$ is a bounded-degree self-map of a projective variety X and $x \in X(K)$ is a point such that $\mathcal{O}_f(x)$ is well-defined. Let $V \subseteq X$ be a closed subvariety. We will prove that $\{n \in \mathbb{N}|\ f^n(x) \in V(K)\}$ is a p-normal set in $\mathbb{N}$. Assume further without loss of generality that $\mathcal{O}_f(x)$ is not dense in X. By substituting x by a proper iterate, we may assume that the closed subsets $\overline{\{f^n(x)|\ n \geq N\}}$ are all the same for any nonnegative integer N. Denote this closed subset as a proper closed subvariety $X_0 \subseteq X$.

Let $X_{11}, \ldots, X_{1d_1}, X_{21}, \ldots, X_{2d_2}, \ldots, X_{r1}, \ldots, X_{rd_r}$ be the irreducible components of X_0 such that $\dim(X_{11}) = \cdots = \dim(X_{1d_1}) > \dim(X_{21}) = \cdots = \dim(X_{2d_2}) > \cdots > \dim(X_{r1}) = \cdots = \dim(X_{rd_r})$. Notice that since $\mathcal{O}_f(x)$ is dense in X_0, we have $\mathcal{O}_f(x) \cap X_{ij}$ is nonempty for each i, j. So $U_{ij} := \mathrm{Dom}(f) \cap X_{ij}$ is a nonempty open subset of X_{ij} for each i, j. We can see that $\bigcup\limits_{i,j} f(U_{ij})$ is a dense subset of X_0.

As a result, one can choose a pair $(\sigma_1(i,j), \sigma_2(i,j))$ for each i, j such that $f(U_{ij}) \subseteq X_{\sigma_1(i,j)\sigma_2(i,j)}$. Using the density of $\bigcup\limits_{i,j} f(U_{ij})$ in X_0, one can show that (by induction on i from 1 to r) $\sigma_1(i,j) = i$ for each i, j and $\sigma_2(i,1), \ldots, \sigma_2(i, d_i)$ is a permutation of $1, \ldots, d_i$ for each $i = 1, 2, \ldots, r$. Furthermore, by the same reason, $f(U_{ij})$ must be dense in $X_{i\sigma_2(i,j)}$ for each i, j. We abbreviate $\sigma_2(i,j)$ as $\sigma(i,j)$.

By the discussion above, we see that f induces dominant rational maps $f_{ij} : X_{ij} \dashrightarrow X_{i\sigma(i,j)}$ for each i, j, and $U_{ij} \subseteq \mathrm{Dom}(f_{ij})$. Suppose $x \in U_{i_0 j_1}(K)$ and let $j_1, j_2, \ldots, j_t, j_{t+1} = j_1$ be a circle under the action of $\sigma(i_0, j)$. We abbreviate $X_{i_0 j_k}$ as X_k and $f_{i_0 j_k}$ as f_k, then we get a circle of dominant rational maps $f_k : X_k \dashrightarrow X_{k+1}$ ($k = 1, \ldots, t$, understood by modulo t for the indices), all induced by f. Denote $g_k : X_k \dashrightarrow X_k$ as the composite $f_{k+t-1} \circ \cdots \circ f_k$ for $k = 1, \ldots, t$, which is a dominant rational self-map of X_k.

One can verify that for each $k = 1, \ldots, t$, $\mathrm{Dom}(f^t) \cap X_k$ is nonempty and $f^t|_{X_k}$ maps into X_k. In fact, one can show that $f^t|_{X_k} : X_k \dashrightarrow X_k$ is same as g_k. So by (the proof of) [JSXZ, Proposition 3.2], we know that g_k are bounded-degree (dominant) self-maps for each $k = 1, \ldots, t$. Moreover, since $x \in U_{i_0 j_1}(K) \subseteq X_1(K)$, we have $f^n(x) = f_n \circ \cdots \circ f_1(x) \in U_{i_0 j_{n+1}}(K) \subseteq X_{n+1}(K)$ for each $n \in \mathbb{N}$. As a result, we have $\mathcal{O}_{g_k}(f^{k-1}(x))$ is well-defined and $g_k^n(f^{k-1}(x)) = f^{nt+k-1}(x)$ for every $k = 1, \ldots, t$ and every $n \in \mathbb{N}$. Notice that $\dim(X_1) = \cdots = \dim(X_t) < \dim(X)$, we know each g_k satisfies the conclusion of Theorem 1.6 in view of the induction hypothesis. So using Remark 1.2, we may conclude that $\{n \in \mathbb{N} |\ f^n(x) \in V(K)\}$ is a p-normal set in $\mathbb{N}$ for any closed subvariety $V \subseteq X$. Hence we finish the proof by induction. $\square$

Next, we introduce our main tool. This result was firstly proved in [HZ96, Section 5] for the case that the base field is $\mathbb{C}$. Inheriting the methods in [HZ96], the second author writes a note [Yan] which deals with the arbitrary characteristic case and copes with the details carefully. One can consult [HZ96, Section 5], [Can14, Theorem 2.5] or [Yan, Corollary 1.3] for references.

Theorem 4.6. *(Weil's regularization theorem) Let $f : X \dashrightarrow X$ be a bounded-degree self-map of a projective variety. Then there exists a projective variety Y, a birational map $\pi : Y \dashrightarrow X$ and a bounded-degree automorphism g of Y, such that $f \circ \pi = \pi \circ g$.*

We need another lemma before proving Theorem 1.6.

Lemma 4.7. *Let X, Y be varieties and let $\pi : Y \dashrightarrow X$ be a dominant rational map. Let f be a dominant rational self-map of X and g be a flat endomorphism of Y such that $f \circ \pi = \pi \circ g$. Let $x \in X(K)$ be a point such that the orbit $\mathcal{O}_f(x)$ is well-defined and dense in X. Suppose that g satisfies the arithmetic pDML property, i.e. $\{n \in \mathbb{N} |\ g^n(y) \in W(K)\}$ is a p-normal set in $\mathbb{N}$ for every point $g \in Y(K)$ and every closed subvariety $W \subseteq Y$. Then for every closed subvariety $V \subseteq X$, we have $\{n \in \mathbb{N} |\ f^n(x) \in V(K)\}$ is a p-normal set in $\mathbb{N}$.*

Proof. Firstly, notice that since the flat endomorphism g of Y must be dominant, no problem will occur when compositing the maps. Let $D = \mathrm{Dom}(\pi)$ and let $\pi_0 : D \to X$ be the morphism which represents π. Then D is an open dense subset of Y and $\pi_0(D)$ is a constructible dense subset of X since π is dominant. So $\pi_0(D)$ contains an open dense subset of X. By substituting x by a proper iterate, we may assume $x \in \pi_0(D)$ without loss of generality because $\mathcal{O}_f(x)$ is dense in X. Then we can choose a point $y \in D(K)$ such that $\pi_0(y) = x$. We firstly prove that $\mathcal{O}_g(y) \subseteq D$.

Let n be a nonnegative integer. We want to prove that $g^n(y) \in D$. By the assumption that $\mathcal{O}_f(x)$ is well-defined, we can see that $x \in \mathrm{Dom}(f^n)$. As a result, we have $y \in \mathrm{Dom}(f^n \circ \pi)$. So $y \in \mathrm{Dom}(\pi \circ g^n)$ because $f^n \circ \pi = \pi \circ g^n$. Now since g^n is a flat morphism, we conclude that $g^n(y) \in D$ by [BLR90, 2.5, Proposition 5] (although the statement of this reference contains a smoothness requirement, one can verify that the proof is still valid without that assumption). Therefore, we deduce that $g^n(y) \in D$ for each nonnegative integer n and thus $\mathcal{O}_g(y) \subseteq D$.

Now for each nonnegative integer n, we have $f^n(x) = f^n(\pi_0(y)) = \pi_0(g^n(y))$ since $x \in \mathrm{Dom}(f^n)$, $g^n(y) \in D$ and $f^n \circ \pi = \pi \circ g^n$. Let W be the closure of $\pi_0^{-1}(V)$ in Y. Then W is a closed subvariety of Y such that $W \cap D = \pi_0^{-1}(V)$. Now combining the fact $\mathcal{O}_g(y) \subseteq D$ with the equality $f^n(x) = \pi_0(g^n(y))$ above, we know that $\{n \in \mathbb{N} | \ f^n(x) \in V(K)\} = \{n \in \mathbb{N} | \ g^n(y) \in W(K)\}$. Hence the result follows. $\square$

Remark 4.8. *The analogue of the lemma above for the 0-DML property (i.e. the statement which asserts that the return set is a finite union of arithmetic progressions in* $\mathbb{N}$*) is also valid. No change is needed to make in the proof.*

Now we can finish the proof of Theorem 1.6.

Proof of Theorem 1.6. Combining Lemma 4.5, Theorem 4.6, Lemma 4.7 and Proposition 4.4 and then we are done. $\square$

5 Geometric version of the pDML conjecture

For the dynamical Mordell–Lang conjecture in characteristic 0, we know that there is an arithmetic version and a geometric version which are equivalent with each other. See [BGT16, Section 3.1.3]. Both of the two versions can be regarded as the dynamical analogue of the corresponding version of the classical Mordell–Lang conjecture in characteristic 0. Since to our knowledge there is still no geometric version of the dynamical p-Mordell–Lang conjecture, we would like to formulate a reasonable one in this article.

We want our conjecture to be a dynamical analogue of the "geometric p-Mordell–Lang theorem", that is, [Hru96, Theorem 1.1]. So firstly we review Hrushovski's result. We translate the original statement from the language of varieties into the language of schemes.

Theorem 5.1. *(pML, geometric version;* [*Hru96*, Theorem 1.1]*) Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$. Let $X \subseteq G$ be an integral closed subvariety and let $\Gamma \subseteq G(K)$ be a subgroup such that $\Gamma \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$ is a finitely generated $\mathbb{Z}_{(p)}$-module. If $X(K) \cap \Gamma$ is dense in X, then there exist*

- *a semiabelian subvariety $G_1 \subseteq G$,*
- *a semiabelian variety G_0 over a finite subfield $\mathbb{F}_q \subseteq K$,*
- *a geometrically integral closed subvariety $X_0 \subseteq G_0$,*
- *a surjective algebraic group homomorphism $f : G_1 \to G_0 \times_{\mathbb{F}_q} K$, and*
- *a point $x_0 \in G(K)$*

such that $X = x_0 + (f^{-1}(X_0 \times_{\mathbb{F}_q} K))_{\mathrm{red}}$.

It is a core philosophy in arithmetic dynamics that the algebraic dynamic systems are analogues of semiabelian varieties in arithmetic geometry. So here we have to find a notion which serves as an analogue of isotrivial semiabelian varieties in positive characteristic. We think that the *bounded-degree self-maps* might be a right answer.

Now we propose our conjecture. We want to focus on the dominant rational self-maps of (quasi-)projective varieties.

Conjecture 5.2. *(pDML, geometric version) Let X be a projective variety over an algebraically closed field K of characteristic $p > 0$ and let $f : X \dashrightarrow X$ be a dominant rational self-map. Let $Y \subseteq X$ be an integral closed subvariety of positive dimension and let $x \in X(K)$ be a point whose orbit $\mathcal{O}_f(x)$ is well-defined. Suppose that $\mathcal{O}_f(x) \cap Y$ is dense in Y, then there exist*

- *a positive integer n_0, an integral closed subvariety $X_1 \subseteq X$, and a dominant self-map $f_1 : X_1 \dashrightarrow X_1$ such that $\mathrm{Dom}(f^{n_0}) \cap X_1$ is nonempty and $f^{n_0}|_{X_1} = i \circ f_1$ in which i is the closed immersion $X_1 \hookrightarrow X$,*
- *a projective variety X_0 over K and a bounded-degree self-map $f_0 : X_0 \dashrightarrow X_0$,*
- *a dominant rational map $F : X_1 \dashrightarrow X_0$ such that $F \circ f_1 = f_0 \circ F$, and*
- *a closed subvariety $Y_0 \subseteq X_0$,*

such that $Y \subseteq F^{-1}(Y_0)$ and $(F^{-1}(Y_0)\backslash Y) \cap \mathcal{O}_f(x)$ is a finite set. Here we interpret $F^{-1}(Y_0)$ as the closure of $(F|_{\mathrm{Dom}(F)})^{-1}(Y_0) \subseteq \mathrm{Dom}(F)$ in X_1.

Remark 5.3.

(i) *One can verify that the above description of the closed subvariety $Y \subseteq X$ includes the case in which Y is f-periodic. The point is that one can let the bounded-degree system (X_0, f_0) be trivial, i.e. $(\mathrm{Spec}(K), \mathrm{id})$.*

(ii) *By changing every "projective" into "quasi-projective" and every rational map into morphism in the statement of Conjecture 5.2, we get another version of the geometric pDML conjecture. Taking Theorem 1.6 into account, this version of the conjecture implies the arithmetic pDML Conjecture 1.3 for the case of endomorphisms of quasi-projective varieties. Although we also believe that the statement of this version should be true, we deliberately remain the rational maps as in Conjecture 5.2 because we think that the rational maps should naturally appear in the picture.*

(iii) *A closed subvariety of X of the form $F^{-1}(Y_0)$ as in Conjecture 5.2 above is of "f-bounded-degree". We think that "f-bounded-degree" should be a reasonable relaxed condition on subvarieties of "f-preperiodic" (as the relationship between bounded-degree and periodic self-maps). One can also substitute the term $F^{-1}(Y_0)$ in Conjecture 5.2 by "an f-bounded-degree closed subvariety of X" and get a weaker version of Conjecture 5.2, which is more similar to the geometric version of 0-DML.*

The careful reader may have noticed that the statement of Conjecture 5.2 is a little bit more complicated than Theorem 5.1. The example below shows that the integral closed subvariety $Y \subseteq X$ may not be f-bounded-degree. So we cannot expect that Y itself is of the form $F^{-1}(Y_0)$.

Example 5.4. *Let the base field K be $\overline{\mathbb{F}_p(t)}$. Let $X = \mathbb{G}_m^4$ and let f be the automorphism of X given by $(x_1, x_2, x_3, x_4) \mapsto (x_1x_2, (t+1)x_2, x_3x_4, tx_4)$. Let $x = (1, 1, 1, 1) \in X(K)$ and let $Y = \{x_2 = x_4 + 1\} \cap \{x_1^2 x_2 = x_3^2 x_4 + 1\}$ be a 2-dimensional integral closed subvariety of X. Then $\mathcal{O}_f(x) \cap Y(K)$ is dense in Y but Y is not of f-bounded-degree.*

Proof. Firstly, we compute that $f^n(x_1, x_2, x_3, x_4) = ((t+1)^{\frac{n(n-1)}{2}} x_1 x_2^n, (t+1)^n x_2, t^{\frac{n(n-1)}{2}} x_3 x_4^n, t^n x_4)$ for all nonnegative integer n. So in particular we have $f^n(x) = ((t+1)^{\frac{n(n-1)}{2}}, (t+1)^n, t^{\frac{n(n-1)}{2}}, t^n)$. As a result, the intersection $\mathcal{O}_f(x) \cap Y(K) = \{f^{p^n}(x) |\ n \in \mathbb{N}\}$. The assertion "$\{f^{p^n}(x) |\ n \in \mathbb{N}\}$ is dense in Y" is equivalent to that $\{((t+1)^{\frac{p^n(p^n-1)}{2}}, t^{\frac{p^n(p^n-1)}{2}}) |\ n \in \mathbb{N}\}$ is dense in $\mathbb{A}^2$, which can be easily verified by using [Ghi19, Theorem 1.3].

However, one can verify that $f^n(Y) = \{(t+1)^{-n} x_2 = t^{-n} x_4 + 1\} \cap \{(t+1)^{n^2} x_1^2 x_2^{-2n+1} = t^{n^2} x_3^2 x_4^{-2n+1} + 1\}$ for each nonnegative n. Substituting $x_2 = (t+1)^n (t^{-n} x_4 + 1)$ into the second equation, we get the equation $t^{n(2n-1)} x_1^2 x_4^{2n-1} - (t+1)^{n(n-1)} (x_4 + t^n)^{2n-1} (t^{n^2} x_3^2 + x_4^{2n-1}) = 0$ in which the left hand side is an irreducible polynomial of degree $4n-2$ for large n. As a result, Y is not of f-bounded-degree. $\square$

In the example above, one can realize the hypersurface $\{x_2 = x_4 + 1\} \subseteq X$ as $F^{-1}(Y_0)$ in the way of Conjecture 5.2. This explains why we modify the naive analogue of Theorem 5.1 into the statement of Conjecture 5.2 in this way.

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