

On the dynamical Mordell–Lang conjecture in positive characteristic

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Abstract

We disprove the original version of the dynamical Mordell–Lang conjecture in positive characteristic and propose a improved version of this p DML conjecture. We prove that this new version holds for bounded-degree self-maps of projective varieties. Moreover, we propose a geometric version of this p DML conjecture.

1 Introduction

In this paper, as a matter of convention, every variety is assumed to be integral but the closed subvarieties can be reducible. For a rational map $f : X \dashrightarrow Y$ between two varieties, we denote $\text{Dom}(f) \subseteq X$ as the domain of definition of f . Let X be a variety over an algebraically closed field K and let f be a rational self-map of X . For a point $x \in X(K)$, we say the orbit $\mathcal{O}_f(x) := \{f^n(x) \mid n \in \mathbb{N}\}$ is well-defined if every iterate $f^n(x)$ lies in $\text{Dom}(f)$. We denote $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$. An arithmetic progression is a set of the form $\{mk + l \mid k \in \mathbb{Z}\}$ for some $m, l \in \mathbb{Z}$ and an arithmetic progression in \mathbb{N} is a set of the form $\{mk + l \mid k \in \mathbb{N}\}$ for some $m, l \in \mathbb{N}$.

The dynamical Mordell–Lang conjecture, which is one of the core problems in the field of arithmetic dynamics, asserts that for any rational self-map f of a variety X over \mathbb{C} , the return set $\{n \in \mathbb{N} \mid f^n(x) \in V(\mathbb{C})\}$ is a finite union of arithmetic progressions in \mathbb{N} where $x \in X(\mathbb{C})$ is a point such that the orbit $\mathcal{O}_f(x)$ is well-defined and $V \subseteq X$ is a closed subvariety. There is an extensive literature on various cases of this 0-DML conjecture (“0” stands for the characteristic of the base field). Two significant cases are as follows:

- (i) If X is a quasi-projective variety over \mathbb{C} and f is an étale endomorphism of X , then the 0-DML conjecture holds for (X, f) . See [BGT10, Theorem 1.3].
- (ii) If $X = \mathbb{A}_{\mathbb{C}}^2$ and f is an endomorphism of X , then the 0-DML conjecture holds for (X, f) . See [Xie17] and [Xie, Theorem 3.2].

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One can consult [BGT16, Xie] and the references therein for more known results. However, we remark that not much is known about the 0-DML conjecture when f is just a *rational* self-map of the variety X . The following problem might reflect the issue in some sense. It seems that the dynamical Mordell–Lang problem is not quite compatible with birational transformations. More precisely, let X, Y be varieties over \mathbb{C} , f, g be dominant rational self-maps of X, Y respectively and $\pi : Y \dashrightarrow X$ be a birational map such that $f \circ \pi = \pi \circ g$. Even if the 0-DML conjecture holds for (Y, g) , generally we do not know how to deduce that the 0-DML conjecture holds for (X, f) .

The statement of the 0-DML conjecture fails when the base field has positive characteristic. See [BGT16, Example 3.4.5.1] for an example. Consequently, Ghioca and Scanlon proposed a dynamical Mordell–Lang conjecture in positive characteristic. See [BGT16, Conjecture 13.2.0.1]. It was conjectured that the return set would be a “ p -normal set” (see Definition 6.1), which was firstly introduced in [Der07, Definition 1.7] for the Skolem–Mahler–Lech problem in positive characteristic. However, the Skolem–Mahler–Lech problem is a very special case of the dynamical Mordell–Lang problem (more precisely, the case of linear recurrence sequences) and it turns out that some sets of a more complicated form are needed for the general case. See Section 6 for a disproof of this old version of the p DML conjecture. As a result, now we introduce a modified version of this conjecture.

Definition 1.1. *Let p be a prime number and let $q = p^e$ for some positive integer e . Suppose that $d \in \mathbb{Z}_+, r \in \mathbb{N}$ and $c_0, c_{ij} \in \mathbb{Q}$ where $(i, j) \in \{1, \dots, d\} \times \{0, \dots, r\}$. Then we define*

$$S_{q,d,r}(c_0; c_{ij}) = \{c_0 + \sum_{i=1}^d \sum_{j=0}^r c_{ij} q^{2^j n_i} \mid n_1, \dots, n_d \in \mathbb{N}\}.$$

We define a widely p -normal set in \mathbb{Z} as a union of finitely many arithmetic progressions (possibly singleton) along with finitely many subsets of \mathbb{Z} of the form $S_{q,d,r}(c_0; c_{ij})$ as above. A widely p -normal set in \mathbb{N} is a subset of \mathbb{N} which is, up to a finite set, equal to the intersection of a widely p -normal set in \mathbb{Z} and \mathbb{N} .

Here, we say two sets S, T are equal up to a finite set if the symmetric difference $(S \setminus T) \cup (T \setminus S)$ is finite, as in [Der07].

In the rest of this article, we will abbreviate “widely p -normal set in \mathbb{Z} ” as “widely p -normal set”.

Remark 1.2. (i) *Notice that in the definition of widely p -normal set above, all of the sets $S_{q,d,r}(c_0; c_{ij})$ involved are required to be contained in \mathbb{Z} . We remark that this condition bounds the denominators of c_0 and c_{ij} . More precisely, we have $(q^{2^j} - 1) \cdot \prod_{s=0, s \neq j}^r (q^{2^s} - q^{2^s}) \cdot c_{ij} \in \mathbb{Z}$ for every pair $(i, j) \in \{1, \dots, d\} \times \{0, \dots, r\}$.*

(ii) *A widely p -normal set intersects \mathbb{N} is a widely p -normal set in \mathbb{N} ; a finite union of widely*

p -normal sets (resp. widely p -normal sets in \mathbb{N}) is still a widely p -normal set (resp. widely p -normal set in \mathbb{N}); $aS + b$ is a widely p -normal set (resp. widely p -normal set in \mathbb{N}) if S is a widely p -normal set (resp. widely p -normal set in \mathbb{N}) and $a, b \in \mathbb{Z}$ (resp. \mathbb{N}). Moreover, although we will not use this statement in this article, we remark that one can show that a finite intersection of widely p -normal sets (resp. widely p -normal sets in \mathbb{N}) is still a widely p -normal set (resp. widely p -normal set in \mathbb{N}) inheriting the method of the proof of [Der07, Lemma 9.5].

(iii) The definition above coincides with the definition of “ p -normal sets” in Definition 6.1 if one demands the r always be 0.

Now we can state our modified version of the dynamical Mordell–Lang conjecture in positive characteristic.

Conjecture 1.3. (*pDML, arithmetic version*) Let X be a variety over an algebraically closed field K of characteristic $p > 0$ and let f be a rational self-map of X . Let $x \in X(K)$ be a closed point such that the orbit $\mathcal{O}_f(x)$ is well-defined and let $V \subseteq X$ be a closed subvariety. Then $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\}$ is a widely p -normal set in \mathbb{N} .

It seems that the necessity for widely p -normal sets is firstly noticed in this article (see Section 6). The previous works towards the p DML problem are mainly as follows: [CGSZ21] shows that for certain cases of endomorphisms of the algebraic torus, the return set is a “ p -normal set in \mathbb{N} ” (see Definition 6.1); and [Xie23, Theorem 1.4, Theorem 1.5], [Yan24] shows that the original statement of the 0-DML conjecture still holds for certain cases of high complexity endomorphisms.

People (for instance, [Xie23] and also Ghioca and Scanlon) guessed that the p -part in the return set occurs only when the self-map f involves or “comes from” some group actions. Otherwise, it is conjectured that the initial statement of the 0-DML conjecture should still be valid for f . We will form a rigorous conjecture towards this perspective in this paper (see Conjecture 5.1). We think that p -sets come from *bounded-degree self-maps*, which is a synonym of “come from group actions” in some sense but has a more dynamical flavor.

We shall briefly introduce how to measure the complexity of a dominant rational self-map $f : X \dashrightarrow X$ in which X is a projective variety over an algebraically closed field. We use the concept of the *degree sequence* of f . Since there are many references of this concept in the literature (see for example [Dan20], [Tru20], [Xie23, Section 2.1] and [Yan, Section 5]), we will just state the definition and some basic properties here.

Let $L \in \text{Pic}(X)$ be a big and nef line bundle. We consider the graph $\Gamma_f \subseteq X \times X$ which is an irreducible closed subvariety. Let $\pi_1, \pi_2 : \Gamma_f \rightarrow X$ be the two projections. Then π_1 is a birational proper morphism and π_2 is surjective proper. We define the *first degree* $\deg_{1,L}(f)$ of f with respect to L as the intersection number $(\pi_2^*(L) \cdot \pi_1^*(L)^{\dim(X)-1})$ on Γ_f . Then we get a sequence $\{\deg_{1,L}(f^n) \mid n \in \mathbb{N}\}$ of positive integers.

For two sequences $\{a_n\}, \{b_n\} \in (\mathbb{R}_{\geq 1})^{\mathbb{N}}$, we say that they have the same speed of growth if $\{\frac{a_n}{b_n} \mid n \in \mathbb{N}\}$ has an upper bound and a positive lower bound. Let $\deg_1(f)$ be the class of the speed of growth of the sequence $\{\deg_{1,L}(f^n) \mid n \in \mathbb{N}\}$, which by [Dan20, Theorem 1(ii)] is irrelevant with the choice of the big and nef line bundle L . Notice that although [Dan20, Theorem 1(ii)] was stated for normal projective variety X , the result also holds for arbitrary irreducible projective variety X because one can pass to the normalization of X . Then we can abuse notation and say that $\deg_1(f)$ is the *degree sequence* of f . We also remark that in fact $\deg_1(f)$ remains the same on different birational models. See [Dan20, top of p. 1269].

Definition 1.4. *Let X be a projective variety over an algebraically closed field. We say a dominant rational self-map $f : X \dashrightarrow X$ is of bounded-degree if $\deg_1(f)$ is a bounded sequence.*

Notice that according to our definition, a bounded-degree self-map is a priori dominant.

Remark 1.5. *If f is a surjective endomorphism of a projective variety X in the definition above, then f is of bounded-degree if and only if the sequence $\{(f^n)^*(L) \cdot L^{\dim(X)-1} \mid n \in \mathbb{N}\}$ is bounded for some (and hence every) ample line bundle $L \in \text{Pic}(X)$.*

Now we can state the main theorem of this paper.

Theorem 1.6. *Let X be a projective variety over an algebraically closed field K of characteristic $p > 0$ and let f be a bounded-degree self-map of X . Let $x \in X(K)$ be a closed point such that the orbit $\mathcal{O}_f(x)$ is well-defined and let $V \subseteq X$ be a closed subvariety. Then $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\}$ is a widely p -normal set in \mathbb{N} .*

Remark 1.7. (i) *We only consider bounded-degree self-maps of projective varieties in this article, but one can also define this notion on quasi-projective varieties since the degree sequence is independent of the choice of birational models as we have mentioned before. Then our result automatically extends to bounded-degree self-maps of quasi-projective varieties over an algebraically closed field of positive characteristic.*

(ii) *We focus on the positive characteristic case in this article, but our method is also valid for the 0-characteristic case. Notice [CS93, Theorem 7] implies that $\{n \in \mathbb{Z} \mid g^n \in X(K)\}$ is a finite union of arithmetic progressions in which G is an algebraic group over an algebraically closed field K of characteristic 0, $g \in G(K)$ is a closed point and $X \subseteq G$ is a closed subvariety. By using this result instead of Theorem 3.1 in the proof of Theorem 1.6, one can show that the set $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\}$ is a finite union of arithmetic progressions in \mathbb{N} if K is an algebraically closed field of characteristic 0 (and the other conditions remain the same).*

At the end of the Introduction, we describe the structure of this paper. In Section 2, we will make some preparations about the Mordell–Lang problem in positive characteristic. Then, in Section 3, we will use those preparations to prove Conjecture 1.3 in the case of translation of algebraic

groups. After that, we will finish the proof of Theorem 1.6 in Section 4, using the philosophy that bounded-degree self-maps come from group actions. We will make further discussions and propose a geometric version of the p DML conjecture in Section 5. Our purpose of proposing this geometric p DML conjecture is to give a reasonable and rigorous explanation about where the “ p -sets” in Conjecture 1.3 comes from by dividing Conjecture 1.3 into “Conjecture 5.1 + Theorem 1.6”. Lastly, in Section 6, we will give some heuristic examples in which the widely p -normal sets may appear in the return set and a rigorous disproof of the original version of the p DML conjecture.

This paper also includes an Appendix about the p -Mordell–Lang problem in which we fix an error and generalize the results in [GY24].

2 Preparations

In this Section, we will make some preparations about the Mordell–Lang problem in positive characteristic. These preparations will be used in Section 3 to prove that Conjecture 1.3 holds for translations of algebraic groups.

2.1 Two theorems towards Mordell–Lang problem in positive characteristic

In this subsection, we introduce the main theorems of [Hru96] and [MS04] towards the Mordell–Lang problem in positive characteristic. The Mordell–Lang conjecture for a semiabelian variety G over an algebraically closed field K of characteristic 0 states that $X(K) \cap \Gamma$ is a finite union of cosets of subgroups of Γ where $X \subseteq G$ is a closed subvariety and $\Gamma \subseteq G(K)$ is a finitely generated subgroup. This conjecture was proved by Faltings [Fal94] in the case of abelian varieties and by Vojta [Voj96] in the general case of semiabelian varieties. But this statement fails in positive characteristic and it turns out that this problem is more subtle in positive characteristic. The groundbreaking work [Hru96, Theorem 1.1] forms and proves a rigorous statement that the counterexamples of the original 0-ML statement in positive characteristic come from isotrivial semiabelian varieties. Then [MS04, Theorem B] describes the form of the intersection set of a closed subvariety and an Frobenius invariant finitely generated subgroup in an isotrivial semiabelian variety. Recently, [Ghi] gives a description of the intersection set of a closed subvariety and an arbitrary finitely generated subgroup in an isotrivial semiabelian variety. We find that the situation of the p ML problem is somehow similar to the p DML problem, that both of them can be divided to two parts: one has a more geometric flavor and another has a more arithmetic flavor. For more discussions, please see Section 5.

We firstly state the main theorem of [Hru96].

Theorem 2.1. (*p ML, geometric version; [Hru96, Theorem 1.1]*) *Let G be a semiabelian variety*

over an algebraically closed field K of characteristic $p > 0$. Let $X \subseteq G$ be an irreducible closed subvariety and let $\Gamma \subseteq G(K)$ be a finitely generated subgroup. If $X(K) \cap \Gamma$ is dense in X , then there exist

- a semiabelian subvariety $G_1 \subseteq G$,
- a semiabelian variety G_0 over a finite subfield $\mathbb{F}_q \subseteq K$,
- a geometrically integral closed subvariety $X_0 \subseteq G_0$,
- a surjective algebraic group homomorphism $f : G_1 \rightarrow G_0 \times_{\mathbb{F}_q} K$, and
- a point $x_0 \in G(K)$

such that $X = x_0 + (f^{-1}(X_0 \times_{\mathbb{F}_q} K))_{\text{red}}$.

Now we state [MS04, Theorem B] concerning the form of the intersection set of a closed subvariety and an Frobenius invariant finitely generated subgroup in an isotrivial semiabelian variety. Firstly, we introduce the definition of “ F -sets”.

Definition 2.2. Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$ which is defined over a finite field \mathbb{F}_q . Let $F = \text{Frob}_q$ be the Frobenius endomorphism of G and let $\Gamma \subseteq G(K)$ be a finitely generated subgroup.

- (i) An F -set is a subset of $G(K)$ of the form $\{\alpha_0 + \sum_{i=1}^d F^{kn_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\}$ where $k, d \in \mathbb{Z}_+$ and $\alpha_0, \alpha_1, \dots, \alpha_d \in G(K)$. An F -set in Γ is an F -set which is contained in Γ .
- (ii) An F -normal set in Γ is a finite union of sets of the form $S + \Lambda$ where S is an F -set in Γ and $\Lambda \subseteq \Gamma$ is a subgroup.

Remark 2.3. In literature, the “ F -set” above may be called as “groupless F -set” and the “ F -normal set” above may be called as “ F -set”. We use this slightly modified terminology here in order to be similar as in Definition 1.1. Moreover, we would like to shorten the name of those F -sets in Definition 2.2(i) because we will mainly play with them later.

Theorem 2.4. ([MS04, Theorem B]) Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$ which is defined over a finite field \mathbb{F}_q . Let $F = \text{Frob}_q$ be the Frobenius endomorphism of G . Let $\Gamma \subseteq G(K)$ be a finitely generated F -invariant subgroup and let $X \subseteq G$ be a closed subvariety. Then $X(K) \cap \Gamma$ is an F -normal set in Γ .

Notice that the F -invariant requirement on the finitely generated subgroup Γ (i.e. $F(\Gamma) \subseteq \Gamma$) in the theorem above cannot be dropped.

2.2 Intersection of F -sets and cyclic subgroups

As we have mentioned above, Theorem 2.4 is only available for the intersection of closed subvarieties with F -invariant finitely generated subgroups. But for our application towards the p DML problem,

we mainly concern about the intersection set of a closed subvariety with a cyclic group. So we will discuss the form of the intersection set of an F -set with an infinite cyclic group in this subsection.

We start with the definition of “widely F -sets”. Their form is similar to the form of “widely p -sets” in Definition 1.1.

Definition 2.5. *Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$ which is defined over a finite field \mathbb{F}_q . Let $F = \text{Frob}_q$ be the Frobenius endomorphism of G . A widely F -set is a subset of $G(K)$ of the form $\{\alpha_0 + \sum_{i=1}^d \sum_{j=0}^r F^{k \cdot 2^j n_i}(\alpha_{ij}) \mid n_1, \dots, n_d \in \mathbb{N}\}$ where $k, d \in \mathbb{Z}_+, r \in \mathbb{N}$ and $\alpha_0, \alpha_{ij} \in G(K)$ where $(i, j) \in \{1, \dots, d\} \times \{0, \dots, r\}$.*

The main result of this subsection is the following Proposition.

Proposition 2.6. *Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$ which is defined over a finite field \mathbb{F}_q . Let $F = \text{Frob}_q$ be the Frobenius endomorphism of G and let $\Gamma \subseteq G(K)$ be a cyclic group. Then the intersection of an F -set with Γ is a finite union of widely F -sets.*

Remark 2.7. (i) *We will see that the common ratio 2 of the geometric series on the exponents comes from the two different absolute values of the roots of the minimal polynomial of the Frobenius endomorphism.*

(ii) *With some efforts, one can show that the conclusion above also holds for the intersection of a (widely) F -set with a finitely generated subgroup of $G(K)$. But since we only need this general statement in the Appendix and its proof is essentially the same as the proof of Proposition 2.6, we will focus on the intersection of an F -set with a cyclic group for simplicity.*

In order to make things clear, we start with a technical definition.

Definition 2.8. *Let $d \in \mathbb{Z}_+$.*

(i) *A good subgroup of $\mathbb{Z}^d = \{(n_1, \dots, n_d) \mid n_1, \dots, n_d \in \mathbb{Z}\}$ is a subgroup defined by some requirements of the following 4 types*

(a) $n_i = 0$,

(b) n_i is a multiple of some positive integer D ,

(c) $n_i = n_j$, and

(d) $n_i = 2 \cdot n_j$

where $1 \leq i, j \leq d$.

(ii) *A big rectangle in \mathbb{N}^d is a set of the form $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid n_1 \geq M_1, \dots, n_d \geq M_d\}$ for some nonnegative integers M_1, \dots, M_d .*

(iii) A good coset in \mathbb{N}^d is the intersection set of a big rectangle in \mathbb{N}^d with a coset of a good subgroup of \mathbb{Z}^d .

The following lemma is easy to verify and so we omit its proof.

Lemma 2.9. (i) The intersection of two good subgroups of \mathbb{Z}^d is also a good subgroup of \mathbb{Z}^d ; the intersection of two good cosets in \mathbb{N}^d is also a good coset in \mathbb{N}^d .

(ii) A good subgroup of \mathbb{Z}^d can be expressed as the form $\sum_{i=1}^k \mathbb{Z} \cdot D_i \eta_i$ where $k \in \mathbb{N}$, $D_1, \dots, D_k \in \mathbb{Z}_+$ and $\eta_1, \dots, \eta_k \in \mathbb{N}^d$ are vectors satisfying

(a) every component of each η_i lies in $\{0\} \cup \{2^m \mid m \in \mathbb{N}\}$, and

(b) the vectors η_i are pairwise orthogonal, i.e. any two of them do not have a common nonzero component.

A nonempty good coset in \mathbb{N}^d can be expressed as the form $\eta_0 + \sum_{i=1}^k \mathbb{N} \cdot D_i \eta_i$ where $\eta_0 \in \mathbb{N}^d$ and $k, D_1, \dots, D_k, \eta_1, \dots, \eta_k$ are as above.

Remark 2.10. In fact, the vectors η_i in part (ii) above also has a property that their components are “continuous”, i.e. η_i will have a component which is 2^a if it has a component which is 2^b for some $b > a$. But we do not need this fact in the following proof.

Now we can prove Proposition 2.6.

Proof of Proposition 2.6. Let $\Gamma \subseteq G(K)$ be a cyclic subgroup generated by $g \in G(K)$. We may assume that g is non-torsion because otherwise Γ will be a finite group and hence the result follows. Recall that an F -set is a set of the form $\{\alpha_0 + \sum_{i=1}^d F^{kn_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\}$ where $k, d \in \mathbb{Z}_+$ and $\alpha_0, \alpha_1, \dots, \alpha_d \in G(K)$. We denote $\Phi = F^k$ and thus Φ is also a Frobenius endomorphism of G , namely, $\Phi = \text{Frob}_{q^k}$. In the following, we will prove that if the intersection set of $\{\alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\}$ and Γ is infinite, then $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid \alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \in \Gamma\}$ is a finite union of good cosets in \mathbb{N}^d and hence the result follows. At this point, we would like to make two remarks: firstly, one can let some α_{ij} be 0 in the definition of widely F -sets; secondly, although the positive integers D_i in Lemma 2.9(ii) can be different, one may consider their least common multiple and then split a set of the form $\{\alpha_0 + \sum_{i=1}^d \sum_{j=0}^r F^{kD_i \cdot 2^j n_i}(\alpha_{ij}) \mid n_1, \dots, n_d \in \mathbb{N}\}$ into finitely many widely F -sets.

Now firstly, recall that Φ admits an equation $P(\Phi) = 0$ where $P(x) = (x - a_1) \cdots (x - a_s) \in \mathbb{Z}[x]$ is a \mathbb{Z} -coefficient polynomial with leading coefficient 1 whose complex roots a_1, \dots, a_s are pairwise distinct and all have absolute value q^k or $q^{\frac{k}{2}}$. Then for each $n \in \mathbb{N}$ and every $\alpha \in G(K)$, we have that $\Phi^n(\alpha) = \sum_{j=0}^{s-1} c_j(n) \Phi^j(\alpha)$ in which for each $0 \leq j \leq s-1$, $c_j(n)$ is the linear recurrence sequence

satisfying $c_j(j) = 1$ and $c_j(0) = \cdots = c_j(j-1) = c_j(j+1) \cdots = c_j(s-1) = 0$ whose characteristic polynomial is $P(x)$. Then for each $0 \leq j \leq s-1$, we can write down the explicit formula that $c_j(n) = b_{j1}a_1^n + \cdots + b_{js}a_s^n$ for some $b_{j1}, \dots, b_{js} \in \mathbb{C}$.

Now let M be the finitely generated subgroup of $G(K)$ generated by $\{\alpha_0\} \cup \{\Phi^j(\alpha_i) \mid 1 \leq i \leq d, 0 \leq j \leq s-1\}$. So $\{\alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\} \subseteq M$. If $M \cap \Gamma = \{0\}$, then the intersection set of $\{\alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\}$ and Γ is contained in $\{0\}$ and hence is a finite union of widely F -sets. So we may assume that $M \cap \Gamma$ is an infinite cyclic group generated by $g_0 \in M$. We want to show that $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid \alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \in \mathbb{Z} \cdot g_0\}$ is a finite union of good cosets in \mathbb{N}^d .

Write $M = M_{\text{tor}} \oplus M_0$ where M_0 is a free \mathbb{Z} -module. We can find positive integers m_0, D and a \mathbb{Z} -basis $\{R_1, \dots, R_l\}$ of M_0 such that $m_0 \cdot g_0 = D \cdot R_1$. Now fix any integer $r \in \{0, \dots, m_0 - 1\}$, we will prove that $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid \alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \in (r + m_0\mathbb{Z}) \cdot g_0\}$ is a finite union of good cosets in \mathbb{N}^d .

Write $\alpha_0 - r \cdot g_0 = T_0 + \sum_{u=1}^l x_u \cdot R_u$ and $\Phi^j(\alpha_i) = T_{ij} + \sum_{u=1}^l y_{iju} \cdot R_u$ for some $T_0, T_{ij} \in M_{\text{tor}}$ and $x_u, y_{iju} \in \mathbb{Z}$ where $1 \leq i \leq d, 0 \leq j \leq s-1$ and $1 \leq u \leq l$. Then for each $1 \leq i \leq d$, we have $\Phi^{n_i}(\alpha_i) = \sum_{j=0}^{s-1} c_j(n_i) \Phi^j(\alpha_i) = \sum_{j=0}^{s-1} c_j(n_i) (T_{ij} + \sum_{u=1}^l y_{iju} \cdot R_u)$. So $(\alpha_0 - r \cdot g_0) + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) = (T_0 + \sum_{u=1}^l x_u \cdot R_u) + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) (T_{ij} + \sum_{u=1}^l y_{iju} \cdot R_u) = (T_0 + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) \cdot T_{ij}) + \sum_{u=1}^l (x_u + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) y_{iju}) \cdot R_u$.

Therefore, we may interpret the condition on $(n_1, \dots, n_d) \in \mathbb{N}^d$ that $\alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \in (r + m_0\mathbb{Z}) \cdot g_0$ as follows (recall that $m_0 \cdot g_0 = D \cdot R_1$):

- (i) $T_0 + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) \cdot T_{ij} = 0$,
- (ii) $x_1 + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) y_{ij1}$ is a multiple of D , and
- (iii) $x_u + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) y_{iju} = 0$ for every $2 \leq u \leq l$.

Now we argue that for each one of those three conditions, the set $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid (n_1, \dots, n_d) \text{ satisfies that condition}\}$ is a finite union of good cosets in \mathbb{N}^d . Then by Lemma 2.9(i), we conclude that the intersection set $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid \alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \in (r + m_0\mathbb{Z}) \cdot g_0\}$ is also a finite union of good cosets in \mathbb{N}^d .

Firstly, we deal with condition (i). Notice that for every $0 \leq j \leq s-1$, the sequence $(c_j(n) \pmod{|M_{\text{tor}}|})_{n \in \mathbb{N}}$ is eventually periodic. So for every $1 \leq i \leq d$, the sequence of torsion points $(\sum_{j=0}^{s-1} c_j(n) \cdot T_{ij})_{n \in \mathbb{N}}$ is eventually periodic. Now one can easily see that the set $\{(n_1, \dots, n_d) \in$

$\mathbb{N}^d \mid (n_1, \dots, n_d) \text{ satisfies condition (i)}\}$ is a finite union of good cosets in \mathbb{N}^d . Indeed, the good subgroups of \mathbb{Z}^d involved here can be defined using only requirements of type (a)(b) in Definition 2.8(i).

Secondly, for condition (ii), the argument is just the same as in that for condition (i). One just substitute “mod $|M_{\text{tor}}|$ ” by “mod D ”.

Lastly, we deal with condition (iii). Using Lemma 2.9(i), we only need to prove that the set $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid x_u + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) y_{iju} = 0\}$ is a finite union of good cosets in \mathbb{N}^d for every fixed integer $u \in \{2, \dots, l\}$. So now we fix u and calculate that $x_u + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) y_{iju} = x_u + \sum_{i=1}^d \sum_{j=0}^{s-1} y_{iju} (\sum_{v=1}^s b_{jv} a_v^{n_i}) = x_u + \sum_{i=1}^d \sum_{v=1}^s (\sum_{j=0}^{s-1} y_{iju} b_{jv}) a_v^{n_i}$. To be clear, we let $C_{iv} = \sum_{j=0}^{s-1} y_{iju} b_{jv}$ and then the condition can be written as $x_u + \sum_{i=1}^d \sum_{v=1}^s C_{iv} a_v^{n_i} = 0$ in which $C_{iv} \in \mathbb{C}$ are constants for $(i, v) \in \{1, \dots, d\} \times \{1, \dots, s\}$.

Now consider the elements $\lambda_i = (1, \dots, 1, a_1, \dots, a_s, 1, \dots, 1) \in \mathbb{G}_m^{ds}(\mathbb{C})$ for $1 \leq i \leq d$ in which a_s lies at the i th coordinate in the vector λ_i . Let $V \subseteq \mathbb{G}_{m, \mathbb{C}}^{ds}$ be the hyperplane defined by the linear equation $x_u + \sum_{i=1}^d \sum_{v=1}^s C_{iv} X_{(i-1)s+v} = 0$ where X_1, \dots, X_{ds} are the coordinates of $\mathbb{G}_{m, \mathbb{C}}^{ds}$. Then for $(n_1, \dots, n_d) \in \mathbb{N}^d$, we interpret the condition $x_u + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) y_{iju} = 0$ as $\sum_{i=1}^d n_i \cdot \lambda_i \in V(\mathbb{C})$. Let $\Lambda \subseteq \mathbb{G}_m^{ds}(\mathbb{C})$ be the free subgroup with \mathbb{Z} -basis $\{\lambda_1, \dots, \lambda_d\}$. Then [Lau84, Théorème 2] says that $\{\sum_{i=1}^d n_i \cdot \lambda_i \mid n_1, \dots, n_d \in \mathbb{Z} \text{ and } \sum_{i=1}^d n_i \cdot \lambda_i \in V(\mathbb{C})\}$ is a finite union of sets of the form $\lambda + (H \cap \Lambda)$ in which $\lambda \in \Lambda$ and $H \subseteq \mathbb{G}_{m, \mathbb{C}}^{ds}$ is an algebraic subgroup defined by some equations of the form $X_i = 1$ or $X_i = X_j$ where $1 \leq i, j \leq ds$. Now recall that a_1, \dots, a_s are complex numbers that have absolute value q^k or $q^{\frac{k}{2}}$, we conclude that $\{(n_1, \dots, n_d) \in \mathbb{Z}^d \mid \sum_{i=1}^d n_i \cdot \lambda_i \in V(\mathbb{C})\}$ is a finite union of cosets of good subgroups of \mathbb{Z}^d . As a result, we have proved that $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid x_u + \sum_{i=1}^d \sum_{j=0}^{s-1} c_j(n_i) y_{iju} = 0\}$ is a finite union of good cosets in \mathbb{N}^d for every fixed integer $u \in \{2, \dots, l\}$.

To sum up, the whole argument above proves that $\{(n_1, \dots, n_d) \in \mathbb{N}^d \mid \alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \in \Gamma\}$ is a finite union of good cosets in \mathbb{N}^d . So taking the argument at the end of the first paragraph in the proof into account, we finish the proof of Proposition 2.6. \square

2.3 The form of widely F -sets contained in an infinite cyclic group

In subsection 2.2, we proved that the intersection of an F -set with a cyclic group is a finite union of widely F -sets. So in this subsection, we will discuss the form of those widely F -sets contained in a cyclic group.

Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$

which is defined over a finite field \mathbb{F}_q . Let $F = \text{Frob}_q$ be the Frobenius endomorphism of G and let $\Gamma \subseteq G(K)$ be an infinite cyclic group. Since our purpose is to describe the form of the intersection of Γ with a closed subvariety $X \subseteq G$, it is natural to consider that whether Γ has a nontrivial F^k -invariant subgroup for some $k \in \mathbb{Z}_+$. If so, then we may apply Theorem 2.4 and hence in this case, we only need to consider the form of F -sets contained in an infinite cyclic group. The result to this end is as follows.

Proposition 2.11. *Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$ which is defined over a finite field \mathbb{F}_q . Let $F = \text{Frob}_q$ be the Frobenius endomorphism of G . Let Γ be the cyclic group generated by a non-torsion point $g \in G(K)$ and let S be an F -set contained in Γ . Then the set $\{n \in \mathbb{Z} \mid n \cdot g \in S\}$ is a finite union of sets of the form $S_{p^m, d, 0}(c_0; c_{i0})$ defined in Definition 1.1. In other words, the “widely” p -sets will not appear in here.*

Proof. Firstly, recall that S is of the form $\{\alpha_0 + \sum_{i=1}^d F^{kn_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\}$ where $k, r \in \mathbb{Z}_+$ and $\alpha_0, \alpha_1, \dots, \alpha_d \in G(K)$. Denote $\Phi = F^k$ and hence Φ admits an equation $P(\Phi) = 0$ in which $P(x) \in \mathbb{Z}[x]$ is a polynomial with leading coefficient 1 such that every integer root of P has the form $\pm p^e$ for some positive integer e . Let us analyze what the condition $S = \{\alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\} \subseteq \Gamma$ means. One can see that $\alpha_0 + \alpha_1 + \dots + \alpha_d \in \Gamma$ and $\Phi(\alpha_i) - \alpha_i \in \Gamma$ for each $1 \leq i \leq d$. Write $\alpha_0 + \alpha_1 + \dots + \alpha_d = c \cdot g$ and $\Phi(\alpha_i) - \alpha_i = l_i \cdot g$ for some $c, l_1, \dots, l_d \in \mathbb{Z}$. We may assume $l_i \neq 0$ for each i since otherwise $\Phi^{n_i}(\alpha_i) = \alpha_i$ for all $n_i \in \mathbb{N}$ and so that this term can be absorbed into the constant term α_0 . Now the condition $S \subseteq \Gamma$ can be read as $l_i \Phi^n(g) \in \Gamma$ for any $1 \leq i \leq d$ and $n \in \mathbb{N}$.

Write $l_i \Phi^n(g) = l_i^{(n)} g$ for some $l_i^{(n)} \in \mathbb{Z}$ (so $l_i^{(0)} = l_i$). Then $l_i^{(n)} g = l_i \Phi^n(g) = \Phi(l_i \Phi^{n-1}(g)) = \Phi(l_i^{(n-1)} g) = l_i^{(n-1)} \Phi(g)$ for any $1 \leq i \leq d$ and $n \in \mathbb{Z}_+$. Since g is not a torsion point, we deduce $(l_i^{(n-1)})^2 = l_i^{(n-2)} l_i^{(n)}$ for any $n \geq 2$. Thus $l_i | l_i^{(1)}$ must hold because $l_i^{(0)} = l_i \neq 0$ and each $l_i^{(n)}$ is an integer. So we can write $l_i^{(1)} = l_i t_i$ for some integer t_i and hence $l_i(\Phi(g) - t_i g) = 0$ for each i .

Since each $\Phi(g) - t_i g$ is a torsion point but g is non-torsion, we can see that $t_1 = \dots = t_d := t \in \mathbb{Z}$. Write $\Phi(g) = tg + h$ for some torsion point $h \in G(K)$. Then $\Phi^n(g) = t^n g + t^{n-1} h + t^{n-2} F(h) + \dots + F^{n-1}(h)$ so that $\Phi^n(g) - t^n g$ is a torsion point for each $n \in \mathbb{Z}_+$. But $P(\Phi) = 0$, so $P(t) \cdot g$ is a torsion point and hence $P(t) = 0$ because g is non-torsion. As a result, t has the form $\pm p^e$ for some positive integer e .

Now, we calculate $\Phi^{n_i}(\alpha_i)$. We have $\Phi^{n_i}(\alpha_i) = \alpha_i + (\Phi(\alpha_i) - \alpha_i) + \dots + (\Phi^{n_i}(\alpha_i) - \Phi^{n_i-1}(\alpha_i)) = \alpha_i + l_i g + \dots + l_i \Phi^{n_i-1}(g) = \alpha_i + l_i^{(0)} g + \dots + l_i^{(n_i-1)} g$. Since $l_i^{(0)} = l_i$, $l_i^{(1)} = l_i t$ and $(l_i^{(n-1)})^2 = l_i^{(n-2)} l_i^{(n)}$ for any $n \geq 2$, we deduce $l_i^{(n)} = l_i t^n$ for each $n \in \mathbb{N}$ by $l_i t \neq 0$. Thus $\Phi^{n_i}(\alpha_i) = \alpha_i + l_i(1 + t + \dots + t^{n_i-1})g = \alpha_i + l_i \frac{t^{n_i}-1}{t-1} \cdot g$.

So we know $\alpha_0 + \sum_{i=1}^d \Phi^{n_i}(\alpha_i) = (\alpha_0 + \alpha_1 + \dots + \alpha_d) + \sum_{i=1}^d l_i \frac{t^{n_i}-1}{t-1} \cdot g = (c + \sum_{i=1}^d l_i \frac{t^{n_i}-1}{t-1}) \cdot g$ for nonnegative integers n_1, \dots, n_d in which $c, l_1, \dots, l_d \in \mathbb{Z}$ and $t = \pm p^e$ with a positive integer e . If

t is positive, we immediately see that $\{n \in \mathbb{Z} \mid n \cdot g \in S\} = \{c + \sum_{i=1}^d l_i \frac{t^{n_i}-1}{t-1} \mid n_1, \dots, n_d \in \mathbb{N}\} = S_{p^e, d, 0}(c - \sum_{i=1}^d \frac{l_i}{p^e-1}, \frac{l_1}{p^e-1}, \dots, \frac{l_d}{p^e-1})$. If t is negative, we write $\{n \in \mathbb{Z} \mid n \cdot g \in S\} = \bigcup_{\epsilon_1, \dots, \epsilon_d \in \{0, 1\}} \{(c + \sum_{i=1}^d l_i \frac{(t+1)(t^{\epsilon_i}(t^2)^{n_i}-1)}{t^2-1}) \cdot g \mid n_1, \dots, n_d \in \mathbb{N}\}$ and then also conclude that it is a finite union of sets of the form $S_{p^{2e}, d, 0}(c_0; c_{i0})$. Thus we are done. \square

Now we have to consider the case that the infinite cyclic group $\Gamma \subseteq G(K)$ has no nontrivial F^k -invariant subgroup for any $k \in \mathbb{Z}_+$. Our result is as follows.

Proposition 2.12. *Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$ which is defined over a finite field \mathbb{F}_q . Let $F = \text{Frob}_q$ be the Frobenius endomorphism of G and let Γ be the cyclic group generated by a non-torsion point $g \in G(K)$. Let S be a subset of Γ of the form $\{\alpha_0 + \sum_{i=1}^d \sum_{j=0}^r F^{2^j n_i}(\alpha_{ij}) \mid n_1, \dots, n_d \in \mathbb{N}\}$ where $d \in \mathbb{Z}_+, r \in \mathbb{N}$ and $\alpha_0, \alpha_{ij} \in G(K)$ where $(i, j) \in \{1, \dots, d\} \times \{0, \dots, r\}$. Suppose that*

(i) *F admits an equation $P(F) = 0$ where $P(x) = (x-q)P_0(x)$ for some polynomial $P_0(x) \in \mathbb{Z}[x]$ with leading coefficient 1 such that every complex root of P_0 has absolute value $q^{\frac{1}{2}}$, and*

(ii) *Γ has no nontrivial F^k -invariant subgroup for any $k \in \mathbb{Z}_+$.*

Then $\{n \in \mathbb{Z} \mid n \cdot g \in S\}$ is a set of the form $S_{q, d, r-1}(c_0; c_{ij})$ defined in Definition 1.1 where $c_0, c_{ij} \in \mathbb{Q}$ for $(i, j) \in \{1, \dots, d\} \times \{0, \dots, r-1\}$.

Remark 2.13. (i) *The condition (i) for the Frobenius endomorphism F is easy to fulfill. One just raise to a bigger finite field \mathbb{F}_{q^k} on which the torus part of G splits and then this condition holds for Frob_{q^k} .*

(ii) *Notice that in the conclusion, the power of 2 is only up to 2^{r-1} while we have 2^r in the widely F -set.*

In the following discussions, we will always inherit the setting as in Proposition 2.12. More precisely, we will always let G, F, Γ, g be objects as in Proposition 2.12 satisfying those two conditions. We start with several lemmas.

Lemma 2.14. *Let $\alpha \in G(K)$ be a point, $(l_n)_{n \in \mathbb{N}}$ be a sequence of integers and $(t_n)_{n \in \mathbb{N}}$ be a sequence of torsion points in $G(K)$. If $F^n(\alpha) = l_n \cdot g + t_n$ for every nonnegative integer n , then $l_n = 0$ for every $n \in \mathbb{N}$ and hence α is a torsion point.*

Proof. For every $n \geq 0$, we have $l_{n+1} \cdot g + t_{n+1} = F^{n+1}(\alpha) = l_n F(g) + F(t_n)$. So $l_{n+1} \cdot g = l_n F(g) + (F(t_n) - t_{n+1})$ and hence $(l_{n+1}^2 - l_n l_{n+2}) \cdot g$ is a torsion point for every $n \geq 0$. But g is non-torsion and hence we conclude that $l_{n+1}^2 = l_n l_{n+2}$ for each n .

If $l_0 \neq 0$, then we may deduce a contradiction as follows. Since $(l_n)_{n \in \mathbb{N}}$ is a sequence of integers, we deduce that there exists an integer a such that $l_n = l_0 a^n$ for every $n \geq 0$ (one may consider the cases that whether $l_1 = 0$ and find that the conclusion always holds). So we have $\alpha = l_0 \cdot g + t_0$ and $F(\alpha) = l_1 \cdot g + t_1 = l_0 a \cdot g + t_1$ and therefore, $F(l_0 g) = F(\alpha) - F(t_0) = a l_0 g + (t_1 - F(t_0))$. But there exists a positive integer D such that $D(t_1 - F(t_0)) = 0$, so we have $F(D l_0 g) = a \cdot D l_0 g$ and thus the nontrivial subgroup of Γ generated by $D l_0 \cdot g$ is F -invariant. So we get a contradiction and conclude that $l_0 = 0$.

Now as $l_0 = 0$, we have α is a torsion point. Therefore, $F^n(\alpha)$ and hence $l_n \cdot g$ is torsion for each n . So $l_n = 0$ for each n and thus we are done. \square

Lemma 2.15. *Let r be a nonnegative integer, $\alpha_0, \dots, \alpha_r \in G(K)$ be closed points and let $(l_n)_{n \in \mathbb{N}}$ be a sequence of integers, $(t_n)_{n \in \mathbb{N}}$ be a sequence of torsion points in $G(K)$. If $\sum_{j=0}^r F^{2^j n}(\alpha_j) = l_n \cdot g + t_n$ for every $n \geq 0$, then there exists $c_0, \dots, c_{r-1} \in \mathbb{Q}$ such that $l_n = \sum_{j=0}^{r-1} c_j q^{2^j n}$ for every $n \geq 0$ (where q is the power of p which corresponds to F , i.e. $F = \text{Frob}_q$).*

Proof. We prove by induction on r . We notice that although the data G, Γ, g are fixed, we may change q, F into a suitable power in the induction procedure. The case when $r = 0$ is exactly Lemma 2.14 as in this case we have proved that $l_n = 0$ for each n . Suppose the result holds for r , now we consider the case of $r + 1$.

Recall we have assumed that F admits an equation $P(F) = 0$ where $P(x) = (x - q)P_0(x)$ for some polynomial $P_0(x) \in \mathbb{Z}[x]$ with leading coefficient 1 such that every complex root of P_0 has absolute value $q^{\frac{1}{2}}$. Write $P_0(x) = (x - z_1) \cdots (x - z_s)$ for some complex numbers z_1, \dots, z_s of absolute value $q^{\frac{1}{2}}$. For every $m \geq 0$, we let $Q_m(x) = (x - q^m)(x - z_1^m) \cdots (x - z_s^m)$ which is a \mathbb{Z} -coefficient polynomial and let $R(x) = Q_2(x)Q_4(x) \cdots Q_{2^{r+1}}(x)$. Then we have $Q_m(F^m) = 0$ for every m and thus $R(F^2) = R(F^4) = \cdots = R(F^{2^{r+1}}) = 0$. So by doing some \mathbb{Z} -coefficient linear combinations, we can see that there are a sequence of integers $(l'_n)_{n \in \mathbb{N}}$ and a sequence of torsion points $(t'_n)_{n \in \mathbb{N}}$ such that $F^n(R(F)(\alpha_0)) = l'_n \cdot g + t'_n$ for every $n \geq 0$. Hence by Lemma 2.14, we conclude that $R(F)(\alpha_0)$ is a torsion point.

Notice that the greatest common divisor of $P(x)$ and $R(x)$ in $\mathbb{Q}[x]$ is 1 or $x - q$ since the only common root of them is q (if there exists one) and the root q has multiplicity 1 in $P(x)$. But now $R(F)(\alpha_0)$ is a torsion point and $P(F)(\alpha_0) = 0$, so we can see that $(F - q)(\alpha_0)$ must be a torsion point. Now denote $l'_n = l_{n+1} - q l_n$ and $\alpha'_j = (F^{2^j} - q)(\alpha_j)$ for every $n \geq 0$ and every $1 \leq j \leq r + 1$, then we can see that there is a certain sequence of torsion points $(t'_n)_{n \in \mathbb{N}}$ such that $\sum_{j=1}^{r+1} F^{2^j n}(\alpha'_j) = l'_n \cdot g + t'_n$ for every n . We want to use the induction hypothesis towards $q^2, F^2 = \text{Frob}_{q^2}$ and $\Gamma = \mathbb{Z} \cdot g$.

We need to check that the two conditions in Proposition 2.12 are still valid: firstly, condition (i) still holds because $F^2 = \text{Frob}_{q^2}$ satisfies $Q_2(F^2) = 0$ where $Q_2(x) = (x - q^2)(x - z_1^2) \cdots (x - z_s^2)$

and $(x - z_1^2) \cdots (x - z_s^2)$ is a \mathbb{Z} -coefficient polynomial with leading coefficient 1 whose complex roots have absolute value q ; and secondly, condition (ii) tautologically holds as Γ has no nontrivial F^{2k} -invariant subgroups for any $k \in \mathbb{Z}_+$. So now we can use the induction hypothesis to conclude that there exists $c'_1, \dots, c'_r \in \mathbb{Q}$ such that $l'_n = \sum_{j=1}^r c'_j q^{2^j n}$ for every $n \geq 0$. As a result, we can calculate that $l_n = q^n l_0 + \sum_{m=0}^{n-1} q^{n-1-m} l'_m = q^n l_0 + q^{n-1} \sum_{j=1}^r (c'_j \sum_{m=0}^{n-1} q^{(2^j-1)m}) = q^n l_0 + q^{n-1} \sum_{j=1}^r c'_j \frac{q^{(2^j-1)n} - 1}{q^{2^j-1} - 1} = l_0 q^n + \sum_{j=1}^r c'_j \frac{q^{2^j n} - q^n}{q^{2^j} - q} = (l_0 - \sum_{j=1}^r \frac{c'_j}{q^{2^j} - q}) q^n + \sum_{j=1}^r \frac{c'_j}{q^{2^j} - q} q^{2^j n} = \sum_{j=0}^r c_j q^{2^j n}$ for some $c_0, \dots, c_r \in \mathbb{Q}$ and thus we have proved that the result also holds in the case of $r + 1$. So we finish the proof by induction. \square

Now we can prove Proposition 2.12.

Proof of Proposition 2.12. Firstly, the condition $S \subseteq \Gamma$ implies that for every $1 \leq i \leq d$, there exists a sequence of integers $(l_{in})_{n \in \mathbb{N}}$ such that $\sum_{j=0}^r F^{2^j n} \circ (F^{2^j} - 1)(\alpha_{ij}) = l_{in} \cdot g$ for every $n \geq 0$. So by Lemma 2.15, we know that there exists $c_{i,0}, \dots, c_{i,r-1} \in \mathbb{Q}$ such that $l_{in} = \sum_{j=0}^{r-1} c_{i,j} q^{2^j n}$ for every $(i, n) \in \{1, \dots, d\} \times \mathbb{N}$. Let $\alpha_0 + \sum_{i=1}^d \sum_{j=0}^r \alpha_{ij} = l_0 \cdot g$ for some $l_0 \in \mathbb{Z}$. Then we can calculate that $\alpha_0 + \sum_{i=1}^d \sum_{j=0}^r F^{2^j n_i}(\alpha_{ij}) = (\alpha_0 + \sum_{i=1}^d \sum_{j=0}^r \alpha_{ij}) + \sum_{i=1}^d \sum_{j=0}^r \sum_{m=0}^{n_i-1} F^{2^j m} \circ (F^{2^j} - 1)(\alpha_{ij}) = l_0 \cdot g + \sum_{i=1}^d \sum_{m=0}^{n_i-1} l_{im} \cdot g = (l_0 + \sum_{i=1}^d \sum_{m=0}^{n_i-1} \sum_{j=0}^{r-1} c_{i,j} q^{2^j m}) \cdot g = (l_0 + \sum_{i=1}^d \sum_{j=0}^{r-1} c_{i,j} \frac{q^{2^j n_i} - 1}{q^{2^j} - 1}) \cdot g$ for every $(n_1, \dots, n_d) \in \mathbb{N}^d$. As a result, we have $\{n \in \mathbb{Z} \mid n \cdot g \in S\} = \{l_0 + \sum_{i=1}^d \sum_{j=0}^{r-1} c_{i,j} \frac{q^{2^j n_i} - 1}{q^{2^j} - 1} \mid n_1, \dots, n_d \in \mathbb{N}\} = S_{q,d,r-1}(l_0 - \sum_{i=1}^d \sum_{j=0}^{r-1} \frac{c_{i,j}}{q^{2^j} - 1}; \frac{c_{i,j}}{q^{2^j} - 1})$. Thus we are done. \square

3 Translation of algebraic groups

In this Section, we prove that Conjecture 1.3 holds for translation of algebraic groups. This is a special case of our main Theorem 1.6 and it is interesting as it connects the p -Mordell–Lang problem and the dynamical p -Mordell–Lang problem. The main result of this section is as follows. We will use the preparations in Section 2 to prove it.

Theorem 3.1. *Let G be an algebraic group over an algebraically closed field K of characteristic $p > 0$. Let $g \in G(K)$ be a closed point and let $X \subseteq G$ be a closed subvariety. Then $\{n \in \mathbb{Z} \mid g^n \in X(K)\}$ is a widely p -normal set as in Definition 1.1.*

The proof of Theorem 3.1 has 4 steps: firstly, we prove that it holds for isotrivial semiabelian varieties; secondly, we prove that it holds for semiabelian varieties; thirdly, we prove that it holds for commutative group varieties; and lastly, we prove that it holds for arbitrary algebraic groups.

Now we start with the first step.

Proposition 3.2. *Theorem 3.1 holds when G is an isotrivial semiabelian variety, i.e. when G is a semiabelian variety defined over $\overline{\mathbb{F}_p}$.*

Proof. We prove by induction on $\dim(X)$. Notice that we only need to deal with the case when X is an irreducible closed subvariety of G . The case when $\dim(X) = 0$ is easy.

Firstly, we argue that we may assume $\text{Stab}_G(X) = \{0\}$ without loss of generality. Here we denote $\text{Stab}_G(X)$ as the smooth algebraic subgroup of G satisfies $\text{Stab}_G(X)(K) = \{a \in G(K) \mid a + X = X\}$. We know that $\text{Stab}_G(X)$ (and in fact every smooth algebraic subgroup of G) is defined over $\overline{\mathbb{F}_p}$ as an algebraic subgroup of G . So $G/\text{Stab}_G(X)$ is also an isotrivial semiabelian variety and now $X/\text{Stab}_G(X) \subseteq G/\text{Stab}_G(X)$ is an irreducible closed subvariety with trivial translation stabilizer. But if we denote $\pi : G \rightarrow G/\text{Stab}_G(X)$ as the quotient map, we find that $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\} = \{n \in \mathbb{Z} \mid n \cdot \pi(g) \in (X/\text{Stab}_G(X))(K)\}$. So we may assume that $\text{Stab}_G(X) = \{0\}$ without loss of generality.

Now we find q which is a power of p such that G is defined over \mathbb{F}_q and the corresponding Frobenius endomorphism $F = \text{Frob}_q$ satisfies the condition (i) in Proposition 2.12 (we can do this, see Remark 2.13(ii)). We may assume $g \in G(K)$ is non-torsion since otherwise $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ is a finite union of arithmetic progressions. We will deal with two cases that whether $\Gamma = \mathbb{Z} \cdot g$ has a nontrivial F^k -invariant subgroup for some positive integer k .

Case 1: There exist $k_0, n_0 \in \mathbb{Z}_+$ such that $n_0\Gamma$ is an F^{k_0} -invariant group.

Now since $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\} = \bigcup_{a=0}^{n_0-1} (a + n_0 \cdot \{n \in \mathbb{Z} \mid (a + n_0n) \cdot g \in X(K)\})$, we only need to show that $\{n \in \mathbb{Z} \mid (a + n_0n) \cdot g \in X(K)\}$ is a widely p -normal set for every $0 \leq a \leq n_0 - 1$ by Remark 1.2(ii). Write $\{n \in \mathbb{Z} \mid (a + n_0n) \cdot g \in X(K)\} = \{n \in \mathbb{Z} \mid n \cdot n_0g \in (-ag + X)(K)\}$ and then we can apply Theorem 2.4 for $F^{k_0} = \text{Frob}_{q^{k_0}}$, the F^{k_0} -invariant group $n_0\Gamma$ and the closed subvariety $-ag + X \subseteq G$. Then we can conclude that $(-ag + X)(K) \cap n_0\Gamma$ is an F^{k_0} -normal set in $n_0\Gamma$. So $\{n \in \mathbb{Z} \mid n \cdot n_0g \in (-ag + X)(K)\}$ is a widely p -normal set by Proposition 2.11 for every $0 \leq a \leq n_0 - 1$ and in fact, the “widely” p -sets will not appear in here. So we have proved that $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ is a widely p -normal set in this case.

Case 2: Γ has no nontrivial F^k -invariant subgroup for any positive integer k .

Now let $\Gamma_0 = \mathbb{Z}[F] \cdot g$ be a finitely generated F -invariant subgroup of $G(K)$. Using Theorem 2.4, we can write $X(K) \cap \Gamma_0 = \bigcup_{i=1}^A (S_i + \Lambda_i) \cup \bigcup_{i=1}^B S'_i$ where $S_1, \dots, S_A, S'_1, \dots, S'_B$ are F -sets in Γ_0 and $\Lambda_1, \dots, \Lambda_A \subseteq \Gamma_0$ are infinite subgroups since one can split the term whose Λ is a finite group into a finite union of F -sets in Γ_0 . Notice that $X(K) \cap \Gamma = \bigcup_{i=1}^A (\overline{S_i + \Lambda_i}(K) \cap \Gamma) \cup \bigcup_{i=1}^B (S'_i \cap \Gamma)$, we have $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\} = \bigcup_{i=1}^A \{n \in \mathbb{Z} \mid n \cdot g \in \overline{S_i + \Lambda_i}(K)\} \cup \bigcup_{i=1}^B \{n \in \mathbb{Z} \mid n \cdot g \in S'_i \cap \Gamma\}$.

For every $1 \leq i \leq A$, we have $\overline{S_i + \Lambda_i} \subsetneq X$ because it is contained in X and it has a nontrivial translation stabilizer. So $\dim(\overline{S_i + \Lambda_i}) < \dim(X)$ because we have assumed that X is irreducible and hence we conclude that $\{n \in \mathbb{Z} \mid n \cdot g \in \overline{S_i + \Lambda_i}(K)\}$ is a widely p -normal set by induction hypothesis.

For every $1 \leq i \leq B$, we have $S'_i \cap \Gamma$ is a finite union of widely F -sets by Proposition 2.6. Now since we have known that the two conditions in Proposition 2.12 holds for $F = \text{Frob}_q$, we can see that they also holds for every power $F^k = \text{Frob}_{q^k}$. So using Proposition 2.12, we conclude that $\{n \in \mathbb{Z} \mid n \cdot g \in S'_i \cap \Gamma\}$ is a widely p -normal set for every $1 \leq i \leq B$.

Combining the arguments above, we deduce that in this case $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ is a widely p -normal set.

All in all, we proved that $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ is a widely p -normal set by induction. \square

Next, we prove that Theorem 3.1 holds for arbitrary semiabelian varieties.

Lemma 3.3. *Theorem 3.1 holds when G is a semiabelian variety.*

Proof. Denote $\Gamma = \mathbb{Z} \cdot g$. Firstly, we may assume that X is an irreducible closed subvariety of G and $\overline{X(K)} \cap \Gamma = X$ without loss of generality. Then using Theorem 2.1, we can see that there exists

- a semiabelian subvariety $G_1 \subseteq G$,
- an isotrivial semiabelian variety G_0 over K ,
- a closed subvariety $X_0 \subseteq G_0$,
- a surjective algebraic group homomorphism $f : G_1 \rightarrow G_0$, and
- a point $x_0 \in G(K)$

such that $X = x_0 + f^{-1}(X_0)$ as a closed subset of G .

Now we have $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\} = \{n \in \mathbb{Z} \mid ng - x_0 \in G_1(K) \text{ and } f(ng - x_0) \in X_0(K)\}$. We may assume that the set $\{n \in \mathbb{Z} \mid ng - x_0 \in G_1(K)\}$ is infinite since otherwise $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ will be a finite set and then we are done. Then $\{n \in \mathbb{Z} \mid ng - x_0 \in G_1(K)\}$ must be an infinite arithmetic progressions. So we may write $\{n \in \mathbb{Z} \mid ng - x_0 \in G_1(K)\} = a + n_0\mathbb{Z}$ where a is an integer which satisfies $ag - x_0 \in G_1(K)$ and n_0 is a positive integer which satisfies $n_0g \in G_1(K)$. Then $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\} = a + n_0 \cdot \{n \in \mathbb{Z} \mid f(ag - x_0) + n \cdot f(n_0g) \in X_0(K)\} = a + n_0 \cdot \{n \in \mathbb{Z} \mid n \cdot f(n_0g) \in (-f(ag - x_0) + X_0)(K)\}$. So we reduce to the case of isotrivial semiabelian varieties and finish the proof by Proposition 3.2 and Remark 1.2(ii). \square

Remark 3.4. *At this point, we remark that the original version of p DML conjecture holds for translation of abelian varieties. More precisely, the set $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ will be a p -normal set as in Definition 6.1 if the ambient algebraic group is an abelian variety. This statement can be proved by running the procedure of the proof once again for abelian varieties. The key point is that every complex root of the minimal polynomial of the Frobenius endomorphism of an isotrivial*

abelian variety has the same absolute value. Therefore, in the setting of Proposition 2.6 we have that the intersection set of an F -set with a cyclic group is also a finite union of F -sets (instead of widely F -sets).

Next, we prove that Theorem 3.1 holds for commutative group varieties. We remark that the notion of group varieties here is a synonym of smooth algebraic groups, i.e. they do not need to be connected.

Lemma 3.5. *Theorem 3.1 holds when G is a commutative group variety.*

Proof. Firstly, by [Bri17, Theorem 5.6.3(i)], we know that G admits an exact sequence $0 \rightarrow S \rightarrow G \rightarrow Q \rightarrow 0$ where S is a semiabelian variety over K and Q is commutative group variety over K which has finite exponent. So in particular $Q(K)$ is a torsion group. As a result, there exists a positive integer n_0 such that $n_0 \cdot g \in S(K)$. Then we may write $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\} = \bigcup_{a=0}^{n_0-1} (a + n_0 \cdot \{n \in \mathbb{Z} \mid n \cdot n_0 g \in (-a + X)(K)\})$. But for every $0 \leq a \leq n_0 - 1$, we can see that $\{n \in \mathbb{Z} \mid n \cdot n_0 g \in (-a + X)(K)\} = \{n \in \mathbb{Z} \mid n \cdot n_0 g \in ((-a + X) \cap S)(K)\}$ is a widely p -normal set by Lemma 3.3. So we conclude that $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ is a widely p -normal set by Remark 1.2(ii). \square

Finally, we prove Theorem 3.1.

Proof of Theorem 3.1. Firstly, one can see that there is a smooth commutative algebraic subgroup $H \subseteq G$ whose underlying set is $\overline{g^{\mathbb{Z}}}$. Denote $X_0 = X \cap H$ which is a closed subvariety of H . Then $\{n \in \mathbb{Z} \mid g^n \in X(K)\} = \{n \in \mathbb{Z} \mid g^n \in X_0(K)\}$ is a widely p -normal set by Lemma 3.5 and thus we have finished the proof. \square

At the end of this Section, we would like to give an example. It is well-known that the set $\{n \in \mathbb{Z} \mid g^n \in X(K)\}$ in Theorem 3.1 can be a “ p -set” when the ambient algebraic group is an algebraic torus. We shall give an explicit example to show that when the ambient algebraic group is an abelian variety, this set can also be something beyond a finite union of arithmetic progressions. The result in this example will be used in Section 6 to give a rigorous disproof of the original version of p DML conjecture.

Example 3.6. (i) Let $p = 5$ and let $K = \overline{\mathbb{F}_p(t)}$. Let E be the elliptic curve $x_1^2 x_2 = x_0^3 + x_2^3$ in \mathbb{P}_K^2 with zero point $O = [0, 1, 0] \in E(K)$. Let $A = E \times E$ be an abelian variety. We embed A into \mathbb{P}_K^8 by Segre embedding, i.e. $[x_0, x_1, x_2] \times [y_0, y_1, y_2] \mapsto [x_0 y_0, x_0 y_1, x_0 y_2, x_1 y_0, x_1 y_1, x_1 y_2, x_2 y_0, x_2 y_1, x_2 y_2]$. Let z_{ij} be the coordinate of \mathbb{P}^8 corresponding to $x_i y_j$ for any $0 \leq i, j \leq 2$. Let $X \subseteq A$ be the closed subvariety $\{z_{02} = z_{20} + z_{22}\} \cap A$. Let $g = (Q_1, Q_2) \in A(K)$ where $Q_1 = (t + 1, \sqrt{(t + 1)^3 + 1})$, $Q_2 = (t, \sqrt{t^3 + 1})$ are points lie in the affine chart of $E(K)$. Denote $S = \{n \in \mathbb{N} \mid n \cdot g \in X(K)\}$. Then we have

(a) $\{p^n \mid n \in \mathbb{N}\} \subseteq S$, and

(b) $S \subseteq \{0\} \cup \{p^k m \mid k \in \mathbb{N}, m \in \mathbb{Z}_+, m \equiv \pm 1 \pmod{2p}\}$.

Thus S cannot be a finite union of arithmetic progressions in \mathbb{N} .

(ii) Let A be an abelian surface over an algebraically closed field K of characteristic $p > 0$. Let $X \subseteq A$ be a closed subvariety and let $g \in A(K)$ be a closed point. Then $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ is a union of finitely many arithmetic progressions (possibly singleton) along with finitely many sets of the form $S_{q,1,0}(\frac{c_0}{q-1}; \frac{c_1}{q-1})$ as in Definition 1.1 where q is a power of p and c_0, c_1 are integers satisfying $q-1 \mid c_0 + c_1$.

Proof. (i) Let F be the Frobenius endomorphism Frob_p of E . Since E is a supersingular elliptic curve, we have $F^2 = [-p] \in \text{End}(E)$. As a result, $p^n \cdot P = (-1)^n \cdot F^{2n}(P)$ for any $n \in \mathbb{N}$ and $P \in E(K)$. Thus we can see that (i) holds.

To prove (ii), we need the explicit formula of the multiplication-by- m map of an elliptic curve described in [Sil86, (III, Ex. 3.7)]. We apply this result to our E .

For any positive integer m , there exist $f_m(x) = x^{m^2} + (\text{lower order terms})$ and $g_m(x) = m^2 x^{m^2-1} + (\text{lower order terms})$ which are coprime polynomials in $\mathbb{F}_p[x]$, such that

$$m \cdot P = \begin{cases} \left(\frac{f_m(x)}{g_m(x)}, y' \right), & g_m(x) \neq 0 \\ O, & g_m(x) = 0 \end{cases}$$

where $P = (x, y)$ lies in the affine chart of $E(K)$. In particular, the points Q_1 and Q_2 are non-torsion.

Now $S = \{0\} \cup \{m \in \mathbb{Z}_+ \mid \frac{f_m(t+1)}{g_m(t+1)} = \frac{f_m(t)}{g_m(t)} + 1\}$. But since $f_m(x)$ and $g_m(x)$ are coprime, this condition on m yields $g_m(t+1) = g_m(t)$. So $g_m(x)$ must be a polynomial of $x^p - x$ and as a result, the number of different roots of $g_m(x)$ in K is a multiple of p .

Denote this number by pd_m and write $m = p^k m'$ in which $p \nmid m'$. Then by the supersingularity of E , we deduce

$$m'^2 = |E[m]| = \begin{cases} 1 + 2pd_m, & 2 \nmid m \\ 4 + 2(pd_m - 3), & 2 \mid m \end{cases}$$

But $p = 5$ cannot be a factor of $m'^2 + 2$, so we have $2 \nmid m$ and $m' \equiv \pm 1 \pmod{p}$. Thus we are done.

(ii) Assume g is non-torsion without loss of generality. We only have to deal with the case that $X \subseteq A$ is an irreducible curve and $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ is an infinite set. We may further assume that $n_0 g + X \neq X$ for every positive integer n_0 because otherwise $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ will be a finite union of arithmetic progressions and hence we are done. Now recall the statement in Remark 3.4 which implies that $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$

is a union of finitely many arithmetic progressions along with finitely many sets of the form $S_{q,d,0}(\frac{c_0}{q-1}; \frac{c_i}{q-1})$ where c_0, c_1, \dots, c_d are integers satisfying $q-1 \mid c_0 + c_1 + \dots + c_d$.

Notice that for every positive integer n_0 , the set $\{n \in \mathbb{Z} \mid ng \in X(K) \text{ and } (n+n_0)g \in X(K)\}$ must be finite because of the assumption $n_0g + X \neq X$. But then this condition forces $\{n \in \mathbb{Z} \mid n \cdot g \in X(K)\}$ to be a union of finitely many singletons along with finitely many sets of the form $S_{q,1,0}(\frac{c_0}{q-1}; \frac{c_1}{q-1})$ where c_0, c_1 are integers satisfying $q-1 \mid c_0 + c_1$. Thus we have finished the proof. \square

Remark 3.7. *We find it hard to completely determine the return set. In part (i) above, we do not know how to prove that $S = \{0\} \cup \{p^n \mid n \in \mathbb{N}\}$ although we believe this is true. Maybe one can follow the procedure in [BGM] to get a rigorous proof.*

4 Bounded-degree self-maps

We will prove Theorem 1.6 in this Section. We shall use some knowledge in [Bri] as well as a regularization theorem (Theorem 4.6) to deduce Theorem 1.6 from Theorem 3.1. More precisely, we will deal with the case of bounded-degree automorphisms in subsection 4.1, and prove Theorem 1.6 in subsection 4.2 by reducing to the case of bounded-degree automorphisms.

Through this Section, we fix an algebraically closed field K of characteristic $p > 0$ and let everything be over this field. We require K to be of positive characteristic only because we use Theorem 3.1 in the proof of Proposition 4.4.

4.1 Bounded-degree automorphisms

In this subsection, let X be a projective variety and let f be a bounded-degree automorphism of X . We denote $N^1(X)$ as the group of line bundles on X up to numerical equivalence, which is a finite free \mathbb{Z} -module. For $L \in \text{Pic}(X)$, we denote by $[L]_{\text{num}}$ the class of L in $N^1(X)$. We will prove that Conjecture 1.3 holds for the bounded-degree system (X, f) at the end of this subsection. Now we start with the following proposition which says that bounded-degree automorphisms come from group actions.

Proposition 4.1. *Let X, f be as above. Then there exists a (not necessarily connected) group variety G , a group action $F : G \times X \rightarrow X$ and a point $g_0 \in G(K)$ such that $f = F_{g_0}$ in which F_{g_0} is the automorphism of X induced by the group action.*

Firstly, we shall show that the action of the bounded-degree automorphism f on $N^1(X)$ is unipotent. We need the following lemma on intersection theory.

Lemma 4.2. *Let X be as above and let $C \subseteq X$ be an integral closed subcurve. Then there exists an ample line bundle $L_0 \in \text{Pic}(X)$ such that for every ample line bundle $L \in \text{Pic}(X)$, we have $L \cdot L_0^{\dim X - 1} \geq L \cdot C$.*

Proof. We prove the assertion by induction on $\dim X$. If $\dim X = 1$, then there is nothing to prove. So we may assume that $\dim X \geq 2$.

Denote $I_C \subseteq \mathcal{O}_X$ as the ideal sheaf of $C \subseteq X$. Pick an ample line bundle $L_1 \in \text{Pic}(X)$ such that $I_C \otimes L_1$ is globally generated. Pick a nonzero global section $s \in \Gamma(X, I_C \otimes L_1) \subseteq \Gamma(X, L_1)$ (notice that $I_C \otimes L_1$ is a subsheaf of L_1). Let $D = (s)_0$ be the divisor of zeros of s , which is an effective Cartier divisor on X such that $L_1 \cong \mathcal{L}(D)$. Let $Y \subseteq X$ be the closed subscheme associated with D . Then $C \subseteq Y$ and Y is of pure codimension 1 in X .

Let $Y_0 \subseteq Y$ be an irreducible component of Y containing C and equip Y_0 with the reduced induced closed subscheme structure. Then Y_0 is a projective variety of dimension $\dim X - 1$. Let $i : Y_0 \hookrightarrow X$ be the closed immersion. By induction hypothesis, there is an ample line bundle $L_2 \in \text{Pic}(Y_0)$ such that $(L' \cdot L_2^{\dim Y_0 - 1})_{Y_0} \geq (L' \cdot C)_{Y_0}$ for every ample line bundle $L' \in \text{Pic}(Y_0)$. Now we choose an ample line bundle $L_0 \in \text{Pic}(X)$ such that both $L_0 - L_1$ and $i^*L_0 - L_2$ are ample. We claim that L_0 has the desired property.

Indeed, for every ample line bundle $L \in \text{Pic}(X)$, we have $(L \cdot L_0^{\dim X - 1}) \geq (L \cdot L_0^{\dim X - 2} \cdot L_1) = (L \cdot L_0^{\dim X - 2} \cdot Y) \geq (L \cdot L_0^{\dim X - 2} \cdot Y_0) = (i^*L \cdot (i^*L_0)^{\dim X - 2})_{Y_0} \geq (i^*L \cdot (L_2)^{\dim X - 2})_{Y_0} \geq (i^*L \cdot C)_{Y_0} = (L \cdot C)$. Thus we finish the proof by induction. \square

Lemma 4.3. *Let X, f be as above. Then there exists a positive integer n_0 such that $(f^{n_0})^* : N^1(X) \rightarrow N^1(X)$ is the identity map.*

Proof. Pick a \mathbb{Z} -basis $\{[L_1]_{\text{num}}, \dots, [L_d]_{\text{num}}\}$ of $N^1(X)$. Then there exists $\{C_1, \dots, C_d\}$ which are \mathbb{Q} -coefficient 1-cycles in X , such that $L_i \cdot C_j = \delta_{ij}$ for all $1 \leq i, j \leq d$ where δ_{ij} is the Kronecker symbol. Let $A \in GL_d(\mathbb{Z})$ be the matrix corresponds to $f^* : N^1(X) \rightarrow N^1(X)$ under this basis. We have to show that there is a positive integer n_0 such that $A^{n_0} = I_d$ which is the identity matrix.

For each nonnegative integer n , we have $A^n = ((f^n)^*(L_1), \dots, (f^n)^*(L_d))^T \cdot (C_1, \dots, C_d)$. But by Remark 1.5 and Lemma 4.2, we can see that each sequence $\{(f^n)^*(L_i) \cdot C_j \mid n \in \mathbb{N}\}$ is bounded. So $\{A^n \mid n \in \mathbb{N}\}$ is a sequence in $GL_d(\mathbb{Z})$ in which each element is bounded. As a result, there are only finitely many different matrices in that sequence. Thus we are done because A is invertible. \square

Next, we need to recall some knowledge of the automorphism groups of projective varieties. For a reference, see [Bri, Section 2].

Let X be our projective variety. Let \mathbf{Aut}_X be the contravariant functor from the category of (locally noetherian) K -schemes to the category of groups, which sends the K -scheme S to the group $\text{Aut}(X \times S/S)$ (the products will always be taken over K). This functor is represented by a locally algebraic group Aut_X over K . Let $\text{Aut}(X) = \text{Aut}_{X, \text{red}}$ be the reduced closed (locally algebraic) subgroup of Aut_X , then there are canonical bijections $\text{Aut}(X/K) = \text{Aut}_X(K) = \text{Aut}(X)(K)$. Let

$\text{Aut}^0(X)$ be the identity component of $\text{Aut}(X)$, which is a connected group variety. Then $\text{Aut}(X)$ acts on $N^1(X)$ and in fact $\text{Aut}^0(X)$ acts trivially on it (see [Bri, Lemma 2.8] and the discussion above it).

Let $L \in \text{Pic}(X)$ be an ample line bundle and let $\text{Aut}(X, [L]_{\text{num}})$ be the stabilizer of $[L]_{\text{num}}$ under the action of $\text{Aut}(X)$. Then [Bri, Theorem 2.10] says that $\text{Aut}(X, [L]_{\text{num}})$ is a closed algebraic subgroup of $\text{Aut}(X)$. Notice that $\text{Aut}(X, [L]_{\text{num}})(K) \subseteq \text{Aut}(X)(K)$ is canonically identified with $\{f \in \text{Aut}(X/K) \mid f^*(L) \equiv L\} \subseteq \text{Aut}(X/K)$ where “ \equiv ” stands for numerically equivalent.

Now we can prove Proposition 4.1.

Proof of Proposition 4.1. Let $g_0 \in \text{Aut}_X(K) = \text{Aut}(X)(K)$ be the closed point which corresponds to the bounded-degree automorphism f . Combining Lemma 4.3 and the discussion above, we can see that there is a positive integer n_0 such that $g_0^{n_0}$ lies in a closed algebraic subgroup of $\text{Aut}(X)$ (as it lies in any $\text{Aut}(X, [L]_{\text{num}})$ where $L \in \text{Pic}(X)$ is an ample line bundle). As a result, if we let $G \subseteq \text{Aut}_X$ be the closed smooth (locally algebraic) subgroup whose underlying space is $\overline{\{g_0^n \mid n \in \mathbb{Z}\}}$, then G is in fact an (algebraic) group variety.

Now notice that there is a natural group action $\sigma : \text{Aut}_X \times X \rightarrow X$ such that $f = \sigma_{g_0}$ (in which σ_{g_0} is the automorphism of X induced by the group action), we may just let $F : G \times X \rightarrow X$ be the group action induced by σ and then one can verify that the conclusion of Proposition 4.1 holds for G, F and $g_0 \in G(K)$. \square

Now we can prove Theorem 1.6 for bounded-degree automorphisms of projective varieties.

Proposition 4.4. *Let X, f be as in the beginning of this subsection. Then f satisfies the arithmetic pDML property, i.e. the conclusion of Theorem 1.6 holds.*

Proof. Let $x \in X(K)$ be a closed point and let $V \subseteq X$ be a closed subvariety. We will prove that $\{n \in \mathbb{Z} \mid f^n(x) \in V(K)\}$ is a widely p -normal set.

Let G, F and $g_0 \in G(K)$ be as in Proposition 4.1. Since $f = F_{g_0}$, we know that $f^n = F_{g_0^n}$ for every integer n where $F_{g_0^n}$ is the automorphism of X induced by F and $g_0^n \in G(K)$. So for each integer n , we have $f^n(x) = F(g_0^n, x)$. Now let $i_x : G \hookrightarrow G \times X$ be the closed immersion given by $g \mapsto (g, x)$ and let $j : G \rightarrow X$ be the composition $F \circ i_x$. Then we have $\{n \in \mathbb{Z} \mid f^n(x) \in V(K)\} = \{n \in \mathbb{Z} \mid g_0^n \in j^{-1}(V)\}$. Thus the result follows from Theorem 3.1. \square

4.2 Proof of Theorem 1.6

We will finish the proof of Theorem 1.6 in this subsection. Firstly, we reduce to the case in which the orbit $\mathcal{O}_f(x)$ is dense in X .

Lemma 4.5. *In order to prove Theorem 1.6, we may assume that the orbit $\mathcal{O}_f(x)$ is dense in X without loss of generality.*

Proof. Assume that we have proved Theorem 1.6 with the additional assumption that $\mathcal{O}_f(x)$ is dense in X , we want to prove Theorem 1.6. We will do induction on $\dim(X)$. The case in which $\dim(X) = 1$ is easy.

Now assume $f : X \dashrightarrow X$ is a bounded-degree self-map of a projective variety X and $x \in X(K)$ is a point such that $\mathcal{O}_f(x)$ is well-defined. Let $V \subseteq X$ be a closed subvariety. We will prove that $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\}$ is a widely p -normal set in \mathbb{N} . Assume further without loss of generality that $\mathcal{O}_f(x)$ is not dense in X . By substituting x by a proper iterate, we may assume that the closed subsets $\overline{\{f^n(x) \mid n \geq N\}}$ are all the same for any nonnegative integer N . Denote this closed subset as a proper closed subvariety $X_0 \subseteq X$.

Let $X_{11}, \dots, X_{1d_1}, X_{21}, \dots, X_{2d_2}, \dots, X_{r1}, \dots, X_{rd_r}$ be the irreducible components of X_0 such that $\dim(X_{11}) = \dots = \dim(X_{1d_1}) > \dim(X_{21}) = \dots = \dim(X_{2d_2}) > \dots > \dim(X_{r1}) = \dots = \dim(X_{rd_r})$. Notice that since $\mathcal{O}_f(x)$ is dense in X_0 , we have $\mathcal{O}_f(x) \cap X_{ij}$ is nonempty for each i, j . So $U_{ij} := \text{Dom}(f) \cap X_{ij}$ is a nonempty open subset of X_{ij} for each i, j . We can see that $\bigcup_{i,j} f(U_{ij})$ is a dense subset of X_0 .

As a result, one can choose a pair $(\sigma_1(i, j), \sigma_2(i, j))$ for each i, j such that $f(U_{ij}) \subseteq X_{\sigma_1(i,j)\sigma_2(i,j)}$. Using the density of $\bigcup_{i,j} f(U_{ij})$ in X_0 , one can show that (by induction on i from 1 to r) $\sigma_1(i, j) = i$ for each i, j and $\sigma_2(i, 1), \dots, \sigma_2(i, d_i)$ is a permutation of $1, \dots, d_i$ for each $i = 1, 2, \dots, r$. Furthermore, by the same reason, $f(U_{ij})$ must be dense in $X_{i\sigma_2(i,j)}$ for each i, j . We abbreviate $\sigma_2(i, j)$ as $\sigma(i, j)$.

By the discussion above, we see that f induces dominant rational maps $f_{ij} : X_{ij} \dashrightarrow X_{i\sigma(i,j)}$ for each i, j , and $U_{ij} \subseteq \text{Dom}(f_{ij})$. Suppose $x \in U_{i_0 j_1}(K)$ and let $j_1, j_2, \dots, j_t, j_{t+1} = j_1$ be a circle under the action of $\sigma(i_0, j)$. We abbreviate $X_{i_0 j_k}$ as X_k and $f_{i_0 j_k}$ as f_k , then we get a circle of dominant rational maps $f_k : X_k \dashrightarrow X_{k+1}$ ($k = 1, \dots, t$, understood by modulo t for the indices), all induced by f . Denote $g_k : X_k \dashrightarrow X_k$ as the composite $f_{k+t-1} \circ \dots \circ f_k$ for $k = 1, \dots, t$, which is a dominant rational self-map of X_k .

One can verify that for each $k = 1, \dots, t$, $\text{Dom}(f^t) \cap X_k$ is nonempty and $f^t|_{X_k}$ maps into X_k . In fact, one can show that $f^t|_{X_k} : X_k \dashrightarrow X_k$ is same as g_k . So by (the proof of) [JSXZ, Proposition 3.2], we know that g_k are bounded-degree (dominant) self-maps for each $k = 1, \dots, t$. Moreover, since $x \in U_{i_0 j_1}(K) \subseteq X_1(K)$, we have $f^n(x) = f_n \circ \dots \circ f_1(x) \in U_{i_0 j_{n+1}}(K) \subseteq X_{n+1}(K)$ for each $n \in \mathbb{N}$. As a result, we have $\mathcal{O}_{g_k}(f^{k-1}(x))$ is well-defined and $g_k^n(f^{k-1}(x)) = f^{nt+k-1}(x)$ for every $k = 1, \dots, t$ and every $n \in \mathbb{N}$. Notice that $\dim(X_1) = \dots = \dim(X_t) < \dim(X)$, we know each g_k satisfies the conclusion of Theorem 1.6 in view of the induction hypothesis. So using Remark 1.2(ii), we may conclude that $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\} = \bigcup_{k=1}^t ((k-1) + t \cdot \{n \in \mathbb{N} \mid f^{nt+k-1}(x) \in V(K)\}) = \bigcup_{k=1}^t ((k-1) + t \cdot \{n \in \mathbb{N} \mid g_k^n(f^{k-1}(x)) \in (V \cap X_k)(K)\})$ is a widely p -normal set in \mathbb{N} for any closed subvariety $V \subseteq X$. Hence we finish the proof by induction. \square

Next, we introduce our main tool. This result was firstly proved in [HZ96, Section 5] for the case that the base field is \mathbb{C} . Inheriting the methods in [HZ96], the second author writes a note

[Yan] which deals with the arbitrary characteristic case and copes with the details carefully. One can consult [HZ96, Section 5], [Can14, Theorem 2.5] or [Yan, Corollary 1.3] for references.

Theorem 4.6. *Let $f : X \dashrightarrow X$ be a bounded-degree self-map of a projective variety. Then there exists a projective variety Y , a birational map $\pi : Y \dashrightarrow X$ and a bounded-degree automorphism g of Y , such that $f \circ \pi = \pi \circ g$.*

We need another lemma before proving Theorem 1.6.

Lemma 4.7. *Let X, Y be varieties and let $\pi : Y \dashrightarrow X$ be a dominant rational map. Let f be a dominant rational self-map of X and g be a flat endomorphism of Y such that $f \circ \pi = \pi \circ g$. Let $x \in X(K)$ be a point such that the orbit $\mathcal{O}_f(x)$ is well-defined and dense in X . Suppose that g satisfies the arithmetic p DML property, i.e. $\{n \in \mathbb{N} \mid g^n(y) \in W(K)\}$ is a widely p -normal set in \mathbb{N} for every point $y \in Y(K)$ and every closed subvariety $W \subseteq Y$. Then for every closed subvariety $V \subseteq X$, we have $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\}$ is a widely p -normal set in \mathbb{N} .*

Proof. Firstly, notice that since the flat endomorphism g of Y must be dominant, no problem will occur when compositing the maps. Let $D = \text{Dom}(\pi)$ and let $\pi_0 : D \rightarrow X$ be the morphism which represents π . Then D is an open dense subset of Y and $\pi_0(D)$ is a constructible dense subset of X since π is dominant. So $\pi_0(D)$ contains an open dense subset of X . By substituting x by a proper iterate, we may assume $x \in \pi_0(D)$ without loss of generality because $\mathcal{O}_f(x)$ is dense in X . Then we can choose a point $y \in D(K)$ such that $\pi_0(y) = x$. We firstly prove that $\mathcal{O}_g(y) \subseteq D$.

Let n be a nonnegative integer. We want to prove that $g^n(y) \in D$. By the assumption that $\mathcal{O}_f(x)$ is well-defined, we can see that $x \in \text{Dom}(f^n)$. As a result, we have $y \in \text{Dom}(f^n \circ \pi)$. So $y \in \text{Dom}(\pi \circ g^n)$ because $f^n \circ \pi = \pi \circ g^n$. Now since g^n is a flat morphism, we conclude that $g^n(y) \in D$ by [BLR90, 2.5, Proposition 5] (although the statement of this reference contains a smoothness requirement, one can verify that the proof is still valid without that assumption). Therefore, we deduce that $g^n(y) \in D$ for each nonnegative integer n and thus $\mathcal{O}_g(y) \subseteq D$.

Now for each nonnegative integer n , we have $f^n(x) = f^n(\pi_0(y)) = \pi_0(g^n(y))$ since $x \in \text{Dom}(f^n)$, $g^n(y) \in D$ and $f^n \circ \pi = \pi \circ g^n$. Let W be the closure of $\pi_0^{-1}(V)$ in Y . Then W is a closed subvariety of Y such that $W \cap D = \pi_0^{-1}(V)$. Now combining the fact $\mathcal{O}_g(y) \subseteq D$ with the equality $f^n(x) = \pi_0(g^n(y))$ above, we know that $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\} = \{n \in \mathbb{N} \mid g^n(y) \in W(K)\}$. Hence the result follows. \square

Remark 4.8. *The analogue of the lemma above for the 0-DML property (i.e. the statement which asserts that the return set is a finite union of arithmetic progressions in \mathbb{N}) is also valid. No change is needed to make in the proof.*

Now we can finish the proof of Theorem 1.6.

Proof of Theorem 1.6. Combining Lemma 4.5, Theorem 4.6, Lemma 4.7 and Proposition 4.4 and then we are done. \square

At the end of this Section, we would like to give a remark on the complexity of the return set.

Remark 4.9. *By running the proof procedure more carefully, one may let all of the “widely p -sets” involved in the return set have the form $\{c_0 + \sum_{i=1}^d \sum_{j=0}^{r_i} c_{ij} q^{2^j n_i} \mid n_1, \dots, n_d \in \mathbb{N}\}$ where q a power of p and $d + r_1 + \dots + r_d$ is not greater than the dimension of the closed subvariety V at least in the situation of Theorem 3.1 or Proposition 4.4. The key point is that by running through the proof in [MS04], one may let all of the F -sets involved in $X(K) \cap \Gamma$ have the form $\{\alpha_0 + \sum_{i=1}^d F^{kn_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\}$ where $d \leq \dim(X)$ in the situation of Theorem 2.4. We guess that with some more efforts, one may prove that this kind of statement also holds in the situation of Theorem 1.6 and moreover one may add this kind of statement in Conjecture 1.3.*

5 Geometric version of the p DML conjecture

For the dynamical Mordell–Lang conjecture in characteristic 0, we know that there is an arithmetic version and a geometric version which are equivalent with each other. See [BGT16, Section 3.1.3]. Both of the two versions can be regarded as the dynamical analogue of the corresponding version of the classical Mordell–Lang conjecture in characteristic 0. Since to our knowledge there is still no geometric version of the dynamical p -Mordell–Lang conjecture, we would like to formulate a reasonable one in this article.

We want our conjecture to be a dynamical analogue of the “geometric p -Mordell–Lang theorem”, that is, [Hru96, Theorem 1.1] (see Theorem 2.1). It is a core philosophy in arithmetic dynamics that the algebraic dynamic systems are analogues of semiabelian varieties in arithmetic geometry. So here we have to find a notion which serves as an analogue of isotrivial semiabelian varieties in positive characteristic. We think that the *bounded-degree self-maps* might be a right answer.

Now we propose our conjecture. We want to focus on the dominant rational self-maps of (quasi-)projective varieties.

Conjecture 5.1. (*p DML, geometric version*) *Let X be a projective variety over an algebraically closed field K of arbitrary characteristic and let $f : X \dashrightarrow X$ be a dominant rational self-map. Let $Y \subseteq X$ be an irreducible closed subvariety of positive dimension and let $x \in X(K)$ be a point whose orbit $\mathcal{O}_f(x)$ is well-defined. Suppose that $\mathcal{O}_f(x) \cap Y$ is dense in Y , then there exist*

- *a positive integer n_0 , an irreducible closed subvariety $X_1 \subseteq X$, and a dominant self-map $f_1 : X_1 \dashrightarrow X_1$ such that $\text{Dom}(f^{n_0}) \cap X_1$ is nonempty and $f^{n_0}|_{X_1} = i \circ f_1$ in which i is the closed immersion $X_1 \hookrightarrow X$,*
- *a projective variety X_0 over K and a bounded-degree self-map $f_0 : X_0 \dashrightarrow X_0$,*
- *a dominant rational map $F : X_1 \dashrightarrow X_0$ such that $F \circ f_1 = f_0 \circ F$, and*
- *a closed subvariety $Y_0 \subseteq X_0$,*

such that $Y \subseteq F^{-1}(Y_0)$ and $(F^{-1}(Y_0) \setminus Y) \cap \mathcal{O}_f(x)$ is a finite set. Here we interpret $F^{-1}(Y_0)$ as the closure of $(F|_{\text{Dom}(F)})^{-1}(Y_0) \subseteq \text{Dom}(F)$ in X_1 .

Remark 5.2. (i) One can verify that the above description of the closed subvariety $Y \subseteq X$ includes the case in which Y is f -periodic. The point is that one can let the bounded-degree system (X_0, f_0) be trivial, i.e. $(\text{Spec}(K), \text{id})$.

(ii) By changing every “projective” into “quasi-projective” and every rational map into morphism in the statement of Conjecture 5.1, we get another version of the geometric pDML conjecture. Taking Theorem 1.6 into account, this version of the conjecture implies the arithmetic pDML Conjecture 1.3 for the case of endomorphisms of quasi-projective varieties. Although we also believe that the statement of this version should be true, we deliberately remain the rational maps as in Conjecture 5.1 because we think that the rational maps should naturally appear in the picture.

(iii) A closed subvariety of X of the form $F^{-1}(Y_0)$ as in Conjecture 5.1 above is of “ f -bounded-degree”. We think that “ f -bounded-degree” should be a reasonable relaxed condition on subvarieties of “ f -preperiodic” (as the relationship between bounded-degree and periodic self-maps). One can also substitute the term $F^{-1}(Y_0)$ in Conjecture 5.1 by “an f -bounded-degree closed subvariety of X ” and get a weaker version of Conjecture 5.1, which is more similar to the geometric version of 0-DML.

Briefly, Conjecture 5.1 says that in the positive characteristic case, counterexamples of the statement of 0-DML conjecture should come from some bounded-degree system which is “a factor of a subsystem” of the original dynamical system (X, f) . But if f is a polarized endomorphism of a projective variety X , then the subsystems of (X, f) are all polarized and hence they cannot admit nontrivial bounded-degree systems as their factor. So the Conjecture 5.3 below is a succinct consequence of Conjecture 5.1. It says that the statement of 0-DML conjecture should hold for polarized dynamical systems in arbitrary characteristic.

Conjecture 5.3. Let X be a projective variety over an algebraically closed field K of arbitrary characteristic and let $f : X \rightarrow X$ be a polarized endomorphism. Then for every point $x \in X(K)$ and every closed subvariety $V \subseteq X$, the return set $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\}$ is a finite union of arithmetic progressions in \mathbb{N} .

The careful reader may have noticed that the statement of Conjecture 5.1 is a little bit more complicated than Theorem 2.1. The example below shows that the irreducible closed subvariety $Y \subseteq X$ may not be f -bounded-degree. So we cannot expect that Y itself is of the form $F^{-1}(Y_0)$.

Example 5.4. Let the base field K be $\overline{\mathbb{F}_p(t)}$. Let $X = \mathbb{G}_m^4$ and let f be the automorphism of X given by $(x_1, x_2, x_3, x_4) \mapsto (x_1 x_2, (t+1)x_2, x_3 x_4, t x_4)$. Let $x = (1, 1, 1, 1) \in X(K)$ and let

$Y = \{x_2 = x_4 + 1\} \cap \{x_1^2 x_2 = x_3^2 x_4 + 1\}$ be a 2-dimensional integral closed subvariety of X . Then $\mathcal{O}_f(x) \cap Y(K)$ is dense in Y but Y is not of f -bounded-degree.

Proof. Firstly, we compute that $f^n(x_1, x_2, x_3, x_4) = ((t+1)^{\frac{n(n-1)}{2}} x_1 x_2^n, (t+1)^n x_2, t^{\frac{n(n-1)}{2}} x_3 x_4^n, t^n x_4)$ for all nonnegative integer n . So in particular we have $f^n(x) = ((t+1)^{\frac{n(n-1)}{2}}, (t+1)^n, t^{\frac{n(n-1)}{2}}, t^n)$. As a result, the intersection $\mathcal{O}_f(x) \cap Y(K) = \{f^{p^n}(x) \mid n \in \mathbb{N}\}$. The assertion “ $\{f^{p^n}(x) \mid n \in \mathbb{N}\}$ is dense in Y ” is equivalent to that $\{((t+1)^{\frac{p^n(p^n-1)}{2}}, t^{\frac{p^n(p^n-1)}{2}}) \mid n \in \mathbb{N}\}$ is dense in \mathbb{A}^2 , which can be easily verified by using [Ghi19, Theorem 1.3].

However, one can verify that $f^n(Y) = \{(t+1)^{-n} x_2 = t^{-n} x_4 + 1\} \cap \{(t+1)^{n^2} x_1^2 x_2^{-2n+1} = t^{n^2} x_3^2 x_4^{-2n+1} + 1\}$ for each nonnegative n . Substituting $x_2 = (t+1)^n(t^{-n} x_4 + 1)$ into the second equation, we get the equation $t^{n(2n-1)} x_1^2 x_4^{2n-1} - (t+1)^{n(n-1)}(x_4 + t^n)^{2n-1}(t^{n^2} x_3^2 + x_4^{2n-1}) = 0$ in which the left hand side is an irreducible polynomial of degree $4n - 2$ for large n . As a result, Y is not of f -bounded-degree. \square

In the example above, one can realize the hypersurface $\{x_2 = x_4 + 1\} \subseteq X$ as $F^{-1}(Y_0)$ in the way of Conjecture 5.1. This explains why we modify the naive analogue of Theorem 2.1 into the statement of Conjecture 5.1 in this way.

6 A disproof of the original version of p DML conjecture

In this Section, we will disprove the original version of the p DML conjecture. So firstly let us review its statement.

Definition 6.1. Let p be a prime. A p -normal set (in \mathbb{Z}) is a union of finitely many arithmetic progressions (possibly singleton) along with finitely many sets of the form $S_{q,d,0}(\frac{c_0}{q-1}; \frac{c_i}{q-1})$ (see Definition 1.1) in which q is a power of p and c_0, c_1, \dots, c_d are integers satisfying $q-1 \mid c_0 + c_1 + \dots + c_d$. A p -normal set in \mathbb{N} is a subset of \mathbb{N} which is, up to a finite set, equal to the intersection of a p -normal set and \mathbb{N} .

As we have mentioned in the Introduction, this definition was firstly introduced in [Der07] for the Skolem–Mahler–Lech problem in positive characteristic. Now the original version of the p DML conjecture [BGT16, Conjecture 13.2.0.1] can be read as follows.

Conjecture 6.2. Let X be a variety over an algebraically closed field K of characteristic $p > 0$ and let f be a rational self-map of X . Let $x \in X(K)$ be a closed point such that the orbit $\mathcal{O}_f(x)$ is well-defined and let $V \subseteq X$ be a closed subvariety. Then $\{n \in \mathbb{N} \mid f^n(x) \in V(K)\}$ is a p -normal set in \mathbb{N} .

We shall prove that this statement fails even in the special case of translations of isotrivial semiabelian varieties (but it is valid in the case of translation of abelian varieties, see Remark 3.4).

We will give some heuristic of counterexamples in subsection 6.1 and then give a rigorous disproof in subsection 6.2.

6.1 Heuristic of counterexamples

In this subsection, we provide some evidence to the existence of counterexamples of Conjecture 6.2 in the case of translation of isotrivial semiabelian varieties. Since we only want to give some heuristic arguments here, we will not be quite rigorous in this subsection.

Example 6.3. *Let G be a semiabelian variety over an algebraically closed field K of characteristic $p > 0$ which is defined over a finite field \mathbb{F}_q . Let $F = \text{Frob}_{q^2}$ be a Frobenius endomorphism of G . Suppose there are points $\alpha, \beta \in G(K)$ such that $F(\alpha) = q^2 \cdot \alpha$ and $F(\beta) = q \cdot \beta$ (this may happen as we may play with tori and supersingular elliptic curves). Let $g = \alpha + \beta$. Let $d \in \mathbb{Z}_+$ and let $c_1, \dots, c_d \in \mathbb{Z}$. If there is a closed subvariety $X \subseteq G$ such that $X(K) \cap (\mathbb{Z}[F] \cdot g) = \{F^{n_1}(c_1\alpha) + \sum_{i=2}^d F^{n_i}(c_i\alpha + c_{i-1}\beta) + F^{n_{d+1}}(c_d\beta) \mid n_1, \dots, n_{d+1} \in \mathbb{N}\} = \{(\sum_{i=1}^d c_i q^{2n_i}) \cdot \alpha + (\sum_{i=1}^d c_i q^{n_{i+1}}) \cdot \beta \mid n_1, \dots, n_{d+1} \in \mathbb{N}\}$, then $X(K) \cap (\mathbb{Z} \cdot g)$ will contain $\{(\sum_{i=1}^d c_i q^{2^{i_n}}) \cdot g \mid n \in \mathbb{N}\}$.*

A more precise (but still only heuristic) example is as below.

Example 6.4. *Let $p, K, (E, O)$ be same as in Example 3.6(i). Let $G = \mathbb{G}_m^2 \times E^2$. Let C_1, C_2, C_3 be curves in G given by $C_1(K) = \{(x+1, x, O, O) \mid x \in K\}$, $C_2(K) = \{(1, 1, (y+1, \pm\sqrt{(y+1)^3+1}), (y, \pm\sqrt{y^3+1})) \mid y \in K\}$ and $C_3(K) = \{(z+1, z, (z+1, \pm\sqrt{(z+1)^3+1}), (z, \pm\sqrt{z^3+1})) \mid z \in K\}$ (here we abuse some notations since in fact the sets $C_i(K)$ will be larger as C_i are closed subcurves of G). Let $X = C_1 + C_2 + C_3$ be a closed subvariety of G and let $g = (t+1, t, (t+1, \sqrt{(t+1)^3+1}), (t, \sqrt{t^3+1})) \in G(K)$. Then one can see that $\{(p^n + p^{2n}) \cdot g \in X(K) \mid n \in \mathbb{N}\} \subseteq X(K)$ because $(t^{p^n} + 1, t^{p^n}, O, O) \in C_1(K)$, $(1, 1, p^{2n} \cdot (t+1, \sqrt{(t+1)^3+1}), p^{2n} \cdot (t, \sqrt{t^3+1})) \in C_2(K)$ and $(t^{p^{2n}} + 1, t^{p^{2n}}, p^n \cdot (t+1, \sqrt{(t+1)^3+1}), p^n \cdot (t, \sqrt{t^3+1})) \in C_3(K)$ for every $n \in \mathbb{N}$.*

We believe that this example should already be a counterexample of Conjecture 6.2. But as we have mentioned in Remark 3.7, we are unable to completely determine the return set in this example. So we will modify this example and give a rigorous disproof of Conjecture 6.2 by contradiction in next subsection.

6.2 A rigorous disproof of Conjecture 6.2

In this subsection, we will prove that Conjecture 6.2 fails for some certain translation of isotrivial semiabelian varieties. We fix the data $p, K, (E, O)$ as in Example 3.6(i) through this subsection. Namely, we let our prime $p = 5$, let the base field K be $\overline{\mathbb{F}_p}(t)$ and let (E, O) be that supersingular elliptic curve as in Example 3.6(i). Our main statement is as follows.

Proposition 6.5. *Let $G = \mathbb{G}_m^3 \times (\mathbb{G}_m^2 \times E^2)$. Let $g = ((t+\alpha, t-\alpha, t), (t+1, t, (t+1, \sqrt{(t+1)^3+1}), (t, \sqrt{t^3+1}))) \in G(K)$ in which $\alpha \in \mathbb{F}_{p^2}^\times$ is a generator of this cyclic group. Let $V_0 \subseteq \mathbb{G}_m^3$ be the closed subvariety given by the equation $x + y = 2z + 2\alpha^2$. Then there exists a closed subvariety $V \subseteq \mathbb{G}_m^2 \times E^2$ such that $\{n \in \mathbb{N} \mid n \cdot g \in (V_0 \times V)(K)\}$ is not a p -normal set in \mathbb{N} .*

Firstly, we need a lemma which is a direct consequence of [Der07, Corollary 9.4]. Although all numbers in the reference are assumed to be rational, one can verify that this condition is not used in the proof.

Lemma 6.6. *Let $d \in \mathbb{N}, q \in \mathbb{R}_{>1}$ and $c_1, c_2, e_0, e_1, \dots, e_d \in \mathbb{R}$. Then the set $\{(n_1, n_2) \in \mathbb{N}^2 \mid \exists m_1, m_2, \dots, m_d \in \mathbb{N} \text{ s.t. } c_1 q^{n_1} + c_2 q^{n_2} = e_0 + \sum_{i=1}^d e_i q^{m_i}\}$ is a finite union of sets of the 5 forms below:*

- (i) *a singleton $\{(n_1, n_2)\}$ for some $n_1, n_2 \in \mathbb{N}$.*
- (ii) *$\{(n + n_1, n_2) \mid n \in \mathbb{N}\}$ for some $n_1, n_2 \in \mathbb{N}$.*
- (iii) *$\{(n_1, n + n_2) \mid n \in \mathbb{N}\}$ for some $n_1, n_2 \in \mathbb{N}$.*
- (iv) *$\{(n + n_1, n + n_2) \mid n \in \mathbb{N}\}$ for some $n_1, n_2 \in \mathbb{N}$.*
- (v) *$\{(n_1 + n_{10}, n_2 + n_{20}) \mid n_1, n_2 \in \mathbb{N}\}$ for some $n_{10}, n_{20} \in \mathbb{N}$.*

Next, we introduce a notation which will be used in the proof of Proposition 6.5. We denote $p_0 = p^2 = 25$.

Definition 6.7. *Let $q \in \{p_0^n \mid n \in \mathbb{Z}_+\}$, $q_1, q_2 \in \{p_0^n \mid n \in \mathbb{N}\}$ and $c_0 \in \mathbb{N}, c_1 \in \mathbb{Z}_+$. We denote $A(q; q_1, q_2)$ as the set $\{q_1 q^{n_1} + q_2 q^{n_2} \mid n_1, n_2 \in \mathbb{N}\}$ and denote $B(q; c_0, c_1)$ as the set $\{c_0 + c_1 q^n \mid n \in \mathbb{N}\}$. We will obey the convention that all of these coefficients must lie in their “domain of definition” (i.e. $q \in \{p_0^n \mid n \in \mathbb{Z}_+\}$, $q_1, q_2 \in \{p_0^n \mid n \in \mathbb{N}\}$ and $c_0 \in \mathbb{N}, c_1 \in \mathbb{Z}_+$) when we use this notation.*

Now we can prove Proposition 6.5.

Proof of Proposition 6.5. Assume by contradiction that $\{n \in \mathbb{N} \mid n \cdot g \in (V_0 \times V)(K)\}$ is a p -normal set in \mathbb{N} for every closed subvariety $V \subseteq \mathbb{G}_m^2 \times E^2$. Notice that $\{n \in \mathbb{N} \mid n \cdot (t+\alpha, t-\alpha, t) \in V_0(K)\} = \{p_0^{n_1} + p_0^{n_2} \mid n_1, n_2 \in \mathbb{N}\}$. Firstly, we prove that up to a finite set, $\{n \in \mathbb{N} \mid n \cdot g \in (V_0 \times V)(K)\}$ is a union of finite sets of the form $A(q; q_1, q_2)$ along with finite sets of the form $B(q; c_0, c_1)$ in which $V \subseteq \mathbb{G}_m^2 \times E^2$ is a closed subvariety and the sets $A(q; q_1, q_2), B(q; c_0, c_1)$ are as in Definition 6.7.

For a closed subvariety $V \subseteq \mathbb{G}_m^2 \times E^2$, we denote $S(V)$ as the set $\{n \in \mathbb{N} \mid n \cdot g \in (V_0 \times V)(K)\}$. So by assumption, all of the sets $S(V)$ are p -normal sets in \mathbb{N} . But since $S(V) \subseteq A(p_0; 1, 1)$, we firstly conclude that up to a finite set, $S(V)$ is a finite union of sets of the form $S_{q,d,0}(\frac{c_0}{q-1}; \frac{c_i}{q-1}) \cap \mathbb{N}$ as in Definition 1.1. Now by considering the “least common power” q_0 of p_0 and all the powers of p (i.e. the number q) involved here and split $S_{q,d,0}(\frac{c_0}{q-1}; \frac{c_i}{q-1})$ into a finite union of sets of the form

$S_{q_0,d,0}(\frac{c'_0}{q_0-1}; \frac{c'_i}{q_0-1})$, we may conclude that there is an element $q_0 \in \{p_0^n \mid n \in \mathbb{Z}_+\}$ such that up to a finite set, $S(V)$ is a finite union of sets of the form $S_{q_0,d,0}(\frac{c_0}{q_0-1}; \frac{c_i}{q_0-1}) \cap \mathbb{N}$. Now using the fact $S(V) \subseteq A(p_0; 1, 1)$ once more, we know that up to a finite set, $S(V)$ is a finite union of sets of the form $S_{q_0,d,0}(\frac{c_0}{q_0-1}; \frac{c_i}{q_0-1}) \cap A(p_0; 1, 1)$. But we have $A(p_0; 1, 1) = \bigcup_{a=0}^{M-1} \bigcup_{b=0}^{M-1} A(q_0; p_0^a, p_0^b)$ if $q_0 = p_0^M$ in which $M \in \mathbb{Z}_+$. So by Lemma 6.6 we may conclude that up to a finite set, $S(V)$ is a union of finite sets of the form $A(q; q_1, q_2)$ along with finite sets of the form $B(q; c_0, c_1)$ for every closed subvariety $V \subseteq \mathbb{G}_m^2 \times E^2$.

Now let C_1, C_2, C_3 be locally closed subvarieties of $\mathbb{G}_m^2 \times E^2$ such that for any closed point $u \in (\mathbb{G}_m^2 \times E^2)(K)$, we have

- (i) $u \in C_1(K)$ if and only if $u = (x+1, x, O, O)$ for some $x \in K \setminus \{0, -1\}$,
- (ii) $u \in C_2(K)$ if and only if $u = (1, 1, (y+1, \pm\sqrt{(y+1)^3+1}), (y, \pm\sqrt{y^3+1}))$ for some $y \in K$,
and
- (iii) $u \in C_3(K)$ if and only if $u = (z+1, z, (z+1, \pm\sqrt{(z+1)^3+1}), (z, \pm\sqrt{z^3+1}))$ for some $z \in K \setminus \{0, -1\}$.

In fact C_1 is a closed subvariety of $\mathbb{G}_m^2 \times E^2$ but C_2, C_3 are just locally closed subvarieties. Here we use “ \pm ” to remind that there are two square roots in K while in the definition of $g \in G(K)$, we just arbitrarily fix one. So for example $\pm\sqrt{y^3+1}$ should be comprehended as “an element in K whose square is y^3+1 ”.

Let X be the image of the composition map $C_1 \times C_2 \times C_3 \hookrightarrow (\mathbb{G}_m^2 \times E^2)^3 \xrightarrow{m} \mathbb{G}_m^2 \times E^2$ where m is the addition map $(u, v, w) \mapsto u + v + w$. Then X is a constructible set in $\mathbb{G}_m^2 \times E^2$ satisfying that for any closed point $u \in (\mathbb{G}_m^2 \times E^2)(K)$, we have $u \in X$ if and only if there exist $x, z \in K \setminus \{0, -1\}$ and $y \in K$ such that $u = ((x+1)(z+1), xz, (y+1, \pm\sqrt{(y+1)^3+1}) + (z+1, \pm\sqrt{(z+1)^3+1}), (y, \pm\sqrt{y^3+1}) + (z, \pm\sqrt{z^3+1}))$.

Write $X = \bigcup_{i=1}^N (V_i \setminus W_i)$ in which $V_1, \dots, V_N, W_1, \dots, W_N \subseteq \mathbb{G}_m^2 \times E^2$ are closed subvarieties satisfying $W_i \subseteq V_i$ for every $1 \leq i \leq N$. So $\bigcup_{i=1}^N (S(V_i) \setminus S(W_i)) = \{n \in \mathbb{N} \mid n \cdot (t+\alpha, t-\alpha, t) \in V_0(K)\} \cap \{n \in \mathbb{N} \mid n \cdot g_0 \in X\}$ in which $g_0 = (t+1, t, (t+1, \sqrt{(t+1)^3+1}), (t, \sqrt{t^3+1})) \in (\mathbb{G}_m^2 \times E^2)(K)$. As a result, we can write $\bigcup_{i=1}^N (S(V_i) \setminus S(W_i)) = A(p_0; 1, 1) \cap \{n \in \mathbb{N} \mid \exists x, z \in K \setminus \{0, -1\}, \exists y \in K, s.t. n \cdot g_0 = ((x+1)(z+1), xz, (y+1, \pm\sqrt{(y+1)^3+1}) + (z+1, \pm\sqrt{(z+1)^3+1}), (y, \pm\sqrt{y^3+1}) + (z, \pm\sqrt{z^3+1}))\}$. Then we know $\{p_0^n + p_0^{2n} \mid n \in \mathbb{N}\} \subseteq \bigcup_{i=1}^N (S(V_i) \setminus S(W_i))$ by the calculation in Example 6.4. We shall prove that there exists $q_0 \in \{p_0^n \mid n \in \mathbb{Z}_+\}$ and $q_{10}, q_{20} \in \{p_0^n \mid n \in \mathbb{N}\}$ such that $\{(q_{10} + q_{20})q_0^n \mid n \in \mathbb{N}\} \subseteq \bigcup_{i=1}^N (S(V_i) \setminus S(W_i))$.

Firstly, fix an element $i_0 \in \{1, \dots, N\}$ such that $S(V_{i_0}) \setminus S(W_{i_0})$ contains infinitely many elements of the form $p_0^n + p_0^{2n}$. Recall that up to a finite set, $S(V)$ is a union of finite sets of the form $A(q; q_1, q_2)$ along with finite sets of the form $B(q; c_0, c_1)$ for every closed subvariety $V \subseteq \mathbb{G}_m^2 \times E^2$. So by using the trick of considering the least common power of all the q involved here, we may fix an element $q_0 \in \{p_0^n \mid n \in \mathbb{Z}_+\}$ and find $E, F \subseteq \mathbb{N}$ such that

- (i) up to a finite set, $S(V_{i_0})$ is equal to E and $S(W_{i_0})$ is equal to F , and
- (ii) both E and F are a union of finite sets of the form $A(q_0; q_1, q_2)$ along with finite sets of the form $B(q_0; c_0, c_1)$.

So in particular the sets $S(V_{i_0}) \setminus S(W_{i_0})$ and $E \setminus F$ are equal up to a finite set and hence $E \setminus F$ contains infinitely many elements of the form $p_0^n + p_0^{2n}$. But a set of the form $B(q_0; c_0, c_1)$ can only contain finitely many elements of the form $p_0^n + p_0^{2n}$. So we conclude that there exists a set $A(q_0; q'_{10}, q'_{20}) \subseteq E$ such that $A(q_0; q'_{10}, q'_{20}) \setminus F$ contains infinitely many elements of the form $p_0^n + p_0^{2n}$.

Now using Lemma 6.6, we can see that this condition forces $A(q_0; q'_{10}, q'_{20}) \cap F$ to be a union of a finite set with finitely many sets of the form $B(q_0; c_0, c_1)$. So for a sufficiently large positive integer C , we have $\{(q'_{10}q_0^C + q'_{20})q_0^n \mid n \in \mathbb{N}\} \subseteq A(q_0; q'_{10}, q'_{20})$ and $\{(q'_{10}q_0^C + q'_{20})q_0^n \mid n \in \mathbb{N}\} \cap F$ is a finite set. So up to a finite set, $\{(q'_{10}q_0^C + q'_{20})q_0^n \mid n \in \mathbb{N}\}$ is contained in $E \setminus F$ and hence it is also contained in $S(V_{i_0}) \setminus S(W_{i_0})$ up to a finite set. Therefore, there exists a positive integer N_0 such that $\{(q'_{10}q_0^{C+N_0} + q'_{20}q_0^{N_0})q_0^n \mid n \in \mathbb{N}\} \subseteq S(V_{i_0}) \setminus S(W_{i_0})$ and thus we have proved that there exists $q_0 \in \{p_0^n \mid n \in \mathbb{Z}_+\}$ and $q_{10}, q_{20} \in \{p_0^n \mid n \in \mathbb{N}\}$ such that $\{(q_{10} + q_{20})q_0^n \mid n \in \mathbb{N}\} \subseteq \bigcup_{i=1}^N (S(V_i) \setminus S(W_i))$.

Now we have $\{(q_{10} + q_{20})q_0^n \mid n \in \mathbb{N}\} \subseteq \{n \in \mathbb{N} \mid \exists x, z \in K \setminus \{0, -1\}, \exists y \in K, s.t. n \cdot g_0 = ((x+1)(z+1), xz, (y+1, \pm\sqrt{(y+1)^3+1}) + (z+1, \pm\sqrt{(z+1)^3+1}), (y, \pm\sqrt{y^3+1}) + (z, \pm\sqrt{z^3+1}))\}$. This means that for every $n \in \mathbb{N}$, the set of equations

$$\left\{ \begin{array}{l} (t+1)^{(q_{10}+q_{20})q_0^n} = (x+1)(z+1) \\ t^{(q_{10}+q_{20})q_0^n} = xz \\ (q_{10} + q_{20})q_0^n \cdot (t+1, \sqrt{(t+1)^3+1}) = (y+1, \pm\sqrt{(y+1)^3+1}) + (z+1, \pm\sqrt{(z+1)^3+1}) \\ (q_{10} + q_{20})q_0^n \cdot (t, \sqrt{t^3+1}) = (y, \pm\sqrt{y^3+1}) + (z, \pm\sqrt{z^3+1}) \end{array} \right.$$

have a solution $(x_n, y_n, z_n) \in K^3$. But since both $q_{10}q_0^n$ and $q_{20}q_0^n$ are powers of p , the first two equations forces $\{x_n, z_n\} = \{t^{q_{10}q_0^n}, t^{q_{20}q_0^n}\}$ and hence we may assume that there exist infinitely many $n \in \mathbb{N}$ such that $x_n = t^{q_{10}q_0^n}$ and $z_n = t^{q_{20}q_0^n}$ without loss of generality.

Now for every such $n \in \mathbb{N}$, the last two equations can be read as

$$\left\{ \begin{array}{l} ((q_{10} + q_{20})q_0^n \pm \sqrt{q_{20}q_0^n}) \cdot (t+1, \sqrt{(t+1)^3+1}) = (y_n+1, \pm\sqrt{(y_n+1)^3+1}) \\ ((q_{10} + q_{20})q_0^n \pm \sqrt{q_{20}q_0^n}) \cdot (t, \sqrt{t^3+1}) = (y_n, \pm\sqrt{y_n^3+1}) \end{array} \right.$$

for some suitable choice of the sign \pm since we have $\text{Frob}_p^2 = [-p]$ on E (notice that $\sqrt{q_{20}q_0^n}$ is a positive integer because q_0, q_{20} are powers of $p_0 = p^2$). But by the explicit formula in [Sil86, (III, Ex. 3.7)], one can see that the two signs \pm before $\sqrt{q_{20}q_0^n}$ in the two equations must be same. So without loss of generality we assume that there exist infinitely many $n \in \mathbb{N}$ such that

$$\begin{cases} ((q_{10} + q_{20})q_0^n + \sqrt{q_{20}q_0^n}) \cdot (t + 1, \sqrt{(t + 1)^3 + 1}) = (y_n + 1, \pm\sqrt{(y_n + 1)^3 + 1}) \\ ((q_{10} + q_{20})q_0^n + \sqrt{q_{20}q_0^n}) \cdot (t, \sqrt{t^3 + 1}) = (y_n, \pm\sqrt{y_n^3 + 1}) \end{cases}$$

Now let $S \subseteq \mathbb{N}$ be the return set in Example 3.6(i). By Example 3.6, we know that

- (i) S contains infinitely many positive integers of the form $(q_{10} + q_{20})q_0^n + \sqrt{q_{20}q_0^n}$,
- (ii) $S \subseteq \{0\} \cup \{p^k m \mid k \in \mathbb{N}, m \in \mathbb{Z}_+, m \equiv \pm 1 \pmod{2p}\}$, and
- (iii) S is a union of finitely many arithmetic progressions in \mathbb{N} along with finitely many sets of the form $S_{q,1,0}(\frac{c_0}{q-1}; \frac{c_1}{q-1}) \cap \mathbb{N}$ where q is a power of p and c_0, c_1 are integers satisfying $q - 1 \mid c_0 + c_1$.

But since $p = 5 > 3$, condition (ii) implies that for any arithmetic progression $\{a + dn \mid n \in \mathbb{N}\} \subseteq S$ in which $a \in \mathbb{N}$ and $d \in \mathbb{Z}_+$, we have $v_p(d) > v_p(a + dn)$ for any $n \in \mathbb{N}$ where $v_p(x)$ is the number such that $p^{v_p(x)} \parallel x$. So every arithmetic progression contained in S can only contain finitely many numbers of the form $(q_{10} + q_{20})q_0^n + \sqrt{q_{20}q_0^n}$. But every set of the form $S_{q,1,0}(\frac{c_0}{q-1}; \frac{c_1}{q-1})$ also can only contain finitely many numbers of the form $(q_{10} + q_{20})q_0^n + \sqrt{q_{20}q_0^n}$. So we get a contradiction.

All in all, we have proved that there exists a closed subvariety $V \subseteq \mathbb{G}_m^2 \times E^2$ such that $\{n \in \mathbb{N} \mid n \cdot g \in (V_0 \times V)(K)\}$ is not a p -normal set in \mathbb{N} . \square

One can see that Proposition 6.5 disproves Conjecture 6.2. In other words, one cannot require all of the r to be 0 in Conjecture 1.3. Moreover, Example 6.3 suggests that one cannot require all of the r to be bounded by an absolute constant in Conjecture 1.3 and we believe that one may prove this statement rigorously by an argument similar to our argument here.

7 Appendix: an arithmetic version of the p -Mordell–Lang problem

The careful reader may have noticed that there are “ p ML, geometric version” (Theorem 2.1), “ p DML, arithmetic version” (Conjecture 1.3) and “ p DML, geometric version” (Conjecture 5.1) above but we have not mentioned “ p ML, arithmetic version” yet. The arithmetic version of the p -Mordell–Lang problem should describe the form of the intersection of a closed subvariety with a finitely generated subgroup in a semiabelian variety of positive characteristic. But it turns out that this intersection set can be quite wild (see [GY24, Section 2]). So one cannot expect the

“ p ML, arithmetic version” to have a succinct statement. Some works to this end have been done in [GY24, Theorem 1.6, Theorem 1.10], but it turns out that there is an error in that work. We fix that error and generalize the results in [GY24] to arbitrary algebraic groups here. Our main result is as follows.

Theorem 7.1. (*p ML, arithmetic version*) *Let G be an algebraic group over an algebraically closed field K of characteristic $p > 0$. Let $X \subseteq G$ be a closed subvariety and let $\Gamma \subseteq G(K)$ be a finitely generated commutative subgroup. Then $X(K) \cap \Gamma$ is a finite union of sets of the form*

$$x_0 + (\pi|_{\Gamma_0})^{-1}(S)$$

where

- $x_0 \in \Gamma$,
- $G_0 \subseteq G$ is an algebraic subgroup which is a semiabelian variety over K , and $\Gamma_0 = G_0(K) \cap \Gamma$,
- H_0 is a semiabelian variety over a finite subfield $\mathbb{F}_q \subseteq K$, and F_0 is the absolute Frobenius endomorphism of H_0 corresponding to \mathbb{F}_q ,

• $H = H_0 \times_{\mathbb{F}_q} K$, and $F = F_0 \times_{\mathbb{F}_q} K$ is the Frobenius endomorphism of H ,

• $\pi : G_0 \rightarrow H$ is a surjective algebraic group homomorphism, and

• S is a subset of $\pi(\Gamma_0)$ of the form $\{\alpha_0 + \sum_{i=1}^d \sum_{j=0}^r F^{2^j n_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\}$ where $d \in \mathbb{Z}_+, r \in \mathbb{N}$

and $\alpha_0, \alpha_1, \dots, \alpha_d \in H(K)$.

Note that since Γ is assumed to be commutative, we use “+” for the multiply operation of elements in Γ .

We shall just briefly sketch the proof of Theorem 7.1 since the details are in fact essentially same as the corresponding arguments in [GY24] and Section 3 above. Firstly, one can prove Theorem 7.1 in the case of semiabelian varieties in a way just the same as the proof of [GY24, Theorem 1.6, Theorem 1.10]. The only thing to change is that one should use Remark 2.7(ii) above instead of [Ghi08, Theorem 3.1] in the proof. This explains where the sets $\{\alpha_0 + \sum_{i=1}^d \sum_{j=0}^r F^{2^j n_i}(\alpha_i) \mid n_1, \dots, n_d \in \mathbb{N}\}$ in Theorem 7.1 come from. In order to prove the statement for general algebraic groups, one may inherit the strategy in Section 3. The key point is that one can reduce the general case to the case of semiabelian varieties by using [Bri17, Theorem 5.6.3(i)] as in the proof of Lemma 3.5 and Theorem 3.1.

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