SPACE SPANNED BY CHARACTERISTIC EXPONENTS

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Abstract. We prove several rigidity results on multiplier and length spectrum. For example, we show that for every non-exceptional rational map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree $d \geq 2$, the $\mathbb{Q}$-vector space generated by the characteristic exponents (that are not $-\infty$) of periodic points of f has infinite dimension. This answers a stronger version of a question of Levy and Tucker. Our result can also be seen as a generalization of recent results of Ji-Xie and of Huguin which proved Milnor's conjecture about rational maps having integer multipliers. We also get a characterization of postcritically finite maps by using its length spectrum. Finally as an application of our result, we get a new proof of the Zariski dense orbit conjecture for endomorphisms on $(\mathbb{P}^1)^N, N \geq 1$.

Contents

1. Introduction

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map over $\mathbb{C}$ of degree $d \geq 2$. Our aim is to study the $\mathbb{Q}$-vector space spanned by the characteristic exponents of periodic points of a rational map on $\mathbb{P}^1(\mathbb{C})$ and prove some rigidity results.

1.1. Multiplier, length and characteristic exponent.

Let $z_0 \in \mathbb{P}^1(\mathbb{C})$ be a periodic point of f with exact period n. Define $n_f(z_0) := n$ be this period. We write $n(z_0)$ for simplicity when the map f is clear. The *multiplier* $\rho_f(z_0)$ of f at z_0 is defined to be the differential $df^n(z_0) \in \mathbb{C}$. We write $\rho(z_0)$ for simplicity when the map f is clear. The *length* of f at z_0 is the norm $|\rho_f(z_0)|$. The multiplier and the length are invariant under conjugacy. The *characteristic exponent* of f at z_0 is defined to be $\chi_f(z_0) := n^{-1}\log|\rho_f(z_0)|$.

Denote by $\mathrm{Per}(f)(\mathbb{C})$ the set of all periodic points in $\mathbb{P}^1(\mathbb{C})$ of f and define $\mathrm{Per}^*(f)(\mathbb{C}) := \{z_0 \in \mathrm{Per}(f)(\mathbb{C}) : \rho_f(z_0) \neq 0\}$. When the base field $\mathbb{C}$ is clear, we also write $\mathrm{Per}(f)$ and $\mathrm{Per}^*(f)$ for simplicity.

The *Lyapunov exponent* (of the maximal entropy measure) of f is defined by

$$\mathcal{L}_f := \int_{\mathbb{P}^1(\mathbb{C})} \log|df| d\mu_f,$$

where μ_f is the unique maximal entropy measure, and the norm of the differential is computed with respect to the spherical metric.

1.2. **Exceptional maps.** In complex dynamics, the exceptional maps defined below are often considered as exceptional examples among all rational maps. We may view them as rational maps on $\mathbb{P}^1(\mathbb{C})$ related to algebraic groups.

Definition 1.1. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism over $\mathbb{C}$ of degree $d \geq 2$.

- It is called *Lattès* if it is semi-conjugate to an endomorphism on an elliptic curve. Further it is called *flexible Lattès* if it is semi-conjugate to the multiplication by an integer n on an elliptic curve for some $|n| \geq 2$. Otherwise, it is called *rigid Lattès*.
- We say that f is of *monomial type* if it semi-conjugate to the map $z \mapsto z^n$ on $\mathbb{P}^1$ for some integer n with $|n| \geq 2$.
- We call f *exceptional* if it is Lattès or of monomial type. An endomorphism f is exceptional if and only if some iterate f^k is exceptional ($k \in \mathbb{Z}_{>0}$).

1.3. **Statement of the main results.** We fix an embedding of the algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$ in $\mathbb{C}$ and identify $\overline{\mathbb{Q}}$ as a subfield of $\mathbb{C}$, hence any number field is a subfield of $\mathbb{C}$. Denote the usual absolute value on $\mathbb{C}$ by $|\cdot|$.

Our first result shows that the definition field of a non-flexible Lattès rational map is determined by its length spectrum.

Theorem 1.2. *Let $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be a rational map of degree at least 2. Assume that f is not a flexible Lattès map and for every $x \in \mathrm{Per}(f)(\mathbb{C})$, $|\rho_f(x)| \in \overline{\mathbb{Q}}$. Then f is defined over $\overline{\mathbb{Q}}$.*

In Theorem 2.1, we indeed proved a more general version of Theorem 1.2, in which $\overline{\mathbb{Q}}$ can be replaced to any algebraically closed subfield of $\mathbb{C}$ which is invariant under the complex conjugation.

McMullen's rigidity of multiplier spectrum [McM87] with a standard spread out argument implies that, for a rational map f of degree at least 2 which is not flexible Lattès, if its multipliers at periodic points are all algebraic, then f is defined over $\overline{\mathbb{Q}}$. Theorem 1.2 is a generalization of this result from multiplier spectrum to length spectrum (which contains less information). The rigidity of length spectrum was proved in [JX23b, Theorem 1.5]. However, the spread out argument does not apply directly in this case as the length spectrum map (and its square) is not algebraic on the moduli space of rational maps. Indeed as shown in [JX23b, Section 8.1], its square is not even real algebraic. In Section 2.3, we introduce a way to do the spread out argument respecting the real structure using Weil restriction. Another difficulty in the length spectrum case is the lack of noetherianity for semi-algebraic subsets. We overcome this difficulty using the notion of admissible subsets introduced in [JX23b].

The following two results concern the $\mathbb{Q}$-vector space spanned by the characteristic exponents of periodic points.

Theorem 1.3. *Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$. Suppose that f is not exceptional. Then the $\mathbb{Q}$-vector space generated by $\{\chi_f(z) : z \in \mathrm{Per}^*(f)\}$ in $\mathbb{R}$ has infinite dimension.*

Theorem 1.4. *Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$. Assume that there exists a number field K such that*

$$\forall z_0 \in \mathrm{Per}(f),\ \exists n = n(z_0) \in \mathbb{Z}_{>0},\ |\rho_f(z_0)|^n \in K. \tag{1.1}$$

Then f is exceptional.

Finitely many nonzero elements $z_1, \dots, z_N$ in a commutative ring R are called multiplicatively independent if for all triples $(m_1, \dots, m_N)$ of integers, $z_1^{m_1} \cdots z_N^{m_N} = 1$ if and only if $m_1 = \cdots = m_N = 0$. A sequence $(z_n)_{n=1}^{\infty}$ in $R \setminus \{0\}$ is called multiplicatively independent if any its finite subsequence is multiplicatively independent. Theorem 1.3 immediately implies the existence of infinitely many multipliers for a non-exceptional f whose absolute values are multiplicatively independent.

Corollary 1.5. *Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$. Suppose that f is not exceptional. Then there exists a sequence $(x_j)_{j=1}^{\infty}$ in $\mathrm{Per}^*(f)$ such that the sequence $(|\rho_f(x_j)|)_{j=1}^{\infty}$ is multiplicatively independent in $\mathbb{R}$.*

1.4. **Motivations and previous results.**

Milnor's conjecture. Milnor [Mil06] has showed that an exceptional rational map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree $d \geq 2$ must have all its multipliers of periodic points in the ring of integers $\mathcal{O}_K$ for some imaginary quadratic number field K, and in fact in $\mathbb{Z}$ when f is not a rigid Lattès map. Milnor conjectured that the converse is also true. Milnor's conjecture was recently proved by Ji and Xie:

Theorem 1.6 ([JX23b, Theorem 1.13])**.** *Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$. Assume that there exists an imaginary quadratic field K such that all multipliers of f belong to $\mathcal{O}_K$. Then f is exceptional.*

See also [BGHR22] for a different proof. Recently Huguin generalized the above result using different approach:

Theorem 1.7 ([Hug23, Theorem 7])**.** *Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$. Assume that there exists a number field K such that all multipliers of f belong to K. Then f is exceptional.*

Since our assumption (1.1) in Theorem 1.4 is weaker than that of Theorem 1.7, Theorem 1.4 is a generalization of Theorem 1.7. Indeed our assumption (1.1) is even weaker than the condition that there is a number field K such that

$$\forall z_0 \in \mathrm{Per}(f),\ \exists n \in \mathbb{Z}_{>0},\ (\rho_f(z_0))^n \in K. \tag{1.2}$$

A question of Levy and Tucker. On the other hand, in the 2014 AIM workshop Postcritically Finite Maps In Complex And Arithmetic Dynamics, Levy [Lev14] and Tucker [Tuc14] asked the following question independently:

Question 1.8. *Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a non-exceptional rational map of degree $d \geq 2$ and let S be the set of all multipliers of periodic points of f. Take the subgroup of $\mathbb{C}^*$ generated by $S \setminus \{0\}$. Is that true that the rank of this group is infinite?*

It is not hard to see that our Corollary 1.5 gives a positive answer to (a generalized version of) Levy and Tucker's question.

1.5. **Sketch of the proofs.** We have explained the proof of Theorem 1.2 before. Here we explain the proofs of Theorem 1.3 and Theorem 1.4.

We first give the idea of the proof of Theorem 1.4. We argue by contradiction and suppose that f is not exceptional. The first step is to reduce to the case where f is defined over $\overline{\mathbb{Q}}$. This can be done using our Theorem 1.2. After enlarging K, we may assume that f is defined over K. In the second step, we combine the arithmetic equidistribution theorem with a result of Zdunik [Zdu14] on the Lyapunov exponent to get a contradiction. This argument is inspired by Huguin's proof of Theorem 1.7. Not like the case of Theorem 1.7, we can not apply the equidistribution theorem to the one dimensional dynamical system $f : \mathbb{P}^1 \to \mathbb{P}^1$ directly. Our idea is to consider the two dimensional endomorphism $F := f \times \overline{f}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ instead. More precisely, applying a result of Zdunik [Zdu14], we get a sequence $(x_n)_{n=1}^{\infty}$ of distinct periodic points such that

$$\lim_{n\to+\infty} \chi_f(x_n) = a > \mathcal{L}_f.$$

Consider the endomorphism $F := f \times \overline{f}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\Gamma := \overline{\{p_n = (z_n, \overline{z_n})\}}^{\mathrm{Zar}} \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. By [GTZ11], the Dynamical Manin-Mumford conjecture holds for F. Hence we may assume that Γ is F-invariant.

Let ν_n be the discrete probability measure equally supported at the union of Galois orbits of iterates of p_n under F. Then ν_n converges weakly to the canonical measure μ on Γ with respect to F by an equidistribution-type theorem (Theorem 3.1), which is a reformulation of [Yua08, Theorem 3.1], see Section 3 for details. Applying $\nu_n \to \mu$ to the continuous test functions $\max\{\log|\det(dF)|, A\}$ $(A \in \mathbb{R})$ and letting $A \to -\infty$, we get

$$2a \leq \int \log|\det(dF)| d\mu,$$

which is impossible since the right hand side equals to $2\mathcal{L}_f < 2a$ by a direct computation.

Next we sketch the proof of Theorem 1.3. According to [DH93], postcritically finite (PCF) maps are defined over $\overline{\mathbb{Q}}$ in the moduli space $\mathcal{M}_d$ of rational maps of degree d, except for the family of flexible Lattès maps. So it suffices to consider the following two cases: 1). f is defined over $\overline{\mathbb{Q}}$, and 2). f is not PCF. For the first case the conclusion follows from Theorem 1.4. For the second case, we need to develop some new techniques, which are presented in Section 5. In Section 5, we consider some pseudo linear algebra (which means that the domain may not be the whole vector space), and the vector space $\mathbb{D}(k)_{\mathbb{Q}} = k^* \otimes_{\mathbb{Z}} \mathbb{Q}$ for a field k of characteristic zero. We will actually prove a theorem (Theorem 5.6) stronger than the non-PCF case of Theorem 1.3, see Section 5 and 6 for details. To prove Theorem 5.6, in Section 6.1 we first deal with the case that f is defined over $\overline{\mathbb{Q}}$. A key ingredient in this step is [BGKT12, Lemma 4.1] which is a consequence of Siegel's theorem on S-integral points. The existence of a no preperiodic critical point is essentially used in here. In Section 6.2, we consider the general case and finish the proof. This is achieved by reducing to the case that f is defined over $\overline{\mathbb{Q}}$ via an algebraic-geometric argument and techniques in Section 5.

1.6. **Applications.**

The Zariski-dense orbit Conjecture. By applying Corollary 1.5 we can give a new proof of a special case of the Zariski-dense orbit conjecture.

Zariski-dense orbit Conjecture (=ZDO)**.** Let k be an algebraically closed field of characteristic 0. Given an irreducible quasiprojective variety X over k and a dominant rational self-map f on X. If we have $\{g \in k(X) : g \circ f = g\} = k$ where $k(X)$ is the function field of X, then there exists $x \in X(k)$ whose forward orbit under f is well-defined and Zariski-dense in X.

Remark 1.9. The converse of ZDO is easy. For some progressions of ZDO, see e.g.[ABR11], [AC08], [MS14], [Xie17] and [Xie22].

As an application of Corollary 1.5, we give a new proof of (the most difficult part of) a special case of ZDO, which was firstly proved in [Xie22, Theorem 1.16].

Theorem 1.10. *Let $X = \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ be the variety of product of N copies of projective line over an algebraically closed field k of characteristic 0. Suppose that $f : X \to X$ is an endomorphism of form $f_1 \times \cdots \times f_N$ where $f_j : \mathbb{P}^1 \to \mathbb{P}^1$ is a non-constant rational map for $1 \leq j \leq N$. The ZDO holds for X and f.*

Remark 1.11. We note that every dominant endomorphism $f : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N$ over an algebraically closed field k of characteristic zero must be of form $f_1 \times \cdots \times f_N$, after replacing f by a suitable positive-integer iterate.

The original proof of Theorem 1.10 in [Xie22] relies on the solution of the (adelic) Zariski dense orbit conjecture on smooth projective surfaces [Xie22, Theorem 1.15], the notion of adelic topology introduced in [Xie22, Section 3] and a classification result on invariant subvarieties of $f : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N$ [Xie22, Proposition 9.2] (see also [MS14] and [GNY18]). When $n = 2$, Pakovich gave another proof [Pak23] using his classification of invariant curves in $\mathbb{P}^1 \times \mathbb{P}^1$ and some height argument. In our new proof, we don't need the ingredients mentioned above.

A characterization of PCF maps. We also show that one can decide whether a rational map $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ of degree $d \geq 2$ is PCF with the information of its multiplier spectrum or length spectrum on periodic points.

Theorem 1.12. *Let $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map of degree $d \geq 2$. Then the followings are equivalent:*

(1) f is PCF;

(2) $\rho_f(x) \in \overline{\mathbb{Q}}$ for all $x \in \mathrm{Per}(f)(\mathbb{C})$ and the $\mathbb{Q}$-subspace $V = V(f)$ of $\mathbb{R}$ is of finite dimension, where V is generated over $\mathbb{Q}$ by $\{\log|N_{K_x/\mathbb{Q}}(\rho_f(x))| : x \in \mathrm{Per}^*(f)(\mathbb{C})\}$;

(3) $|\rho_f(x)| \in \overline{\mathbb{Q}}$ for all $x \in \mathrm{Per}(f)(\mathbb{C})$ and the $\mathbb{Q}$-subspace $W = W(f)$ of $\mathbb{R}$ is of finite dimension, where W is generated over $\mathbb{Q}$ by $\{\log|N_{L_x/\mathbb{Q}}(|\rho_f(x)|)| : x \in \mathrm{Per}^*(f)(\mathbb{C})\}$.

Here K_x (resp. L_x) is any number field containing $\rho_f(x)$ (resp. $|\rho_f(x)|$) and $N_{K_x/\mathbb{Q}}$ (resp. $N_{L_x/\mathbb{Q}}$) is the norm map for the extension $K_x/\mathbb{Q}$ (resp. $L_x/\mathbb{Q}$), i.e. the determinant of the $\mathbb{Q}$-linear transformation induced by multiplication by $\rho_f(x)$ (resp. $|\rho_f(x)|$).

Clearly, the subspaces V, W above is independent of the choices of the fields K_x, L_x, respectively.

The proofs of Theorem 1.10 and Theorem 1.12 will be given in Section 6.

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2. Rational maps with algebraic lengths

Let K be an algebraically closed subfield of $\mathbb{C}$ which is invariant under the complex conjugate τ i.e. $\tau(K) = K$. The aim of this section is the following result.

Theorem 2.1. *Let $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Assume that f is not a flexible Lattès map and for every $x \in \mathrm{Per}(f)(\mathbb{C})$, $|\rho_f(x)| \in K$. Then f is defined over K.*

Applying Theorem 2.1 to the case $K = \overline{\mathbb{Q}}$, we get Theorem 1.2.

2.1. Weil restriction.

Recall that K is an algebraically closed field of $\mathbb{C}$ such that $\tau(K) = K$. Set $L := K^{\tau} = K \cap \mathbb{R}$. For example, if $K = \mathbb{C}$, then $L = \mathbb{R}$. We need the following easy lemma.

Lemma 2.2. *We have $K = L + iL$, in particular $[K : L] = 2$.*

Proof of Lemma 2.2. Since K is algebraically closed, $i \in K$. In particular, $K \neq L$. For every $u \in K$, we may write

$$u = \frac{u + \tau(u)}{2} + \frac{u - \tau(u)}{2i} i$$

and both $\frac{u+\tau(u)}{2}$ and $\frac{u-\tau(u)}{2i}$ are contained in L. This concludes the proof. □

We briefly recall the notion of Weil restriction. See [Poo17, Section 4.6] and [BLR90, Section 7.6] for more information.

Denote by $Var_{/K}$ (resp. $Var_{/L}$) the category of varieties over K (resp. L). For every variety X over K, there is a unique variety $R(X)$ over L represents the functor $Var_{/L} \to Sets$ sending $V \in Var_{/L}$ to $\mathrm{Hom}(V \otimes_L K, X)$. It is called the *Weil restriction of* X. The functor $X \mapsto R(X)$ is called the Weil restriction. One has the canonical morphism $\psi_K : X(K) \to R(X)(L)$. When $K = \mathbb{C}$, this map is a real analytic diffeomorphism. One may view $X(K)$ as an L-algebraic variety via ψ_K.

Definition 2.3. The *L-Zariski topology* on $X(K)$ is the restriction of the Zariski topology on $R(X)$ via ψ_K. A subset Y of $X(K)$ is *L-algebraic* if it is closed in the L-Zariski topology. When $K = \mathbb{C}$, the L-Zariski topology is exactly the real Zariski topology as in [JX23b, Section 8.1.1].

By (iii) of Proposition 2.5 below, the L-Zariski topology is stronger than the Zariski topology on $X(K)$.

When $K = \mathbb{C}$, roughly speaking, the Weil restriction is just constructed by splitting a complex variable z into two real variables x, y via $z = x + iy$. For the convenience of the reader, in the following example, we show the concrete construction of $R(X)$ when X is affine.

Example 2.4. First assume that $X = \mathbb{A}^N_K$. Then $R(X) = \mathbb{A}^{2N}_L$. The map

$$\psi_K : \mathbb{A}^N_L(L) = K^N \to \mathbb{A}^{2N}_L(L) = \mathbb{R}^{2N}$$

sends $(z_1, \ldots, z_N)$ to $(x_1, y_1, x_2, y_2, \ldots, x_N, y_N)$ where $z_j = x_j + iy_j$.

Consider the algebra $\mathbb{B} := K[I]/(I^2 + 1) \simeq K \oplus IK$. Every $f \in K[z_1, \ldots, z_N]$ defines an element

$$F := f(x_1 + Iy_1, \ldots, x_N + Iy_N) \in \mathbb{B}[x_1, y_1, \ldots, x_N, y_N].$$

Since

$$\mathbb{B}[x_1, y_1, \ldots, x_N, y_N] = K[x_1, y_1, \ldots, x_N, y_N] \oplus IK[x_1, y_1, \ldots, x_N, y_N],$$

F can be uniquely decomposed to

$$F = r(f) + Ii(f)$$

where $r(f), i(f) \in K[x_1, y_1, \ldots, x_N, y_N]$.

More generally, if X is the closed subvariety of $\mathbb{A}^N_K = \mathrm{Spec} K[z_1, \ldots, z_M]$ defined by the ideal $(f_1, \ldots, f_s)$, then $R(X)$ is the closed subvariety of

$$R(\mathbb{A}^N_K) = \mathbb{A}^{2N}_L = \mathrm{Spec}\, L[x_1, y_1, \ldots, x_N, y_N]$$

defined by the ideal generated by $r(f_1), i(f_1), \ldots, r(f_s), i(f_s)$.

We list some basic properties of Weil restriction without proof.

Propsition 2.5. *Let $X, Y \in Var_{/K}$, then we have the following properties:*

- *if X is irreducible, then $R(X)$ is irreducible;*

- $\dim R(X) = 2\dim X$;
- *if $f: Y \to X$ is a closed (resp. open) immersion, then the induced morphism $R(f): R(Y) \to R(X)$ is a closed (resp. open) immersion.*

We still denote by τ the restriction of τ to K. Denote by X^τ the base change of X by the field extension $\tau : K \to K$. This induces a morphism of schemes (over $\mathbb{Z}$) $\tau : X^\tau \to X$. It is not a morphism of schemes over K. It is clear that $(X^\tau)^\tau = X$.

Example 2.6. If X is the subvariety of $\mathbb{A}^N_K = \mathrm{Spec} K[z_1, \dots, z_N]$ defined by the equations $\sum_I a_{i,I} z^I = 0, i = 1, \dots, s$ Then X^τ is the subvariety of $\mathbb{A}^N_K$ defined by $\sum_I \tau(a_{i,I}) z^I = 0, i = 1, \dots, s$. The map $\tau : X = (X^\tau)^\tau \to X^\tau$ sends a point $(z_1, \dots, z_N) \in X(K)$ to $(\tau(z_1), \dots, \tau(z_N)) \in X^\tau(K)$.

The following result due to Weil is useful for computing the Weil restriction.

Propsition 2.7. [Poo17, Exercise 4.7] *We have a canonical isomorphism*

$$R(X) \otimes_L K \simeq X \times X^\tau.$$

Under this isomorphism,

$$R(X)(L) = \{(z_1, z_2) \in X(K) \times X^\tau(K) |\ z_2 = \tau(z_1)\}$$

and ψ_K sends $z \in X(K)$ to $(z, \tau(z)) \in R(X)(L)$.

2.2. **Admissible subsets.** In this section,we recall the notion of admissible subsets on real algebraic varieties introduced in [JX23b].

Let X be a variety over $\mathbb{R}$.

Definition 2.8. [JX23b, Section 8.2] A closed subset V of $X(\mathbb{R})$ is called *admissible* if there is a morphism $f : Y \to X$ of real algebraic varieties and a Zariski closed subset $V' \subseteq Y$ such that $V = f(V'(\mathbb{R}))$ and f is étale at every point in $V'(\mathbb{R})$.

In particular, every algebraic subset of $X(\mathbb{R})$ is admissible.

Remark 2.9. Denote by J the non-étale locus for f in V. We have $J \cap V(\mathbb{R}) = \emptyset$. Since we may replace V by $V \setminus J$, in the above definition we may further assume that f is étale.

Propsition 2.10. [JX23b, Remarks 8.14, 8.15 and Proposition 8.16] *We have the following basic properties:*

1. *Let Y be a Zariski closed subset of X. If V is admissible as a subset of $X(\mathbb{R})$, then $V \cap Y$ is admissible as a subset of $Y(\mathbb{R})$.*
2. *An admissible subset is semialgebraic.*
3. *Let V_1, V_2 be two admissible closed subsets of $X(\mathbb{R})$. Then $V_1 \cap V_2$ is admissible.*

The following theorem shows that admissible subsets satisfy the descending chain condition.

Theorem 2.11. [JX23b, Theorem 8.17] *Let $V_n, n \geq 0$ be a sequence of decreasing admissible subsets of $X(\mathbb{R})$. Then there is $N \geq 0$ such that $V_n = V_N$ for all $n \geq N$.*

2.3. **Transcendental points.** Let X_K be a variety over K and $X := X_K \otimes_K \mathbb{C}$. We think that X_K as a model of X over K.

Denote by $\pi_K : X \to X_K$ the natural projection. For any point $x \in X(\mathbb{C})$, define $Z(x)_K$ to be the Zariski closure of $\pi_K(x)$ and $Z(x) := \pi_K^{-1}(Z(x)_K)$. It is clear that $Z(x)$ is irreducible. We call $Z(x)$ the $\mathbb{C}/K$-closure of x w.r.t the model X_K. We say that x is *transcendental* if $\dim Z(x) \geq 1$ and call $\dim Z(x)$ the *transcendental degree* of x.

The notion of transcendental points (on curves) was introduced in [XY23, Section 4.1] and it plays important role in [XY23] on the geometric Bombieri-Lang conjecture and [JX23a] on the dynamical André-Oort conjecture . Roughly speaking, a very general point in $Z(x)$ satisfies the same algebraic properties as x. In this paper, we study lengths of periodic points in whose definition we need the norm map $|\cdot| : \mathbb{C} \to \mathbb{R}_{\geq 0}$ which is not algebraic. However $|\cdot|^2 : \mathbb{C} \to \mathbb{R}$ is real algebraic. For this reason we need to generalize the above notions to respect the real structure.

The Weil restriction $R(X)$ of X w.r.t. $\mathbb{C}/\mathbb{R}$ is a real algebraic variety. We have $R(X) = R(X_K) \otimes_L \mathbb{R}$. Denote by $\pi_L : R(X) \to R(X_K)$ the natural projection. For every $x \in X(\mathbb{C})$, let $Y(x)_L$ be the Zariski closure of $\pi_L(\psi_{\mathbb{C}}(x))$ and $Y(x) := \pi_L^{-1}(Y(x)_L)$. Set $Z^{\mathbb{R}}(x) := \psi_{\mathbb{C}}^{-1}(Y(x)(\mathbb{R}))$ which is a real Zariski closed subset of $X(\mathbb{C})$.

We now give a more concrete description of $Z(x)$ and $Z^{\mathbb{R}}(x)$. Let U_K be an affine open neighborhood of $\pi_K(x)$. Set $U := \pi_K^{-1}(U_K) = U_K \otimes_K \mathbb{C}$. We have a natural embedding $\pi_K^* : \mathcal{O}(U_K) \hookrightarrow \mathcal{O}(U)$. We can view elements in $\mathcal{O}^K(U) := \pi_K^*(\mathcal{O}(U_K))$ as the algebraic functions on $U(\mathbb{C})$ defined over K. Then we have

$$Z(x) \cap U = \{y \in U |\ h(y) = 0 \text{ for every } h \in \mathcal{O}^K(U) \text{ with } h(x) = 0\}$$

and $Z(x)$ is the Zariski closure of $Z(x) \cap U$.

As $\mathcal{O}(R(U)_{\mathbb{C}}) = \mathcal{O}(R(U)) \otimes_{\mathbb{R}} \mathbb{C}$, every $h \in \mathcal{O}(R(U)_{\mathbb{C}})$ can be viewed as a $\mathbb{C}$-valued algebraic function on $R(U)(\mathbb{R})$. Every $h \in \mathcal{O}(R(U)_{\mathbb{C}})$ induces a function $h \circ \psi_{\mathbb{C}}$ on $U(\mathbb{C})$. The functions of this form are exactly the $\mathbb{C}$-valued real algebraic functions on $U(\mathbb{C})$. Denote by $\mathcal{C}^{\mathbb{R}-\mathrm{alg}}(U)$ the $\mathbb{R}$-algebra of $\mathbb{C}$-valued real algebraic functions on $U(\mathbb{C})$. Since algebraic functions are real algebraic, we have a natural embedding $\mathcal{O}(U) \subseteq \mathcal{C}^{\mathbb{R}-\mathrm{alg}}(U)$. By Proposition 2.7, we have

$$\mathcal{C}^{\mathbb{R}-\mathrm{alg}}(U) \simeq \mathcal{O}(U) \otimes_{\mathbb{C}} \tau(\mathcal{O}(U)).$$

Let $\mathcal{O}^L(R(U)) := \pi_L^*(\mathcal{O}(R(U_K)))$ be the set of algebraic functions defined over L on $R(U)$. Let $\mathcal{C}^{\mathbb{R}-\mathrm{alg},L}(U)$ the image of $\mathcal{O}^L(R(U)) \otimes_L K$ in $\mathcal{C}^{\mathbb{R}-\mathrm{alg}}(U)$, which is the set of $\mathbb{C}$-valued real algebraic functions on $U(\mathbb{C})$ defined over L. It is clear that $\mathcal{O}^K(U) \subseteq \mathcal{C}^{\mathbb{R}-\mathrm{alg},L}(U)$. By Proposition 2.7, we have

$$\mathcal{C}^{\mathbb{R}-\mathrm{alg},L}(U) \simeq \mathcal{O}^K(U) \otimes_K \tau(\mathcal{O}^K(U)).$$

We have

$$Z^{\mathbb{R}}(x) \cap U(\mathbb{C}) = \{y \in U(\mathbb{C}) |\ h(y) = 0 \text{ for every } h \in \mathcal{C}^{\mathbb{R}-\mathrm{alg},L}(U) \text{ with } h(x) = 0\}$$

and $Z^{\mathbb{R}}(x)$ is the real Zariski closure of $Z^{\mathbb{R}}(x)\cap U(\mathbb{C})$. This implies the following lemma.

Lemma 2.12. *Let $f_K : X'_K \to X_K$ be a morphisms between K-varieties. Set $X' := X'_K \otimes_K \mathbb{C}$ and let $f : X' \to X$ be the morphism induced by f. Let $x' \in X'(\mathbb{C})$ and $x \in X(\mathbb{C})$ with $f(x') = x$. Then we have $f(Z^{\mathbb{R}}(x')) \subseteq Z^{\mathbb{R}}(x)$.*

Lemma 2.13. *We have $Z^{\mathbb{R}}(x) \subseteq Z(x)$ and $Z^{\mathbb{R}}(x)$ is Zariski dense in $Z(x)$. In particular, if x is transcendental, then $\dim_{\mathbb{R}} Z^{\mathbb{R}}(x) > 1$.*

Proof. It is clear that $Z^{\mathbb{R}}(x) \subseteq Z(x)$. After replacing X_K by an affine open neighborhood of $\pi_K(x)$. We may assume that X_K, X are affine. Let $h \in \mathcal{O}(X)$ such that $h(Z^{\mathbb{R}}(x)) = 0$. Let $e_j, j \in J$ be a K-basis of $\mathbb{C}$. We may assume that $0 \in J$ and $1 = e_0$. Write $h \otimes_{\mathbb{C}} 1 = \sum_{j\in J} g_j e_j$. Then $g_j \in \mathcal{C}^{\mathbb{R}-\mathrm{alg},L}(X)$ and $g_j(x) = 0$.

Let $f_n, n \in N$ be a K-basis of $\mathcal{O}^K(X)$. We may assume that $0 \in N$ and $1 = f_0$. Write

$$g_j = \sum_{m,n\in N} b_{j,m,n} f_m \otimes \tau(f_n).$$

The we get

$$h \otimes_{\mathbb{C}} 1 = \sum_{j\in J,m,n\in N} b_{j,m,n} e_j f_m \otimes \tau(f_n).$$

As $e_j f_m \otimes \tau(f_n), j \in J, m, n \in N$ forms a K-basis of $\mathcal{C}^{\mathbb{R}-\mathrm{alg}}(X)$, we have

$$b_{j,m,n} = 0$$

for every $n \neq 0$. So $g_j = \sum_{m\in N} b_{j,m,0} f_m \otimes 1 \in \mathcal{O}^K(X)$. Since $g_j(x) = 0$, $g_j|_{Z(x)} = 0$. Then we have $h|_{Z(x)} = 0$ which concludes the proof. □

Lemma 2.14. *Assume that X_K is affine. Let $h \in \mathcal{C}^{\mathbb{R}-\mathrm{alg},L}(X)$. For $x \in X(\mathbb{C})$, if $h(x) \in K$, then h is constant on $Z^{\mathbb{R}}(x)$.*

Proof. Write $h = g \circ \psi_{\mathbb{C}}$ where $g \in \mathcal{O}^L(R(X)) \otimes_L K$. Write $g = \pi_L^*(g_1) + \pi_L^*(g_2)i$ where $g_1, g_2 \in \mathcal{O}(R(X_K))$. Since $h(x) \in K$, $\pi_L^*(g_1)(\psi(x)), \pi_L^*(g_2)(\psi(x)) \in L$. The map $\pi_L|_{\psi(x)} : \psi(x) \to Y(x)_L$ induces an embedding $\mathcal{O}(Y(x)_L) \hookrightarrow L$. The image of $g_i|_{Y(x)_L}, i = 1, 2$ are contained in L. Hence $g_i|_{Y(x)_L}, i = 1, 2$ are contained in L. This implies that $\pi_L^*(g_1), \pi_L^*(g_2)$ are constant on $Y(x)(L)$, hence h is constant on $Z^{\mathbb{R}}(x)$. This concludes the proof. □

2.4. **Moduli space of rational maps.** For $d \geq 2$, let Rat_d be the space of degree d endomorphisms on $\mathbb{P}^1$. It is a smooth quasi-projective variety of dimension $2d+1$ [Sil12]. Let $FL_d \subseteq \mathrm{Rat}_d$ be the locus of flexible Lattès maps, which is Zariski closed in Rat_d. The group $\mathrm{PGL}_2 = \mathrm{Aut}(\mathbb{P}^1)$ acts on Rat_d by conjugacy. The geometric quotient

$$\mathcal{M}_d := \mathrm{Rat}_d/\mathrm{PGL}_2$$

is the (coarse) *moduli space* of endomorphisms of degree d [Sil12]. The moduli space $\mathcal{M}_d = \mathrm{Spec}(\mathcal{O}(\mathrm{Rat}_d)^{\mathrm{PGL}_2})$ is an affine variety of dimension $2d-2$ [Sil07, Theorem 4.36(c)]. Let $\Psi : \mathrm{Rat}_d \to \mathcal{M}_d$ be the quotient morphism. Set

$[FL_d] := \Psi(FL_d)$. The above construction works over any algebraically closed field of characteristic 0 and commutes with base changes.

For every $n \in \mathbb{Z}_{>0}$, let $\mathrm{Per}_n(f_{\mathrm{Rat}_d})$ be the closed subvariety of $\mathrm{Rat}_d \times \mathbb{P}^1$ of the n-periodic points of f_{Rat_d}. Let $\phi_n : \mathrm{Per}_n(f_{\mathrm{Rat}_d}) \to \mathrm{Rat}_d$ be the first projection. It is a finite map of degree $d^n + 1$. Let $\lambda_n : \mathrm{Per}_n(f_{\mathrm{Rat}_d}) \to \mathbb{A}^1$ be the morphism $(f_t, x) \mapsto df_t^n(x) \in \mathbb{A}^1$. View $\mathrm{Per}_n(f_{\mathrm{Rat}_d})$ as the moduli space of endomorphisms of degree d with a marked n-periodic point. We also denote it by $\mathrm{Rat}_d[n]$ or $\mathrm{Rat}_d^1[n]$.

Let $s_1, \dots, s_n$ be a sequence of elements in $\mathbb{Z}_{\geq 0}$ with $s_1 \leq \cdots \leq s_n$ and $s_i \leq d^{i!}+1$. We construct the space $R_d(s_1, \dots, s_n)$ of rational functions of degree d with s_n marked $n!$-periodic points (counting with multiplicities) and in which there are s_{n-1} $(n-1)!$-periodic points (counting with multiplicities) ...and in which there are s_1 1-periodic points (counting with multiplicities) as follows: Consider the fiber product $(\mathrm{Rat}_d[n!])^{s_n}_{/\mathrm{Rat}_d}$ of s_n copies of $\mathrm{Rat}_d[n!]$ over Rat_d. For $i \neq j \in \{1, \dots, d^{n!}+1\}$, let $\pi_{i,j} : (\mathrm{Rat}_d[n!])^{s_n}_{/\mathrm{Rat}_d} \to (\mathrm{Rat}_d[n!])^2_{/\mathrm{Rat}_d}$ be the projection to the i, j coordinates. The diagonal $\Delta \subseteq (\mathrm{Rat}_d[n!])^2_{/\mathrm{Rat}_d}$ is an irreducible component of $(\mathrm{Rat}_d[n!])^2_{/\mathrm{Rat}_d}$. Consider the open subset

$$U := (\mathrm{Rat}_d[n!])^{s_n}_{/\mathrm{Rat}_d} \setminus (\cup_{i \neq j \in \{1,\dots,d^{n!}+1\}} \pi_{i,j}^{-1}(\Delta)).$$

Let U' be the subset of U of points $(f, x_1, \dots, x_{s_n})$ satisfying $f^{m!}(x_i) = x_i$ for every $m = 1, \dots, n$ and $i = 1, \dots, s_m$. This set is open and closed in U. We then define $R_d(s_1, \dots, s_n)$ to be the Zariski closure of U' in $(\mathrm{Rat}_d[n!])^{s_n}_{/\mathrm{Rat}_d}$. For $m \leq n$, define $\phi_{n,m} : R_d(s_1, \dots, s_n) \to R_d(s_1, \dots, s_m)$ the morphism $(f, x_1, \dots, x_{s_n}) \mapsto (f, x_1, \dots, x_{s_m})$. Moreover, denote by $\phi_{n,0} : R_d(s_1, \dots, s_n) \to \mathrm{Rat}_d$ the morphism $(f, x_1, \dots, x_{s_n}) \mapsto f$. For $m_1 \leq m_2 \leq n$, we have $\phi_{m_2,m_1} \circ \phi_{n,m_2} = \phi_{n,m_1}$. Let $\lambda_{s_1,\dots,s_n} : R_d(s_1, \dots, s_n) \to \mathbb{A}^{s_n}$ the morphism defined by

$$(f, x_1, \dots, x_{s_n}) \mapsto (df^{n!}(x_1), \dots, df^{n!}(x_{s_n})).$$

Since ϕ_n is étale at every point $x \in \mathrm{Per}_n(f_{\mathrm{Rat}_d}) \setminus \lambda^{-1}_{s_1,\dots,s_n}(1)$, $\phi_{n,0}$ is étale at every point $x \in (\lambda_{s_1,\dots,s_n})^{-1}((\mathbb{A}^1 \setminus \{1\})^{s_n})$.

Define $\mathcal{M}_d(s_1, \dots, s_n) := R_d(s_1, \dots, s_n)/\mathrm{PGL}_2$ to be the moduli space of endomorphisms of degree d on $\mathbb{P}^1$ with s_n marked $n!$-periodic points (counting with multiplicities) and in which there are s_{n-1} $(n-1)!$-periodic points (counting with multiplicities) ...and in which there are s_1 1-periodic points (counting with multiplicities). The morphisms $\phi_{n,m}$, $\lambda_{s_1,\dots,s_n}$ descent to $[\phi_{n,m}] : \mathcal{M}_d(s_1, \dots, s_n) \to \mathcal{M}_d(s_1, \dots, s_m)$ when $m = 1, \dots, n$, $[\phi_{n,0}] : \mathcal{M}_d(s_1, \dots, s_n) \to \mathcal{M}_d$ and $[\lambda_{s_1,\dots,s_n}] : \mathcal{M}_d(s_1, \dots, s_n) \to \mathbb{A}^{s_n}$. Then $[\phi_{n,0}]$ is étale at every point $x \in [\lambda_{s_1,\dots,s_m}]^{-1}((\mathbb{A}^1 \setminus \{1\})^{s_n})$.

2.5. **Length maps.** For $d \geq 2$, let $s_1, \dots, s_n$ be a sequence of elements in $\mathbb{Z}_{\geq 0}$ with $s_1 \leq \cdots \leq s_n$ and $s_i \leq d^{i!} + 1$. Let

$$|\lambda_{s_1,\dots,s_n}| : \mathcal{M}_d(s_1, \dots, s_n)(\mathbb{C}) \to \mathbb{R}^{s_n}_{\geq 0}$$

be the composition of

$$[\lambda_{s_1,\dots,s_n}] : \mathcal{M}_d(s_1, \dots, s_n)(\mathbb{C}) \to \mathbb{C}^{s_n}$$

and the norm map

$$(a_1, \ldots, a_{s_n}) \in \mathbb{C}^{s_n} \mapsto (|a_1|, \ldots, |a_{s_n}|) \in \mathbb{R}^{s_n}_{\geq 0}.$$

Define

$$q_{s_1,\ldots,s_n} : \mathcal{M}_d(s_1, \ldots, s_n)(\mathbb{C}) \to \mathbb{R}^{s_n}_{\geq 0}$$

be the composition of

$$|\lambda_{s_1,\ldots,s_n}| : \mathcal{M}_d(s_1, \ldots, s_n)(\mathbb{C}) \to \mathbb{R}^{s_n}_{\geq 0}$$

and the map

$$(a_1, \ldots, a_{s_n}) \in \mathbb{R}^{s_n}_{\geq 0} \mapsto (a_1^2, \ldots, a_{s_n}^2) \in \mathbb{R}^{s_n}_{\geq 0}.$$

It is clear that

$$q_{s_1,\ldots,s_n} \in \mathcal{C}^{\mathbb{R}-\mathrm{alg},L}(\mathcal{M}_d(s_1, \ldots, s_n)(\mathbb{C})).$$

Here the model of $\mathcal{M}_d(s_1, \ldots, s_n)_{\mathbb{C}}$ over K is taken to be $\mathcal{M}_d(s_1, \ldots, s_n)_K$.

By Lemma 2.14, for every $x \in \mathcal{M}_d(s_1, \ldots, s_n)(\mathbb{C})$, if $q_{s_1,\ldots,s_n}(x) \in L^{s_n}$, then $q_{s_1,\ldots,s_n}|_{V^{\mathbb{R}}(x)}$ is constant. Hence for every $x \in \mathcal{M}_d(s_1, \ldots, s_n)(\mathbb{C})$, if $|\lambda_{s_1,\ldots,s_n}|(x) \in L^{s_n}$, then $|\lambda_{s_1,\ldots,s_n}||_{V^{\mathbb{R}}(x)}$ is constant.

2.6. **Rigidity of length spectrum.** In this section, we recall the rigidity of length spectrum proved by Ji and Xie [JX23b].

Let f be an endomorphism of $\mathbb{P}^1(\mathbb{C})$ of degree $d \geq 2$. As in [JX23b, Section 8.3] the *length spectrum* $L(f) = \{L(f)_n, n \geq 1\}$ of f is a sequence of finite multisets[1], where $L(f)_n := L_n(f)$ is the multiset of norms of multipliers of all fixed points of f^n. In particular, $L(f)$ is a multiset of non-negative real numbers of cardinality $d^n + 1$. For every $n \geq 0$, let $RL(f)_n$ be the sub-multiset of $L(f)_n$ consisting of all elements > 1. We call $RL(f) := \{RL(f)_n, n \geq 1\}$ the *repelling length spectrum* of f and $RL^*(f) := \{RL^*(f)_n := RL(f)_{n!}, n \geq 1\}$ the *main repelling length spectrum* of f. We have $d^n + 1 \geq \#RL(f)_n \geq d^n + 1 - M$ for some $M \geq 0$. It is clear that the difference $d^{n!} + 1 - \#RL^*(f)_n$ is increasing and bounded. As $L(f), RL(f)$ and $RL^*(f)$ are invariant under conjugacy, they descent on $\mathcal{M}_d(\mathbb{C})$. For every $[f] \in \mathcal{M}_d(\mathbb{C})$, define $L([f]) := L(f), RL([f]) := RL(f)$ and $RL^*([f]) := RL^*(f)$ for any f in the class $[f]$.

Let Ω be the set of sequences $A_n, n \geq 1$ of multisets consisting of real numbers of norm strictly larger than 1 satisfying $\#A_n \leq d^{n!} + 1$ and for every $a \in A_n$ with multiplicity m, $a^{n+1} \in A_{n+1}$ with multiplicity at least m. For $A, B \in \Omega$, we write $A \subseteq B$ if $A_n \subseteq B_n$ for every $n \geq 1$. An element $A = (A_n) \in \Omega$ is called *big* if $d^{n!} + 1 - \#A_n$ is bounded. For every endomorphism f of $\mathbb{P}^1(\mathbb{C})$ of degree d, we have $RL^*(f) \in \Omega$ and it is big.

Theorem 2.15. [JX23b, Theorem 8.25] *If $A \in \Omega$ is big, then the set*

$$\{f \in \mathcal{M}_d(\mathbb{C}) \setminus [FL_d]|\ A \subseteq RL^*(f)\}$$

is finite.

[1]A multiset is a set except allowing multiple instances for each of its elements. The number of the instances of an element is called the multiplicity. For example: $\{a, a, b, c, c, c\}$ is a multiset of cardinality 6, the multiplicities for a, b, c are 2,1,3, respectively.

2.7. **Proof of Theorem 2.1.** Let $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ be a rational map of degree $d \geq 2$. Assume that f is not a flexible Lattès map and for every $x \in \mathrm{Per}(f)(\mathbb{C})$, $|\rho_f(x)| \in K$. We want to show that $[f] \in \mathcal{M}_d(\mathbb{C})$ is not transcendental over K for the model $(\mathcal{M}_d)_K$. Now assume that $[f]$ is transcendental.

Set $A := RL^*(f) \in \Omega$, which is big. Set $s_n := \# A_n$. We may pick a sequence of periodic points $x_i, i \geq 1$ such that for every $n \geq 1$, $x_1, \ldots, x_{s_n}$ are fixed by $f^{n!}$ and $A_n = \{|\rho(x_i)|^{n!}, i = 1, \ldots, s_n\}$. Let $[f_n] \in \mathcal{M}(s_1, \ldots, s_n)(\mathbb{C})$ be the point presented by $(f, x_1, \ldots, x_{s_n})$. It is clear that $[\phi_{n,0}]([f_n]) = [f]$ for every $n \geq 1$. Since $[f]$ is transcendental, for every $n \geq 1$, $[f_n]$ is transcendental. By Lemma 2.13, $\dim_{\mathbb{R}} Z^{\mathbb{R}}(f_n) \geq 1$ for every $n \geq 1$. Our assumption implies that $|\lambda_{s_1,\ldots,s_n}|([f_n]) \in L^{s_n}$. The last paragraph of Section 2.5 shows that $|\lambda_{s_1,\ldots,s_n}|$ is constant on $Z^{\mathbb{R}}(f_n)$. As $|\lambda_{s_1,\ldots,s_n}|([f_n]) \in (1, +\infty)^{s_n}$, $[\phi_{n,0}]$ is étale in a neighborhood of $Z^{\mathbb{R}}(f_n)$. Since $[\phi_{n,0}]$ is a finite map, $V_n := [\phi_{n,0}](Z^{\mathbb{R}}(f_n))$ is closed in $\mathcal{M}_d(\mathbb{C})$. Then V_n is an admissible subset of $\mathcal{M}_d(\mathbb{C})$. Moreover, by Lemma 2.12, $V_n, n \geq 1$ is decreasing. By Theorem 2.11, there is $N \geq 1$ such that $V_n = V_N$ for $n \geq N$. Then for every $g \in V_N$, we have $A \subseteq RL^*([g])$. Since $[f] \notin [FL_d]$, $Z^{\mathbb{R}}([f_N])$ is real irreducible and $\dim_{\mathbb{R}} Z^{\mathbb{R}}([f_N]) \geq 1$, $V_N \cap (\mathcal{M}_d(\mathbb{C}) \setminus [FL_d])$ is infinite. This contradicts to Theorem 2.15. This concludes the proof. $\square$

3. An equidistribution theorem

The following equidistribution-type theorem is a reformulation of [Yua08, Theorem 3.1]. We only state it in the case where the canonical height of X is 0, since this case often appear in the dynamical settings. Our statement is slightly stronger than [Yua08, Theorem 3.1] as our S_n may contain several Galois orbits. We follow the terminology in [Yua08].

Theorem 3.1. *Let K be a number field and X be a projective variety over K. Fix an embedding of K into $\mathbb{C}$. Let $\overline{\mathcal{L}}$ be a metrized line bundle on X such that $\mathcal{L}$ is ample and the metric is semipositive. Let $\mu := \deg_{\mathcal{L}}(X)^{-1} c_1(\overline{\mathcal{L}})_{\mathbb{C}}^{\dim X}$ be the canonical probability measure on $X(\mathbb{C})$ associated to $\overline{\mathcal{L}}$. For $n \in \mathbb{Z}_{>0}$, let S_n be a countable subset of $X(\overline{K})$ which is $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$-invariant. For $y \in S_n$, given real numbers $a_{n,y} \geq 0$ such that $\sum_{y \in S_n} a_{n,y} = 1$ and $a_{n,y} = a_{n,\sigma y}$ for all $y \in S_n$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$. Assume that $(S_n)_{n=1}^{\infty}$ satisfies the following two conditions:*

(1) (small) $\sum_{y \in S_n} a_{n,y} h_{\overline{\mathcal{L}}}(y) \to 0$ as $n \to +\infty$, here $h_{\overline{\mathcal{L}}}$ is the height function associated with $\overline{\mathcal{L}}$;

(2) (generic) for any proper subvariety $V \subsetneqq X$ of X, $\sum_{y \in S_n \cap V} a_{n,y} \to 0$ as $n \to +\infty$.

Then the measure $\mu_n := \sum_{y \in S_n} a_{n,y} \delta_y$ converges weakly to μ on $X(\mathbb{C})$ as $n \to +\infty$ where δ_y denotes the Dirac measure at the point y, i.e., for all continuous function g on $X(\mathbb{C})$, we have

$$\lim_{n \to \infty} \sum_{y \in S_n} a_{n,y} g(y) = \int_{X(\mathbb{C})} g d\mu. \tag{3.3}$$

Proof. Our proof is a small modification of the one for [Yua08, Theorem 3.1].

Let d be the dimension of X. We say that a continuous function f on $X(\mathbb{C})$ is smooth if there exists an embedding of $X(\mathbb{C})$ into a projection manifold Y such that f can be extended to a smooth function on Y. As in [Zha98], by the Stone-Weierstrass theorem, continuous functions on $X(\mathbb{C})$ can be approximated uniformly by smooth functions. Then we suffice to prove (3.3) for all smooth real-valued function f on $X(\mathbb{C})$. Fix such a function f. Let v_0 be the archimedean place of K corresponding to the fixed embedding $K \hookrightarrow \mathbb{C}$. For a real function g on $X(\mathbb{C})$ and a metrized line bundle $\overline{\mathcal{G}} = (\mathcal{G}, \|\cdot\|)$ on X, we define the twist $\overline{\mathcal{G}}(g) := (\mathcal{M}, \|\cdot\|')$ to be the line bundle $\mathcal{G}$ on X with the metric $\|s\|'_{v_0} = \|s\|_{v_0} e^{-g}$ and $\|s\|'_v = \|s\|_v$ for any $v \neq v_0$. Let $\epsilon > 0$. By the adelic Minkowski's theorem (cf. [BG06, Appendix C]) and [Yua08, Lemma 3.3], for a fixed place $\omega_0 \in \mathcal{M}_K$ and $N \in \mathbb{Z}_{>0}$, there exists a nonzero small section $s_N \in \Gamma(X, N\mathcal{L})$ such that

$$\log\|s_N\|'_{\omega_0} \leq -\frac{\hat{c}_1(\overline{\mathcal{L}}(\epsilon f))^{d+1} + O(\epsilon^2)}{(d+1)\deg_{\mathcal{L}}(X)} N + o(N) = (-h_{\overline{\mathcal{L}}(\epsilon f)}(X) + O(\epsilon^2))N + o(N)$$

and $\log\|s_N\|'_\omega \leq 0$ for all $\omega \neq \omega_0$, where $\|\cdot\|'_\omega$ denotes the metric of $N\overline{\mathcal{L}}(\epsilon f)$. For a point y, denote by $\overline{y}$ its Zariski closure. For $N, n \in \mathbb{Z}_{>0}$, denote the vanishing locus of s_N by $V_N \subsetneqq X$, using the condition of $\overline{\mathcal{L}}$, we have

$$\sum_{y \in S_n} a_{n,y} h_{\overline{\mathcal{L}}(\epsilon f)}(y) \geq \sum_{y \in S_n, \overline{y} \in V_N} a_{n,y} \deg(y)^{-1} \left(\sum_v \sum_{z \in O(y)} (-N^{-1} \log\|s_N(z)\|'_v) \right) + 0$$

$$\geq \left(\sum_{y \in S_n, \overline{y} \notin V_N} a_{n,y} \right) (h_{\overline{\mathcal{L}}(\epsilon f)}(X) + O(\epsilon^2) + o_N(1)).$$

Let $n \to +\infty$, the generic condition (2) implies that

$$\liminf_{n \to +\infty} \sum_{y \in S_n} a_{n,y} h_{\overline{\mathcal{L}}(\epsilon f)}(y) \geq h_{\overline{\mathcal{L}}(\epsilon f)}(X) + O(\epsilon^2) + o_N(1).$$

Let $N \to +\infty$, then

$$\liminf_{n \to +\infty} \sum_{y \in S_n} a_{n,y} h_{\overline{\mathcal{L}}(\epsilon f)}(y) \geq h_{\overline{\mathcal{L}}(\epsilon f)}(X) + O(\epsilon^2).$$

By the definition, it is easy to see that

$$\sum_{y \in S_n} a_{n,y} h_{\overline{\mathcal{L}}(\epsilon f)}(y) = \sum_{y \in S_n} a_{n,y} h_{\overline{\mathcal{L}}}(y) + \epsilon \int_{X(\mathbb{C})} f d\mu_n$$

and

$$h_{\overline{\mathcal{L}}(\epsilon f)}(X) = h_{\overline{\mathcal{L}}}(X) + \epsilon \frac{1}{\deg_{\mathcal{L}}(X)} \int_{X(\mathbb{C})} f c_1(\overline{\mathcal{L}})^d_{\mathbb{C}} + O(\epsilon^2).$$

With the small condition (1), dividing ϵ and setting $\epsilon \to 0^+$, we get

$$\liminf_{n \to +\infty} \int_{X(\mathbb{C})} f d\mu_n \geq \frac{1}{\deg_{\mathcal{L}}(X)} \int_{X(\mathbb{C})} f c_1(\overline{\mathcal{L}})^d_{\mathbb{C}} = \int_{X(\mathbb{C})} f d\mu.$$

Replacing f by $-f$ in the above inequality, we get the other direction and thus

$$\lim_{n\to+\infty}\int_{X(\mathbb{C})} f d\mu_n = \int_{X(\mathbb{C})} f d\mu.$$

$\square$

Remark 3.2. The same idea also applies for a non-archimedean place or the algebraic case, which gives the full analogy of [Yua08, Theorem 3.1 and 3.2].

In order to check the generic condition in Theorem 3.1, we need the following lemma. The proof uses the ergodic theory with respect to the constructible topology (on algebraic varieties) introduced by Xie in [Xie23].

Lemma 3.3. *Let K be a number field and X be a projective variety over K. Given a dominant endomorphism $f : X \to X$ and a sequence $(x_n)_{n=1}^{\infty}$ of periodic points in $X(\overline{K})$ under f. Assume that $(x_n)_{n=1}^{\infty}$ is generic in X, i.e., there does not exist a proper Zariski closed subset $Z \subsetneqq X$ containing all x_n except for finitely many. Then for every proper subvariety $V \subsetneqq X$, we have*

$$\frac{\#(V\cap O_f(x_n))}{\#O_f(x_n)} \to 0, \text{ as } n\to+\infty, \tag{3.4}$$

where $O_f(x_n)$ is the (forward) orbit of x_n under f.

Proof. Clearly, we suffice to show that for any subsequence $(n_k)_k$ of $(n)_{n=1}^{\infty}$, there exists a subsubsequence $(n_{k_l})_l$ such that

$$\frac{\#(V\cap O_f(x_{n_{k_l}}))}{\#O_f(x_{n_{k_l}})} \to 0, \text{ as } l\to+\infty.$$

Given a proper subvariety $V \subsetneqq X$ and fix V. Let $|X|$ be X equipped with the constructible topology (i.e. the topology of X generated by all its Zariski closed and open subsets) and $\mathcal{M}^1(|X|)$ be the space of all probability Radon measures on $|X|$ with the topology of weak convergence relative to all continuous functions on $|X|$. Then $\mathcal{M}^1(|X|)$ is sequentially compact (cf. [Xie23, Corollary 1.14]). For $n \in \mathbb{Z}_{>0}$, set

$$m_n = (\#O_f(x_n))^{-1}\sum_{z\in O_f(x_n)}\delta_z.$$

By the sequentially compactness of $\mathcal{M}^1(|X|)$, we suffice to show that for any subsequence $(n_k)_k$ of $(n)_{n=1}^{\infty}$ with $m_{n_k} \to m$ as $k\to+\infty$ in $\mathcal{M}^1(|X|)$ for some $m \in \mathcal{M}^1(|X|)$, we have

$$\frac{\#(V\cap O_f(x_{n_k}))}{\#O_f(x_{n_k})} \to 0, \text{ as } k\to+\infty.$$

Without loss of generality, we may assume that (m_n) itself converges to a measure $m \in \mathcal{M}^1(|X|)$; and we suffice to show (3.4) in this case. As $f_*m_n = m_n$, we see that $f_*m = m$. Then according to [Xie23, Lemma 5.3], m must be of form $m = \sum_{y\in S} a_y\delta_{O_f(y)}$, where S is a countable set of periodic elements in $|X|$ under f, $a_y \in \mathbb{R}_{\geq 0}$ with $\sum_{y\in S} a_y = 1$, and $\delta_{O_f(y)} = (\#O_f(y))^{-1}\sum_{z\in O_f(y)}\delta_z$ for $y \in S$.

Denote the characteristic function of $V \subsetneqq |X|$ by 1_V, then 1_V is continuous with respect to the constructible topology. As $m_n \to m$, we get

$$\frac{\#(V \cap O_f(x_n))}{\#O_f(x_n)} = \int 1_V dm_n \to \int 1_V dm, \text{ as } n \to +\infty.$$

Suppose that (3.4) fails. Then there must be a $y \in S$ with $a_y > 0$ and $V \cap O_f(y) \neq \emptyset$. Denote the exact period of y under f by k. Let Y be the Zariski closure of $\{y\}$. Then $Y \subseteq \cup_{j=0}^{k-1} f^{\circ j}(V)$, hence Y is also a proper Zariski closed subset of X. Note that

$$\frac{\#(Y \cap O_f(x_n))}{\#O_f(x_n)} = \int 1_Y dm_n \to \int 1_Y dm \geq \frac{a_y}{k} > 0, \text{ as } n \to +\infty.$$

Hence for every sufficiently large integer $n \gg 1$, we have $x_n \in \cup_{j=0}^{\infty} f^{\circ j}(Y) = \cup_{j=0}^{k-1} f^{\circ j}(Y)$; but $\cup_{j=0}^{k-1} f^{\circ j}(Y)$ has dimension strictly smaller than $\dim X$ by the noetherian condition, contradicting the assumption that $(x_n)_{n=1}^{\infty}$ is generic in X. $\square$

4. Proofs of Theorem 1.4 and the defined over $\overline{\mathbb{Q}}$ case of Theorem 1.3

Proof of Theorem 1.4. Assume that f is not exceptional. By Theorem 1.2, our assumption implies that f is defined over $\overline{\mathbb{Q}}$ (after a conjugate over $\mathbb{C}$), hence over a number field K. After replacing K by a finite extension of K, we may assume that both f and $\overline{f}$ are defined over K. Here we denote by $\overline{f}$ the rational map obtained from f via replacing the coefficients by their complex conjugates. According to [Hug23, Theorem 9 and Lemma 11] (cf. [Zdu14]), there exists a sequence $(x_n)_{n=1}^{\infty}$ of distinct points in $\mathrm{Per}^*(f)$ such that

$$a := \lim_{n\to\infty} \chi_f(x_n) > \mathcal{L}_f,$$

where the limit exists and is finite.

Clearly, $\mathcal{L}_f = \mathcal{L}_{\overline{f}}$. For an arbitrary $x \in \mathrm{Per}(f)$, we have $\overline{x} \in \mathrm{Per}(\overline{f})$, $n_f(x) = n_{\overline{f}}(\overline{x})$ and $\rho_f(x) = \overline{\rho_{\overline{f}}(\overline{x})}$, hence $\chi_f(x) = \chi_{\overline{f}}(\overline{x})$.

Consider the morphism $F := f \times \overline{f} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ over K. For $n \in \mathbb{Z}_{>0}$, set $p_n = (x_n, \overline{x_n}) \in \mathrm{Per}(F)$. Let Γ be the Zariski closure of $\{p_n : n \in \mathbb{Z}_{>0}\}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. As $(p_n)_{n=1}^{\infty}$ is pairwise distinct, by the noetherian condition, we have $\dim \Gamma \geq 1$. After taking a subsequence, we may assume that Γ is irreducible and that $(p_n)_{n=1}^{\infty}$ is generic in Γ.

There are 2 cases: $\dim \Gamma = 2$ or 1. When $\dim \Gamma = 2$, then $\Gamma = \mathbb{P}^1 \times \mathbb{P}^1$ and the canonical probability measure on Γ relative to F is $\mu := \mu_f \times \mu_{\overline{f}}$, where μ_f and $\mu_{\overline{f}}$ are the canonical measures on $\mathbb{P}^1$ relative to f and $\overline{f}$, respectively. When $\dim \Gamma = 1$, by the dynamical Manin-Mumford problem for F on $\mathbb{P}^1 \times \mathbb{P}^1$, proved in [GTZ11], Γ is periodic under F. After replacing f by f^m for some suitable $m \in \mathbb{Z}_{>0}$, we may assume that Γ is F-invariant. Still denote by μ the canonical probability measure on Γ relative to F. In all cases, let $\pi_j : \Gamma \to \mathbb{P}^1$ be the j-th projection on Γ for $j = 1, 2$. Then we have

$$\deg(\pi_1)\mu = \pi_1^* \mu_f, \ \deg(\pi_2)\mu = \pi_2^* \mu_{\overline{f}}.$$

For $n \in \mathbb{Z}_{>0}$, set

$$\nu_n = \frac{1}{n_f(x_n)[K_n : K]} \sum_{j=0}^{n_f(x_n)-1} \sum_{\tau \in \mathrm{Gal}(K_n/K)} \delta_{F^{\circ j}(\tau(p_n))},$$

where K_n is the Galois closure of $K(x_n)$ over K in $\overline{\mathbb{Q}}$ and $K(\infty) := K$.

Claim: ν_n converges weakly to μ as $n \to +\infty$.

We prove the claim using Theorem 3.1. Let $\mathcal{L}$ be the line bundle $\pi_1^*\mathcal{O}_{\mathbb{P}^1}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^1}(1)$ on Γ. Then $F|_\Gamma^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$. By [Zha95], there exists a unique semipositive metric over $\mathcal{L}$ making $F|_\Gamma^*\mathcal{L} \cong \mathcal{L}^{\otimes d}$ an isometry; denote $\mathcal{L}$ with this metric by $\overline{\mathcal{L}}$. We need to check the conditions (1) and (2) in Theorem 3.1. The condition (1) is trivial, since $h_{\overline{\mathcal{L}}}(\Gamma) = 0$ and the height of any periodic algebraic point relative to $\overline{\mathcal{L}}$ is zero. For the condition (2), let V be an arbitrary proper subvariety of Γ and fix V. By consider the finitely many images of V under Galois transformations, the generic condition (2) follows from Lemma 3.3. Thus the claim is true.

Let $n \in \mathbb{Z}_{>0}$, take $m \in \mathbb{Z}_{>0}$ such that $|\rho_f(x_n)|^m \in K$ by the assumption, and write $l = n_f(x_n)$. For every $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ and $0 \le j \le l-1$, we have

$$\begin{aligned} &\det(dF^{\circ l}(F^{\circ j}(\tau(p_n))))^m = \det(dF^{\circ l}(\tau(p_n)))^m = \tau(\det(dF^{\circ l}(p_n))^m) \\ =&\tau(\rho_f(x_n)^m \rho_{\overline{f}}(\overline{x_n})^m) = \tau(|\rho_f(x_n)|^{2m}) = |\rho_f(x_n)|^{2m}, \end{aligned}$$

hence $|\det(dF^l(F^{\circ j}(\tau(p_n))))| = |\rho_f(x_n)|^2$. Then by the definition of ν_n, we have

$$\begin{aligned} &\int \log|\det(dF)| d\nu_n = \frac{1}{l} \int \log|\det(dF^{\circ l})| d\nu_n \\ =&\frac{1}{l^2[K_n : K]} \sum_{j=0}^{l-1} \sum_{\tau \in \mathrm{Gal}(K_n/K)} \log|\det(dF^{\circ l}(F^{\circ j}(\tau(p_n))))| \\ =&\frac{1}{l^2[K_n : K]} \sum_{j=0}^{l-1} \sum_{\tau \in \mathrm{Gal}(K_n/K)} \log|\rho_f(x_n)|^2 \\ =&\frac{2}{l} \log|\rho_f(x_n)| = 2\chi_f(x_n). \end{aligned}$$

For any $A \in \mathbb{R}$, since the function $\max\{\log|\det(dF)|, A\}$ is continuous, we have

$$\begin{aligned} &2a = 2 \lim_{n\to\infty} \chi_f(x_n) = \lim_{n\to\infty} \int \log|\det(dF)| d\nu_n \\ \le& \lim_{n\to\infty} \int \max\{\log|\det(dF)|, A\} d\nu_n \\ =& \int \max\{\log|\det(dF)|, A\} d\mu. \end{aligned}$$

Let $A \to -\infty$, by the monotone convergence theorem, we have

$$2\mathcal{L}_f < 2a \le \int \log|\det(dF)| d\mu. \tag{4.5}$$

When $\dim \Gamma = 2$, it is clear that

$$\int \log|\det(dF)|d\mu = \int_{\mathbb{P}^1\times\mathbb{P}^1} \log|\det(d(f\times\overline{f}))|d(\mu_f\times\mu_{\overline{f}}) = \mathcal{L}_f + \mathcal{L}_{\overline{f}} = 2\mathcal{L}_f,$$

contradicting (4.5). When $\dim \Gamma = 1$, then

$$\begin{aligned} &\int \log|\det(dF)|d\mu = \int_\Gamma \log|\det(d(f\times\overline{f}))|d\mu \\ =&\int_\Gamma \log|\det(df)\circ\pi_1|d\mu + \int_\Gamma \log|\det(d\overline{f})\circ\pi_2|d\mu \\ =&\int_\Gamma \log|\det(df)\circ\pi_1|d\frac{\pi_1^*\mu_f}{\deg(\pi_1)} + \int_\Gamma \log|\det(d\overline{f})\circ\pi_2|d\frac{\pi_2^*\mu_{\overline{f}}}{\deg(\pi_2)} \\ =&\int_{\mathbb{P}^1} \log|\det(df)|d\mu_f + \int_{\mathbb{P}^1} \log|\det(d\overline{f})|d\mu_{\overline{f}} \\ =&\mathcal{L}_f + \mathcal{L}_{\overline{f}} = 2\mathcal{L}_f, \end{aligned}$$

contradicting (4.5). Therefore, f must be exceptional. We have finished the proof. □

Proof of Theorem 1.3 when f is defined over $\overline{\mathbb{Q}}$. Assume that f is defined over $\overline{\mathbb{Q}}$, hence f is defined over a number field K. Suppose that the Theorem 1.3 does not hold for f. Let V be the $\mathbb{Q}$-span of $\chi_f(\mathrm{Per}^*(f))$ in $\mathbb{R}$, then $\dim_{\mathbb{Q}} V < \infty$. We can take $M \in \mathbb{Z}_{>0}$ and $x_1,\dots,x_M \in \mathrm{Per}^*(f)$ such that $\chi_f(x_1),\dots,\chi_f(x_M)$ generate V over $\mathbb{Q}$. By enlarging K, we may assume that $|\rho_f(x_1)|,\dots,|\rho_f(x_M)| \in K$. Then for every $z_0 \in \mathrm{Per}^*(f)$, $\chi_f(z_0)$ is a linear combination of $\chi_f(x_1),\dots,\chi_f(x_M)$ over $\mathbb{Q}$; then it is easy to see that there exists $n\in\mathbb{Z}_{>0}, n_1,\dots,n_M \in \mathbb{Z}$ such that

$$|\rho_f(z_0)|^n = |\rho_f(x_1)|^{n_1}\cdots|\rho_f(x_M)|^{n_M} \in K,$$

which contradicts Theorem 1.4 since f is not exceptional by the assumption. □

5. Some linear algebras

5.1. **Pseudo linear algebra.** Let V, W be two $\mathbb{R}$-linear spaces. A pseudo morphism $f: V\to W$ is a pair (V_f, f) where V_f is a linear subspace of V and $f: V_f \to W$ is an $\mathbb{R}$-linear map. If $x\in V\setminus V_f$, we write $f(x)=\infty$. When $W=\mathbb{R}$, we say that f is a pseudo linear function.

Denote by $\mathrm{PHom}(V,W)$ the set of pseudo morphisms from V to W. For $f,g\in \mathrm{PHom}(V,W)$, we define $f+g$ to be the pair $(V_f\cap V_g, f|_{V_f\cap V_g} + g|_{V_f\cap V_g})$. Then $\mathrm{PHom}(V,W)$ is a commutative semigroup with $+$ as the operation. We denote by 0 the pair $(V,0)$. We have $0+f=f$ for all $f\in\mathrm{PHom}(V,W)$. For every $a\in\mathbb{R}$, we define af to be the pair (V_f, af). We note that $f+(-f)=(V_f,0)$, which is not 0 if $V_f\neq V$. We have an natural embedding $\mathrm{Hom}(V,W)\hookrightarrow \mathrm{PHom}(V,W)$.

For $f\in\mathrm{PHom}(U,V)$ and $g\in\mathrm{PHom}(V,W)$, we define their composition $g\circ f$ to be $(U_f\cap f^{-1}(V_g), g\circ f|_{U_f\cap f^{-1}(V_g)})\in\mathrm{PHom}(U,W)$. Observe that if $f(v)=\infty$, then $g\circ f(v)=\infty$.

Fix a subset O of V. Denote the set of positive real numbers by $\mathbb{R}^+$, and set $\mathbb{R}_{\geq 0} := \mathbb{R}^+\cup\{0\}$.

Definition 5.1. A sequence $(f_i)_{i=1}^{\infty}$ in $\mathrm{PHom}(V, W)$ is said to be an O-sequence if the following conditions are satisfied:

(i) $f_i(O) \subseteq \mathbb{R}_{\geq 0} \cup \{\infty\}$ for $i \geq 1$; (ii) for every $\lambda \in O$, the set $\{i \geq 1 : f_i(\lambda) \neq 0\}$ is finite.

Clearly, an infinite subsequence of an O-sequence is still an O-sequence.

Definition 5.2. Let $(\lambda_i)_{i=1}^{\infty}$ be a sequence in O and $(f_i)_{i=1}^{\infty}$ be a sequence in $\mathrm{PHom}(V, \mathbb{R})$. We say that $((\lambda_i)_{i=1}^{\infty}, (f_i)_{i=1}^{\infty})$ is an upper triangle O-system (resp. weak upper triangle O-system) if the following conditions hold:

(i) $(f_i)_{i=1}^{\infty}$ is an O-sequence;
(ii) $f_i(\lambda_i) \in \mathbb{R}^+$ (resp. $f_i(\lambda_i) \in \mathbb{R}^+ \cup \{\infty\}$) for $i \geq 1$;
(iii) $f_j(\lambda_i) = 0$ for $j > i \geq 1$.

Clearly, an upper triangle O-system is a weak upper triangle O-system.

Lemma 5.3. *Let $((\lambda_i)_{i=1}^{\infty}, (f_i)_{i=1}^{\infty})$ be a weak upper triangle O-system. Then $(\lambda_i)_{i=1}^{\infty}$ are linearly independent over $\mathbb{R}$.*

Proof. Since $f_1(\lambda_1) \neq 0$, we see that $\lambda_1 \neq 0$. Then we only need to show that for all $l \geq 2$, λ_l is not contained in $\mathrm{span}_{\mathbb{R}}\{\lambda_i : i \leq l-1\}$. Otherwise, $\lambda_l = \sum_{i=1}^{l-1} a_i \lambda_i$ for some $l \geq 2, a_i \in \mathbb{R}, 1 \leq i \leq l-1$. Then $f_l(\lambda_l) = \sum_{i=1}^{l-1} a_i f_l(\lambda_i) = 0$, which contradicts to our assumption. $\square$

Let $\tau : V \to V$ be an involution (i.e. $\tau^2 = \mathrm{id}$).

Lemma 5.4. *Assume that $\tau(O) \subseteq O$. Let $((\lambda_i)_{i=1}^{\infty}, (f_i)_{i=1}^{\infty})$ be an upper triangle O-system. Then there exists a strictly increasing sequence $(m_i)_{i=1}^{\infty}$ in $\mathbb{Z}_{>0}$ such that the pair $((\lambda_{m_i} + \tau(\lambda_{m_i}))_{i=1}^{\infty}, (f_{m_i})_{i=1}^{\infty})$ is a weak upper triangle O'-system, where $O' = \{\lambda_{m_i} + \tau(\lambda_{m_i}) : i \in \mathbb{Z}_{>0}\}$.*

Proof. It is clear that $(f_i)_{i=1}^{\infty}$ is also an O'-sequence. We construct $(m_i)_{i=1}^{\infty}$ recursively. Set $m_1 := 1$. As $\tau(\lambda_1) \in O$, we have $f_1(\tau(\lambda_1)) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. Since $f_1(\lambda_1) \in \mathbb{R}^+$, we have

$$f_{m_1}(\lambda_{m_1} + \tau(\lambda_{m_1})) = f_1(\lambda_1) + f_1(\tau(\lambda_1)) \in \mathbb{R}^+ \cup \{\infty\}.$$

Assume that we have constructed $m_1, \ldots, m_l$ satisfying the conditions for weak upper triangle systems. Since $(f_i)_{i=1}^{\infty}$ is an O-system and $\tau(\lambda_1), \cdots, \tau(\lambda_l) \in O$, there exists $m_{l+1} > m_l$ such that $f_{m_{l+1}}(\tau(\lambda_i)) = 0$ for all $i = 1, \ldots, l$. Then for all $i = 1, \ldots, l$, we have

$$f_{m_{l+1}}(\lambda_{m_i} + \tau(\lambda_{m_i})) = f_{m_{l+1}}(\lambda_{m_i}) + f_{m_{l+1}}(\tau(\lambda_{m_i})) = 0;$$

also,

$$f_{m_{l+1}}(\lambda_{m_{l+1}} + \tau(\lambda_{m_{l+1}})) = f_{m_{l+1}}(\lambda_{m_{l+1}}) + f_{m_{l+1}}(\tau(\lambda_{m_{l+1}})) \in \mathbb{R}^+ \cup \{\infty\}$$

and

$$f_{m_i}(\lambda_{m_{l+1}} + \tau(\lambda_{m_{l+1}})) = f_{m_i}(\lambda_{m_{l+1}}) + f_{m_i}(\tau(\lambda_{m_{l+1}})) \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

We conclude the proof. $\square$

By Lemma 5.4 and Lemma 5.3 we get the following result.

Corollary 5.5. *Assume that $\tau(O) \subseteq O$. Let $((\lambda_i)_{i=1}^{\infty}, (f_i)_{i=1}^{\infty})$ be an upper triangle O-system. Then* $\dim_{\mathbb{R}}\mathrm{span}_{\mathbb{R}}\{2^{-1}(\lambda_i + \tau(\lambda_i)) : i \geq 1\} = \infty$.

Note that the discussion in this subsection also applies with $\mathbb{R}$ replaced by any ordered field F.

5.2. **Linear algebra for multiplication.** For every field k of characteristic 0, denote by μ_k the subgroup of roots of unity in k. Denote by $\mathrm{rog} : k^* \to \mathbb{D}(k) := k^*/\mu_k$ the quotient map. Extend rog to a map $\mathrm{rog} : k \to \mathbb{D}(k) \cup \{\infty\}$ by sending 0 to ∞. Here we use the notation rog since it is an analogy of the classical log function to some extent. The embedding $k \hookrightarrow \overline{k}$ gives a natural embedding $\mathbb{D}(k) \hookrightarrow \mathbb{D}(\overline{k})$ as multiplicative abelian groups. Write $\mathbb{D}(k)_{\mathbb{Q}} := \mathbb{D}(k) \otimes_{\mathbb{Z}} \mathbb{Q}$, where $\mathbb{D}(k)$ is as a multiplicative commutative group, hence a $\mathbb{Z}$-module; then $\mathbb{D}(k)_{\mathbb{Q}}$ is the subspace of $\mathbb{D}(\overline{k})$ spanned by $\mathbb{D}(k)$ over $\mathbb{Q}$. Write $\mathbb{D}(k)_{\mathbb{R}} := \mathbb{D}(k) \otimes_{\mathbb{Z}} \mathbb{R}$.

Let $A \subseteq k$ be an integral domain with $\mathrm{Frac}(A) = k$. Define $\mathbb{D}(A) := \mathrm{rog}(A \setminus \{0\}) \subseteq \mathbb{D}(k)$, which is a subsemigroup of $\mathbb{D}(k)$. For every prime ideal p of A, the surjective projection $A \to A/p$ induces a surjective morphism $s_p : \mathbb{D}(A) \cup \{\infty\} \to \mathbb{D}(A/p) \cup \{\infty\}$. In fact, we may view s_p as a pseudo morphism

$$s_p : \mathbb{D}(k)_{\mathbb{R}} \to \mathbb{D}(\mathrm{Frac}(A/p))_{\mathbb{R}}$$

with domain $V_{s_p} := (A \setminus p) \otimes_{\mathbb{Z}} \mathbb{R}$.

5.3. **Norms.** Let k be a field of characteristic 0. For every finite field extension $\widetilde{k}$ over k , denote by $N_{\widetilde{k}/k} : \widetilde{k} \to k$ the norm map. We define a morphism $\mathbf{n}_k : \mathbb{D}(\overline{k})_{\mathbb{Q}} \to \mathbb{D}(k)_{\mathbb{Q}}$ by

$$\mathbf{n}_k : \mathrm{rog}(x) \mapsto [l : k]^{-1}\mathrm{rog}(N_{l/k}(x)),$$

where l is any finite extension over k containing x. We may check that $\mathbf{n}_k$ is well defined and is $\mathbb{Q}$-linear. We also denote by $\mathbf{n}_k$ its $\mathbb{R}$-linear extension $\mathbf{n}_k : \mathbb{D}(\overline{k})_{\mathbb{R}} \to \mathbb{D}(k)_{\mathbb{R}}$. When the field k is clear, we also write $\mathbf{n}$ for $\mathbf{n}_k$.

5.4. **Valuations.** Assume that K is a number field. Denote by $\mathcal{M}_K$ the set of all places of K. For every $v \in \mathcal{M}_K$, denote by $v : \mathbb{D}(K)_{\mathbb{R}} \to \mathbb{R}$ the $\mathbb{R}$-linear map given by

$$\mathrm{rog}(x) \mapsto -\log(|x|_v),\ x \in K^*.$$

It is easy to check that this map is well-defined and $\mathbb{R}$-linear. We also denote by $v : \mathbb{D}(K)_{\mathbb{Q}} \to \mathbb{R}$ its restriction. For every $a \in \mathbb{D}(K)_{\mathbb{R}}$, the set $\{v \in \mathcal{M}_K : v(a) \neq 0\}$ is finite.

Let S be a finite subset of $\mathcal{M}_K$ containing all the archimedean places. Let $\mathcal{O}_{K,S}$ be the ring of S-integers in K. Let $\mathcal{O}$ be the integral closure of $\mathcal{O}_{K,S}$ in $\overline{K}$. For every $v \in \mathcal{M}_K \setminus S$ and $\lambda \in \mathcal{O}$, we have $v \circ \mathbf{n}(\lambda) \geq 0$. Write $\mathcal{M}_K \setminus S = \{v_1, v_2, \dots\}$. Then $(v_i)_{i=1}^{\infty} \subseteq \mathrm{Hom}(\mathbb{D}(K)_{\mathbb{R}}, \mathbb{R})$ is an $\mathcal{O}_{K,S}$-sequence and $(v_i \circ \mathbf{n})_{i=1}^{\infty} \subseteq \mathrm{Hom}(\mathbb{D}(\overline{K})_{\mathbb{R}}, \mathbb{R})$ is an $\mathcal{O}$-sequence.

5.5. **Complex conjugation and absolute value.** Denote by $\tau : \mathbb{C} \to \mathbb{C}$ the complex conjugation. Then $\mathbb{R}$ is the fixed field $\mathbb{C}^\tau$ of τ. As $\mathbb{Q}$-vector spaces, we have an identification $\mathbb{D}(\mathbb{R}) = \mathbb{R}^*/\{\pm 1\} \to \mathbb{R}, \mathrm{rog}(a) \mapsto \log|a|$, where the latter log is the classical one on $\mathbb{R}^+$. Using this identification, the absolute value on $\mathbb{C}$ can be viewed as the norm $\mathbf{n}_{\mathbb{R}} : \mathbb{D}(\mathbb{C}) \to \mathbb{D}(\mathbb{R})$ sending $\mathrm{rog}(x)$ to $2^{-1}(\mathrm{rog}(x) + \mathrm{rog}(\tau(x)))$.

Let $\mathbf{k}$ be an algebraically closed subfield of $\mathbb{C}$ stable under the complex conjugation. Still denote by $\tau \in \mathrm{Gal}(\mathbf{k}/\mathbb{Q})$ the restriction of the complex conjugation on $\mathbf{k}$. Note that τ is an involution. Denote by $\mathbf{k}^\tau$ the τ-fixed subfield of $\mathbf{k}$. Then the restriction of the absolute value $\mathbf{n}_{\mathbb{R}}$ on $\mathbf{k}$ is $\mathbf{n}_{\mathbf{k}^\tau} : \mathbb{D}(\mathbf{k}) \to \mathbb{D}(\mathbf{k}^\tau), \mathrm{rog}(x) \mapsto 2^{-1}(\mathrm{rog}(x) + \mathrm{rog}(\tau(x)))$.

We shall prove the following result.

Theorem 5.6. *Assume that* $\mathbf{k}$ *is an algebraically closed field of characteristic* 0. *Let* $\tau \in \mathrm{Gal}(\mathbf{k}/\mathbb{Q})$ *be an element with* $\tau^2 = \mathrm{id}$. *If* $f : \mathbb{P}^1 \to \mathbb{P}^1$ *is an endomorphism over* $\mathbf{k}$ *of degree at least* 2 *which is not PCF, then the* $\mathbb{Q}$*-subspace in* $\mathbb{D}(\mathbf{k}^\tau)_{\mathbb{Q}}$ *spanned by* $\{\mathbf{n}_{\mathbf{k}^\tau}(\mathrm{rog}(\rho_f(x))) : x \in \mathrm{Per}^*(f)(\mathbf{k})\}$ *is of infinite dimension.*

Take $\mathbf{k} = \mathbb{C}$ and let τ be the complex conjugation, then Theorem 5.6 implies Theorem 1.3 in the case that f is a non-PCF map.

Remark 5.7. Setting $\tau = \mathrm{id}$, then from Theorem 5.6 we get the following result:

Assume that $\mathbf{k}$ is an algebraically closed field of characteristic 0. If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is an endomorphism over $\mathbf{k}$ of degree at least 2 which is not PCF, then the $\mathbb{Q}$-subspace in $\mathbb{D}(\mathbf{k})_{\mathbb{Q}}$ spanned by $\{\mathrm{rog}(\rho_f(x)) : x \in \mathrm{Per}^*(f)(\mathbf{k})\}$ is of infinite dimension.

6. Proofs of Theorem 5.6 and Theorem 1.3

6.1. **Proof of Theorem 5.6: the case $\mathbf{k} = \overline{\mathbb{Q}}$.** Let τ be an element in $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with $\tau^2 = \mathrm{id}$.

Denote by $\mathcal{C}_f$ the set of critical points of f. Since f is not postcritically finite, there exists $o \in \mathcal{C}_f$ such that the (forward) orbit $O_f(o)$ of o is infinite. We fix this critical point o. Let X be the union of all (forward) orbits of periodic critical points of f. Then X is finite.

Pick a number field K satisfying $\tau(K) = K$ and such that f, o and all points in X are defined over K.

Denote by $\mathcal{M}_K$ the set of places of K. Let $B \subseteq \mathcal{M}_K$ be a finite set containing all the archimedean places, satisfying $\tau(B) = B$ and such that for every $v \in \mathcal{M}_K \backslash B, f$ has good reduction at v. Then we have $\tau(\mathcal{O}_{K,B}) = \mathcal{O}_{K,B}$. For $x \in \mathrm{Per}(f)(\mathbf{k})$, set $\lambda(x) = (n_f(x))^{-1}\mathrm{rog}(\rho_f(x)) \in \mathbb{D}(\mathbf{k})_{\mathbb{R}} \cup \{\infty\}$. Recall that $n_f(x)$ is the exact periods of x and $\rho_f(x)$ is the multiplier of x. Then for all $x \in \mathrm{Per}^*(f)(\mathbf{k})$, we have $\mathbf{n}_K(\lambda(x)) \in \mathbb{D}(\mathcal{O}_{K,B})$.

Denote by $\mathbb{C}_v$ the completion of the algebraically closure of K_v. Every embedding $\sigma : \mathbf{k} \hookrightarrow \mathbb{C}_v$ gives a bijection $\sigma : \mathrm{Per}(f)(\mathbf{k}) \to \mathrm{Per}(f)(\mathbb{C}_v)$. Observe that for every $x \in \mathrm{Per}(f)(\mathbf{k})$, we have $\sigma(\lambda(x)) = \lambda(\sigma(x))$.

For every $v \in \mathcal{M}_K \setminus B$ and $x \in \mathbb{P}^1(\mathbb{C}_v)$, denote by $\tilde{x} \in \mathbb{P}^1(\overline{\widetilde{K_v}})$ the reduction of x in the special fiber at v and $f_v : \mathbb{P}^1_{\overline{\widetilde{K_v}}} \to \mathbb{P}^1_{\overline{\widetilde{K_v}}}$ the reduction of f. After enlarging B, we may assume that $\tilde{o} \notin X_v$ where X_v is the reduction of X in $\mathbb{P}^1(\widetilde{K_v}) \subseteq \mathbb{P}^1(\overline{\widetilde{K_v}})$

Observe that for every $v \in \mathcal{M}_K \setminus B, x \in \mathrm{Per}(f)(\mathbf{k})$ of exact period $n \geq 1$ and any embedding $\sigma : \mathbf{k} \hookrightarrow \mathbb{C}_v$, we have $v(\lambda(x)) \geq 0$. Moreover the followings are equivalent:

(i) $v(\lambda(x)) > 0$;

(ii) there exists an embedding $\sigma : \mathbf{k} \hookrightarrow \mathbb{C}_v$ such that $(f_v^n)'(\widetilde{\sigma(x)}) = 0$;

(iii) there exists an embedding $\sigma : \mathbf{k} \hookrightarrow \mathbb{C}_v, q \in \mathcal{C}_f$ and $m \geq 0$, such that $\widetilde{\sigma(q)}$ is periodic for f_v and $\widetilde{\sigma(x)} = f_v^m(\widetilde{\sigma(q)})$.

For $v \in \mathcal{M}_K \setminus B$, denote by P_v the union of all orbits of periodic critical points of f_v. Then P_v is finite. For every $v \in \mathcal{M}_K \setminus B, q \in P_v$, there exists a unique periodic point $y \in \mathrm{Per}(f)(\mathbb{C}_v \cap \mathbf{k})$ such that $\tilde{y} = q$. Then there exists a unique $\mathrm{Gal}(\mathbf{k}/K)$-orbit $O(q)$ in $\mathbf{k}$ such that for some (then every) $x \in O(q)$, there exists an embedding $\sigma : \mathbf{k} \hookrightarrow \mathbb{C}_v$ such that $\widetilde{\sigma(x)} = q$ (here $O(q)$ is the orbit of q). In particular, we have $X_v \subseteq P_v$ and $\cup_{q \in X_v} O(q) = X$. It follows that the set

$$Q_v := \{x \in \mathrm{Per}(f)(\mathbf{k}) : v(\lambda(x)) > 0\} = \bigcup_{q \in P_v} O(q)$$

is finite. Moreover, $Q_v = X$ if and only if $P_v = X_v$.

Lemma 6.1. *The set $S := \{v \in \mathcal{M}_K \setminus B : P_v \setminus X_v \neq \emptyset\}$ is infinite.*

Proof. By [BGKT12, Lemma 4.1], there are infinitely may $v \in \mathcal{M}_K \setminus B$, for which there exists $n \in \mathbb{Z}_{>0}$ such that $f_v^n(\tilde{o}) = \tilde{o}$. For such v, we have $\tilde{o} \in P_v \setminus X_v$, which concludes the proof. □

Lemma 6.2. *There exists a sequence $(x_i)_{i=1}^{\infty}$ in $\mathrm{Per}^*(f)(\mathbf{k})$ and a sequence $(v_j)_{j=1}^{\infty}$ in $\mathcal{M}_K \setminus B$ such that $v_i(\lambda(x_i)) > 0$ for $i \geq 1$ and $v_j(\lambda(x_i)) = 0$ for $j \neq i$. In particular, $((\lambda(x_i))_{i=1}^{\infty}, (v_i)_{i=1}^{\infty})$ is an upper triangle $\mathbb{D}(\mathcal{O}_{K,B})$-system for $\mathbb{D}(K)_{\mathbb{R}}$.*

Proof. We construct these two sequences recursively.

By Lemma 6.1, S is infinite. Pick $v_1 \in S$, then there exists $x_1 \in Q_{v_1} \setminus X \subseteq \mathrm{Per}^*(f)(\mathbf{k})$. We have $v_1(\lambda(x_1)) > 0$.

Assume that we have constructs $x_1, \dots, x_m \in \mathrm{Per}^*(f)(\mathbf{k})$ and $v_1, \dots, v_m \in \mathcal{M}_K \setminus B$ such that $v_j(\lambda(x_i)) \geq 0$ and the quality holds if and only if $j \neq i$. The set $\cup_{i=1}^m Q_{v_i} \setminus X$ is finite. Then there exists a finite set $T_m \subseteq \mathcal{M}_K$ such that for all $x \in \cup_{i=1}^m Q_{v_i} \setminus X$, and $v \in \mathcal{M}_K \setminus T_m$, we have $v(x) = 0$. By Lemma 6.1, there exists $v_{m+1} \in S \setminus (\{v_1, \dots, v_m\} \cup T_m)$. Then we have $v_{m+1}(x_i) = 0$ for $i = 1, \dots, m$. Pick $x_{m+1} \in Q_{v_{m+1}} \setminus X$. We have $v_{m+1}(x_{m+1}) > 0$. It follows that $x_{m+1} \notin \cup_{i=1}^m Q_{v_i}$. Then $v_i(x_{m+1}) = 0$ for $i = 1, \dots, m$. We conclude the proof of Lemma 6.2. □

Then we conclude the proof by Corollary 5.5.

6.2. **Proof of Theorem 5.6: the general case.** Denote by $\mathcal{C}_f$ the set of critical points of f. Since f is not PCF, there is an $o \in \mathcal{C}_f$ which is not preperiodic. We fix this critical point o. Fix a subfield K of $\mathbf{k}$ such that $K/\mathbb{Q}$ is finite generated, $\tau(K) = K$, and o, f are defined over K. Without loss of generality, we may assume that $\mathbf{k} = \overline{K}$.

Take a finite generated $\mathbb{Z}$-subalgebra A of K with $\mathrm{Frac}(A) = K$ and $\tau(A) = A$. After shrinking $\mathrm{Spec}(A)$, we may assume that there exists an endomorphism $f_A : \mathbb{P}^1_A \to \mathbb{P}^1_A$ over A whose restriction $f_K : \mathbb{P}^1_K \to \mathbb{P}^1_K$ over the generic fiber $\mathbb{P}^1_K$ satisfies $f = f_K \otimes_K \mathbf{k}$.

For every $c \in \mathrm{Spec}(A \otimes_{\mathbb{Z}} \overline{\mathbb{Q}})(\overline{\mathbb{Q}})$, denote by f_c the specialization of f_A at c, and o_c the specialization of o at c. Then o_c is a critical point of f_c. By [GX18, Lemma 3.3], there exists $c \in \mathrm{Spec}(A \otimes_{\mathbb{Z}} \overline{\mathbb{Q}})(\overline{\mathbb{Q}})$ such that the orbit of o_c is infinite. In particular, f_c is not PCF. There exists a number field $L \subseteq \overline{K}$ such that c is defined over L. Denote by A_1 the algebra generated by $A, \mathcal{O}_L$ and $\tau(\mathcal{O}_L)$; we may replace $\mathrm{Spec}(A)$ by some Zariski open set of $\mathrm{Spec}(A_1)$ for which $f_A : \mathbb{P}^1_A \to \mathbb{P}^1_A$ is still everywhere well-defined. We may view $\mathrm{Spec}(A)$ as an $\mathcal{O}_L$-scheme, and pick a point $c \in \mathrm{Spec}(A \otimes_{\mathcal{O}_L} L)$ such that the orbit of o_c is infinite. After shrinking $\mathrm{Spec}(A)$, the Zariski closure of c in $\mathrm{Spec}(A)$ is isomorphic to $\mathrm{Spec}(\mathcal{O}_{L,S})$ for a finite set of places $S \subseteq \mathcal{M}_L$ containing all archimedean places. It corresponds to a prime ideal p of A.

Denote by $s_p : \mathbb{D}(K)_{\mathbb{R}} \dashrightarrow \mathbb{D}(L)_{\mathbb{R}}$ the pseudo morphism as in Section 5. We have $s_p(\mathbb{D}(A)) \subseteq \mathbb{D}(\mathcal{O}_{L,S}) \cup \{\infty\}$. Then for every $v \in \mathcal{M}_L \setminus S$ and $\lambda \in \mathbb{D}(A)$, we have $v(\lambda) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$. Moreover, for every $\lambda \in \mathbb{D}(A)$, if $s_p(\lambda) \neq \infty$, then there are only finitely many $v \in \mathcal{M}_L \setminus S$ for which $v(s_p(\lambda)) \neq 0$.

For every $y \in \mathrm{Per}(f)(\overline{K})$, denote by y_c the set of $x \in \mathrm{Per}(f_c)(\overline{L})$ with whose image is contained in the image of y in $\mathbb{P}^1_A$. For every $y \in \mathrm{Per}(f)(\overline{K})$, y_c is finite and nonempty. On the other hand, for every $x \in \mathrm{Per}(f_c)(\overline{L})$, the set of $y \in \mathrm{Per}(f)(\overline{K})$ with $x \in y_c$ is finite and nonempty. Moreover, if $x \in y_c$, then

$$s_p(\mathbf{n}_K(\lambda(y))) = \mathbf{n}_L(\lambda(x)).$$

Since the set of $x \in \mathrm{Per}(f_c)(\overline{L})$ with $\mathbf{n}_L(\lambda(x)) = \infty$ is finite, the set

$$W_c := \{y \in \mathrm{Per}(f)(\overline{K}) : s_p(\mathbf{n}_K(\lambda(y))) = \infty\}$$

is also finite. Similarly $W_{\tau(c)} := \{y \in \mathrm{Per}(f)(\overline{K}) : s_{\tau(p)}(\mathbf{n}_K(\lambda(y))) = \infty\}$ is finite.

By Lemma 6.2, there exists $(y_i)_{i=1}^{\infty}$ in $\mathrm{Per}(f_c)(\overline{L})$ and $(v_i)_{i=1}^{\infty}$ in $\mathcal{M}_L \setminus S$ such that $((\mathbf{n}_L(\lambda(y_i)))_{i=1}^{\infty}, (v_i)_{i=1}^{\infty})$ is an upper triangle $\mathbb{D}(\mathcal{O}_{L,S})$-system for $\mathbb{D}(L)_{\mathbb{R}}$. For every $i \in \mathbb{Z}_{>0}$, there exists $x_i \in \mathrm{Per}(f)(\overline{K})$ such that the image of y_i is contained in the Zariski closure of the image of x_i in $\mathbb{P}^1_A$. We have

$$s_p(\mathbf{n}_K(\lambda(x_i))) = \mathbf{n}_L(\lambda(y_i)).$$

After removing finite terms, we may assume that $y_i \notin W_c \cup W_{\tau(c)}$ for all $i \geq 1$. It follows that $\mathbf{n}_L(\lambda(y_i)) \in \mathrm{rog}(A \setminus (p \cup \tau(p)))$ for $i \geq 1$. Observe that $(v_i \circ s_p)_{i=1}^{\infty}$ is a $\mathrm{rog}(A \setminus (p \cup \tau(p)))$-sequence. It follows that $((\mathbf{n}_L(\lambda(y_i)))_{i=1}^{\infty}, (v_i \circ s_p)_{i=1}^{\infty})$ is an upper triangle $\mathrm{rog}(A \setminus (p \cup \tau(p)))$-system for $\mathbb{D}(K)_{\mathbb{R}}$. Since $\mathrm{rog}(A \setminus (p \cup \tau(p)))$ is invariant under τ, we conclude the proof by Corollary 5.5.

6.3. **Proof of Theorem 1.3.** There are two cases:

1. The case f is PCF. In this case according to [DH93], PCF maps are defined over $\overline{\mathbb{Q}}$ in the moduli space $\mathcal{M}_d$ of rational maps of degree d, except for the family of flexible Lattès maps. So after a conjugacy by an elements in $\mathrm{PGL}_2(\mathbb{C})$, f is defined over $\overline{\mathbb{Q}}$, and Theorem 1.3 was already proved in the end of Section 4.

2. The case f is not PCF. Then Theorem 1.3 is a consequence of Theorem 5.6. This finishes the proof of Theorem 1.3.

7. Proofs of the Applications

7.1. **Proof of Theorem 1.10.** Without loss of generality, we may assume that k is of finite transcendence degree over $\mathbb{Q}$. Fix an embedding of k into $\mathbb{C}$. We view f as an endomorphism on X defined over $\mathbb{C}$. According to [Xie22, Theorem 3.34], we may assume that all $f_j : \mathbb{P}^1 \to \mathbb{P}^1$ has degree at least 2 for $1 \leq j \leq N$.

Assume first that all f_j are not exceptional, $1 \leq j \leq N$. Corollary 1.5 implies that we can take $x_j \in \mathrm{Per}^*(f_j)(\mathbb{C})$ for $1 \leq j \leq N$ such that $\rho_{f_1}(x_1), \cdots, \rho_{f_N}(x_N)$ are multiplicatively independent in $\mathbb{C}$. After replacing f by an iterate, we may assume that $f_j(x_j) = x_j$ for $1 \leq j \leq N$, and the multipliers $(\rho_{f_j}(x_j) = f_j'(x_j))_{j=1}^N$ are still multiplicatively independent. Denote $x = (x_1, \ldots, x_N) \in X(k)$. Then x is a fixed point of f (smooth in the fixed locus of f) such that the eigenvalues of $df|_x$ are nonzero and multiplicatively independent. Then the conclusion follows from [ABR11].

Assume that all f_j are exceptional, $1 \leq j \leq N$. This case is easy, and we just refer to the proof in the first several paragraphs of [Xie22, Section 9.3].

We may assume that $0 \leq s \leq N$ such that $f_1, \cdots, f_s$ are not exceptional and $f_{s+1}, \cdots, f_N$ are exceptional. Let $l(f) = \min\{s, N-s\} \geq 0$. Then we have done in the case $l(f) = 0$. Then an induction on $l(f)$ will prove this corollary, as shown in the last several paragraphs of [Xie22, Section 9.3].

7.2. **Proof of Theorem 1.12.** Using the terminology and notations in Section 5, it is clear that (2) and (3) is equivalent to the following $(2)'$ and $(3)'$, respectively.

$(2)'$ $\rho_f(x) \in \overline{\mathbb{Q}}$ for all $x \in \mathrm{Per}(f)(\mathbb{C})$ and the $\mathbb{Q}$-subspace of $\mathbb{D}(\mathbb{Q})_{\mathbb{Q}}$ generated by $\mathbf{n}_{\mathbb{Q}}(\mathrm{rog}(\rho_f(x)))$ for $x \in \mathrm{Per}^*(f)(\mathbb{C})$ is of finite dimension over $\mathbb{Q}$.

$(3)'$ $|\rho_f(x)| \in \overline{\mathbb{Q}}$ for all $x \in \mathrm{Per}(f)(\mathbb{C})$ and the $\mathbb{Q}$-subspace of $\mathbb{D}(\mathbb{Q})_{\mathbb{Q}}$ generated by $\mathbf{n}_{\mathbb{Q}}(\mathrm{rog}(|\rho_f(x)|))$ for $x \in \mathrm{Per}^*(f)(\mathbb{C})$ is of finite dimension over $\mathbb{Q}$.

Now we prove that $(1), (2)', (3)'$ are equivalent.

$(1) \Rightarrow (2)'$ and $(3)'$:

Suppose that f is PCF. By [DH93], PCF maps are defined over $\overline{\mathbb{Q}}$ in $\mathcal{M}_d$, except for the family of flexible Lattès maps. If f is flexible Lattès, then according to [Mil06, Lemma 5.6], $\rho_f(x) \in \mathbb{Z}$ for all $x \in \mathrm{Per}(f)(\mathbb{C})$. If f is defined over $\overline{\mathbb{Q}}$, then clearly $\rho_f(x) \in \overline{\mathbb{Q}}$ for all $x \in \mathrm{Per}(f)(\mathbb{C})$. Thus, we always have $\rho_f(x), |\rho_f(x)| \in \overline{\mathbb{Q}}$ for all $x \in \mathrm{Per}(f)(\mathbb{C})$.

Suppose that $(2)'$ is false, then

$$\dim_{\mathbb{Q}} \mathrm{span}_{\mathbb{Q}} \{\mathbf{n}_{\mathbb{Q}}(\mathrm{rog}(\rho_f(x))) : x \in \mathrm{Per}^*(f)(\mathbb{C})\} = \infty.$$

By [Mil06, Corollary 3.9], f cannot be a flexible Lattès map, hence f is defined over $\overline{\mathbb{Q}}$, and over a number field K. We use the notation and ideas in the case of Section 6.1 where $\tau = \mathrm{Id}$. Let $B \subseteq \mathcal{M}_K$ be a finite set containing all the archimedean places such that for every $v \in \mathcal{M}_K \setminus B$, f has good reduction at v. For every $v \in \mathcal{M}_K \setminus B$, the reduction f_v are still PCF and its critical orbits from those of f. Then as in Section 6.1, it is easy to see that the set

$$\mathcal{W} := \{x \in \mathrm{Per}^*(f)(\mathbb{C}) : v(\mathbf{n}_K(\lambda(x))) = 0, \forall v \in \mathcal{M}_K \setminus B\}$$

is co-finite in $\mathrm{Per}^*(f)(\mathbb{C})$. It is well-known that $\mathrm{rank}(\mathcal{O}_{K,B}^\times) = \#B - 1 < \infty$ (cf. [Nar04, Theorem 3.12]). Note that $\mathbf{n}_K(\mathrm{rog}(\rho_f(x))) \in \mathbb{D}(\mathcal{O}_{K,B})$ for all $x \in \mathrm{Per}^*(f)(\mathbb{C})$. Then we deduce that

$$\dim_{\mathbb{Q}}\mathrm{span}_{\mathbb{Q}}\{\mathbf{n}_K(\mathrm{rog}(\rho_f(x))) : x \in \mathrm{Per}^*(f)(\mathbb{C})\} < \infty,$$

which implies $\dim_{\mathbb{Q}}\mathrm{span}_{\mathbb{Q}}\{\mathbf{n}_{\mathbb{Q}}(\mathrm{rog}(\rho_f(x))) : x \in \mathrm{Per}^*(f)(\mathbb{C})\} < \infty$, contradicting the assumption. Thus $(2)'$ must hold.

$(3)'$ follows similar to the above paragraph, corresponding to the case where τ is the complex conjugate of Section 6.1.

$(2)' \Rightarrow (1)$:

Suppose that f is not PCF. In particularly, f is not flexible Lattès. Denote by Z the set of conjugacy classes $[g] \in \mathcal{M}_d(\mathbb{C})$ such that f and g have the same multiplier spectrum. By [Sil98, Theorem 4.5] and $(2)'$, Z is Zariski closed in $\mathcal{M}_d(\mathbb{C})$ and it is defined over $\overline{\mathbb{Q}}$. By [McM87, Corollary 2.3], Z consists of finitely many points and possibly a curve of flexible Lattès maps. Since f is not flexible Latteś, then we may assume that f is defined over $\overline{\mathbb{Q}}$, hence over a number field K. By the argument in Section 6.1 of the case $\tau = \mathrm{Id}$, we have

$$\dim_{\mathbb{Q}}\mathrm{span}_{\mathbb{Q}}\{\mathbf{n}_K(\mathrm{rog}(\rho_f(x))) : x \in \mathrm{Per}^*(f)(\mathbb{C}), \mathbf{n}_K(\mathrm{rog}(\rho_f(x))) \in \mathbb{D}(\mathcal{O}_{K,B})\} = \infty,$$

where $B \subseteq \mathcal{M}_K$ is a finite set containing all the archimedean places such that for every $v \in \mathcal{M}_K \setminus B$, f has good reduction at v. After enlarging B, we may assume that B is invariant under every $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. Indeed, a small modification of the proof of Lemma 6.2 shows that there exists a sequence $(x_i)_{i=1}^\infty$ in $\mathrm{Per}^*(f)(\mathbf{k})$ and a sequence $(v_j)_{j=1}^\infty$ in $\mathcal{M}_K \setminus B$ satisfy the following conditions:

$$v_i(\mathbf{n}_K(\lambda(x_i))) > 0 \text{ for all } i \geq 1;$$
$$\sigma(v_j)(\mathbf{n}_K(\lambda(x_i))) = 0 \text{ for all } i \neq j \text{ and } \sigma \in \mathrm{Gal}(K/\mathbb{Q}).$$

For $i \geq 1$, let p_i be the prime number below v_i, let $\widetilde{B}$ be the restriction of B to $\mathcal{M}_{\mathbb{Q}}$. Then it is easy to see that the pair $((\mathbf{n}_{\mathbb{Q}}(\lambda(x_i)))_{i=1}^\infty, (v_{p_i})_{i=1}^\infty)$ is an upper triangle $\mathbb{D}(\mathcal{M}_{\mathbb{Q},\widetilde{B}})$-system for $\mathbb{D}(\mathbb{Q})_{\mathbb{Q}}$. By Corollary 5.5, this contradicts $(2)'$. Thus, f must be PCF.

$(3)' \Rightarrow (1)$:

Assume that f is not PCF. We use the notation and ideas in the case of Section 6.2 with τ the complex conjugate. As in the proof of Section 6.2, we get a number field L and a finite set $S \subseteq \mathcal{M}_L$. We may assume that S is invariant under every $\sigma \in \mathrm{Gal}(L/\mathbb{Q})$. By the argument in $(2)' \Rightarrow (1)$, there exists a pair $((\mathbf{n}_L(\lambda(y_i)))_{i=1}^\infty, (v_i)_{i=1}^\infty)$ satisfy the following conditions:

$$v_i(\mathbf{n}_L(\lambda(x_i))) > 0 \text{ for all } i \geq 1;$$
$$\sigma(v_j)(\mathbf{n}_L(\lambda(x_i))) = 0 \text{ for all } i \neq j \text{ and } \sigma \in \mathrm{Gal}(L/\mathbb{Q}).$$

Then we can deduce a contradiction similar to the proof of $(2)' \Rightarrow (1)$, hence f is PCF.

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