ALGEBRAIC DYNAMICS AND RECURSIVE INEQUALITIES

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In memory of the late Professor Nessim Sibony

Abstract. We get three basic results in algebraic dynamics: (1). We give the first algorithm to compute the dynamical degrees to arbitrary precision. (2). We prove that for a family of dominant rational self-maps, the dynamical degrees are lower semi-continuous with respect to the Zariski topology. This implies a conjecture of Call and Silverman. (3). We prove that the set of periodic points of a cohomologically hyperbolic rational self-map is Zariski dense.

Moreover, we show that, after a large iterate, every degree sequence grows almost at a uniform rate. This property is not satisfied for general submultiplicative sequences. Finally, we prove the Kawaguchi-Silverman conjecture for a class of self-maps of projective surfaces including all the birational ones.

In fact, for every dominant rational self-map, we find a family of recursive inequalities of some dynamically meaningful cycles. Our proofs are based on these inequalities.

Contents

1. Introduction

Let $\mathbf{k}$ be a field. Let X be a projective variety of dimension d over $\mathbf{k}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map. The aim of algebraic dynamics is to study algebraic and dynamical properties of the iterates of f.

Date: May 11, 2024.
The author is supported by the NSFC Grant No.12271007.

1.1. **Dynamical degrees.** The most fundamental dynamical invariants associated to an algebraic dynamical system is arguably its dynamical degrees.

Let L be a big and nef line bundle in $\mathrm{Pic}(X)$. Denote by $I(f)$ the indeterminacy locus of f. Let X' be the graph of f in $X \times X$ i.e. the Zariski closure of $\{(x, f(x)) |\ x \text{ is a closed point in } X \setminus I(f)\}$ and let π_1, π_2 be the projection to the first and the second factors. For $i = 0, \ldots, d$, the *i-th degree* of f is

$$\deg_{i,L} f := ((\pi_2^* L)^i \cdot (\pi_1^* L)^{d-i}).$$

Using the terminology from Section 2.2 and 2.1, we may write

$$\deg_{i,L} f = ((f^* L)^i \cdot L^{d-i})$$

without specifying the birational model X'. The i-th *dynamical degree* of f is

$$\lambda_i(f) := \lim_{n\to\infty} (\deg_{i,L} f^n)^{1/n} \geq 1.$$

The existence of the above limit is non-trivial. It was proved by Russakovskii and Shiffman [RS97] when $X = \mathbb{P}^d_{\mathbb{C}}$, and by Dinh and Sibony [DS05] when X is projective over $\mathbb{C}$. As shown in [DS04] by Dinh and Sibony, the dynamical degrees can be defined even for meromorphisms on Kähler manifolds. It was proved by Truong [Tru20] and Dang [Dan20] in arbitrary characteristic. The methods of Truong [Tru20] and Dang [Dan20] are different. Truong's method is based on Jong's alterations and Roberts' effective version of Chow's moving lemma. This method can be viewed as an algebraic mimic of Dinh-Sibony's proof using positively closed currents [DS04, DS05]. Dang's method is based on Siu's inequality. The definition of $\lambda_i(f)$ does not depend on the choice of L [DS04, DS05, Tru20, Dan20]. Moreover, if $\pi : X \dashrightarrow Y$ is a generically finite and dominant rational map between varieties and $g\colon Y \dashrightarrow Y$ is a rational self-map such that $g \circ \pi = \pi \circ f$, then $\lambda_i(f) = \lambda_i(g)$ for all i. This can be shown by combining [Dan20, Theorem 1] with the projection formula. Another way to prove it is to apply the product formula for relative dynamical degrees directly (c.f. [DN11], [Dan20] and [Tru20, Theorem 1.3]).

Roughly speaking, the dynamical degrees measure the algebraic complexity of f. It controls the topological complexity of f. When X is a smooth projective variety over $\mathbb{C}$ and f is an endomorphism, fundamental results of Gromov [Gro03] and Yomdin [Yom87] show that

$$h_{top}(f^{\mathrm{an}}) = \max_{0\leq i\leq d}\{\lambda_i(f)\},$$

where $h_{top}(f^{\mathrm{an}})$ is the topological entropy of the holomorphic endomorphism $f^{\mathrm{an}} : X(\mathbb{C}) \to X(\mathbb{C})$ induced by f. Dinh-Sibony [DS05] showed that the upper bound

$$h_{top}(f^{\mathrm{an}}) \leq \max_{0\leq i\leq d}\{\lambda_i(f)\} \tag{1.1}$$

still holds for arbitrary rational self-maps over $\mathbb{C}$. However, (1.1) can be strict in general [Gue05a]. Recently Favre, Truong and the author proved (1.1) in the non-archimedean case [FTX22]. In the non-archimedean case, (1.1) can be strict even for endomorphisms [FRL10, FTX22].

When $\mathbf{k} = \overline{\mathbb{Q}}$, the dynamical degrees also control the arithmetic complexity of f, which is measured by the notion of arithmetic degree (c.f. Section 8). Further, the Kawaguchi-Silverman conjecture (=Conjecture 8.5) asserts that for any point $x \in X(\mathbf{k})$ with Zariski dense orbit, the arithmetic degree $\alpha_f(x)$ for (X, f, x) equals $\lambda_1(f)$. This conjecture has attracted a lot of attention. For the recent development, see [Mat23] and the references therein. See [Son23, LS23] for the higher arithmetic degrees and their relations to the higher dynamical degrees. In Section 8, we will prove the Kawaguchi-Silverman conjecture for a class of self-maps of projective surfaces including all the birational ones.

When f is an endomorphism, the dynamical degrees control the action of f^* on the cohomology of X. When X is a smooth projective variety over $\mathbb{C}$, Dinh [Din05] proved that

$$\lambda_i(f) = \rho(f^* : H^{2i}(X(\mathbb{C}), \mathbb{R}) \to H^{2i}(X(\mathbb{C}), \mathbb{R})) \tag{1.2}$$

where $H^{2i}(X(\mathbb{C}), \mathbb{R})$ is the singular cohomology of degree $2i$ and $\rho(f^*)$ is the spectral radius of the linear operator f^*. In positive characteristic, Truong proposed a conjecture saying that (1.2) still holds if one replaces the singular cohomology by the $\mathbb{Q}_l$-cohomology with $l \neq \operatorname{char} \mathbf{k}$ [Tru16]. This conjecture is wildly open. Indeed, the case for Frobenius endomorphisms implies Deligne's famous theorem for Weil's Riemann hypothesis [Del74]. However, it was proved by Esnault and Srinivas [ES13] for surface automorphisms and by Truong [Tru16] for any dominant endomorphisms of smooth projective varieties, that

$$\max_{0 \le i \le d} \lambda_i(f) = \max_{0 \le i \le 2d} \rho(f^* : H^i(X, \mathbb{Q}_l) \to H^i(X, \mathbb{Q}_l))$$

with respect to any field embedding $\mathbb{Q}_l \hookrightarrow \mathbb{C}$. Truong's proof indeed relies on Deligne's theorem.

Cohomologically hyperbolic self-maps. We introduce the notion of cohomological Lyapunov exponents as follows: For $i = 1, \ldots, d$, define the i-th *cohomological Lyapunov exponent* of f to be

$$\mu_i(f) := \lambda_i(f)/\lambda_{i-1}(f).$$

Define $\mu_{d+1}(f) := 0$ for convenience. As the sequence of dynamical degrees is log-concave [DS05, Tru20, Dan20], the sequence $\mu_i(f), i = 1 \ldots, d$ is decreasing.

For $i = 1, \ldots, d$, we say that f is *i-cohomologically hyperbolic* if $\lambda_i(f)$ is strictly larger than other dynamical degrees i.e.

$$\mu_i(f) > 1 \text{ and } \mu_{i+1}(f) < 1.$$

We say that f is *cohomologically hyperbolic* if it is i-cohomologically hyperbolic for some $i = 1, \ldots, d$, in other words, $\mu_j(f) \neq 1$ for every $j = 1, \ldots, d$.

Cohomological hyperbolicity can be viewed as a cohomological version of the important notion of hyperbolic dynamics in differentiable dynamical systems. Indeed, when $\mathbf{k} = \mathbb{C}$, very few algebraic dynamical system could be Anosov (which is a strong version of hyperbolicity) c.f. [Ghy95, Can04, XZ24]. However people expect that a cohomologically hyperbolic self-map looks like a hyperbolic map, hence shares some properties of hyperbolic maps.

1.2. **Algorithm to compute the dynamical degrees.** A basic problem is to compute the dynamical degrees to any given precision. More precisely,

Question 1.1. For any given number $l \in \mathbb{Z}_{>0}$, is there an algorithm (that stops in finite time) to compute a number $\tilde{\lambda}$ such that $\lambda_i(f) \in (\tilde{\lambda}, \tilde{\lambda} + \frac{1}{2^l})$?

Let L be an ample (or big and nef) line bundle on X. By the definition of the dynamical degree, for n sufficiently large, we have

$$\lambda_i(f) \in ((\deg_{i,L} f^n)^{1/n} - \frac{1}{2^{l+1}}, (\deg_{i,L} f^n)^{1/n} + \frac{1}{2^{l+1}}).$$

But this does not answer Question 1.1, as we do not know how large n we need.

Question 1.1 is also interesting in cryptography. See [SB21, Section 2] for interesting discussions.

Our result. Strictly speaking, the answer to Question 1.1 depends on the input i.e. how we represent X and f.

Example 1.2. Let $X := \mathbb{P}^1_{\mathbb{C}}$. Define $f_n : X \to X, n \geq 1$ as follows: Let $T_n, n \geq 0$ be all the Turing machines. Define $a_n := 0$ if T_n will halt, and $a_n := 1$ if T_n will not halt. Define $f_n(z) := a_n z^2 + z$. As $\lambda_1(f_n) = \deg f_n$, $\lambda_1(f_n) = 1$ if T_n will halt, and $\lambda_1(f_n) = 2$ if T_n will not halt. As the Halting problem is unsolvable, there is no algorithm to compute $\lambda_1(f_n)$ for all $n \geq 0$.

To describe our input, we need the notion of *mixed degrees*: Let L be an ample (or big and nef) line bundle on X. Let $s \geq 1$, consider two sequence of non-negative integers: $m_1 > \cdots > m_s \geq 0$ and $r_1, \dots, r_s \geq 0$ with $\sum_{i=1}^s r_i = d$. The mixed degree $(L_{m_1}^{r_1} \cdots L_{m_s}^{r_s})$ is easier to define using the terminology in Section 2.2 and 2.1. Here we define it in a more direct way. Let X' be the graph in $X^{s+1} = X \times (X^s)$ of the morphism $X \to X^s$ sending x to $(f^{m_1}(x), \dots, f^{m_s}(x))$. Let π_i be the projection to the $(i+1)$-th factor. Then we define

$$(L_{m_1}^{r_1} \cdots L_{m_s}^{r_s}) := ((\pi_1^* L)^{r_1} \cdots (\pi_s^* L)^{r_s}) \in \mathbb{Z}_{>0}.$$

We note that (X, f, L) can be defined on a finitely generated field. The following remark shows that the mixed degrees are computable if we represent (X, f) in a reasonable form.

Remark 1.3. Assume that $\mathbf{k}$ is a finitely generated field. We represent X and f as follows: Write $\mathbf{k}$ as

$$\mathbf{k} := \mathrm{Frac}\,(\mathbb{Z}[t_1, \dots, t_l]/P)$$

where $P = (G_1, \dots, G_m)$ is a prime ideal of $\mathbb{Z}[t_1, \dots, t_l]$. Write X as the subvariety of $\mathbb{P}^N_{\mathbf{k}}$ defined by a homogenous prime ideal $(H_1, \dots, H_s)$. The rational map $f : X \dashrightarrow X$ extends to a rational self-map $F : \mathbb{P}^N \dashrightarrow \mathbb{P}^N$ sending $[x_0 : \cdots : x_N]$ to $[F_0 : \cdots : F_N]$ where $F_0, \dots, F_N$ are homogenous polynomials of the same degree in $\mathbf{k}[x_0, \dots, x_N]$.

We represent X, f using the following datas as inputs:

(i) the polynomials $G_1, \dots, G_m$ with integer coefficients;

(ii) the polynomials $H_1, \dots, H_s, F_0, \dots, F_N$, whose coefficients are represented as rational functions in $t_1, \dots, t_l$ with integer coefficients.

In this case, we may ask L to be the restriction of $\mathcal{O}_{\mathbb{P}^N}(1)$ to X. For every mixed degree $(L_{m_1}^{r_1} \cdots L_{m_s}^{r_s})$, there is an algorithm compute its exact value. So our assumption is satisfied and we may use Theorem 1.4 to compute the dynamical degrees of f.

In Section 5, we affirmatively answer Question 1.1, under the assumption that all the mixed degrees $(L_{m_1}^{r_1} \cdots L_{m_s}^{r_s})$ are computable.

Theorem 1.4. *For any given number $l \in \mathbb{Z}_{>0}$, there is an explicit algorithm to output a number $\tilde{\lambda}$ such that $\lambda_i \in (\tilde{\lambda}, \tilde{\lambda} + \frac{1}{2^l})$, using finitely many mixed degrees.*

Strategy of the proof. By Siu's inequality in arbitrary codimension proved in [Dan20, Corollary 3.4.6] and [JL23, Theorem 3.5] (c.f. Theorem 2.2), one can show that the sequence

$$(\frac{\binom{d}{i}}{(L^d)} \deg_i(f^n))^{1/n}, n \geq 0$$

tends to $\lambda_i(f)$ from above. This controls the dynamical degrees from above. So we only need to control the dynamical degrees from below.

The key ingredient is the construction of recursive inequalities for degree sequences, in Section 3. Indeed, in Section 3.3, we construct a family of recursive inequalities of some dynamically meaningful cycles via our estimates of mixed degrees. We combine them to get the recursive inequalities of the dynamical degrees. As shown in Lemma 3.1, for a sequence of real numbers satisfying certain recursive inequalities, one can get a lower bound of the growth rate, if it satisfies certain initial condition. A small problem is that the recursive inequalities we construct involve the dynamical degrees of f, which is not known. Our idea is to define some conditions on X, f, L which involve some extra parameters. Roughly speaking, some parameters take values in $\mathbb{Q}$ and a parameter m is in $\mathbb{Z}_{\geq 0}$. The truth value for the conditions only depend on finitely many mixed degrees. If the rational parameters are sufficiently close to the dynamical degrees and m is sufficiently large, these conditions are satisfied for our X, f, L. Once some the condition for certain parameters is satisfied, we get a lower bound using the according parameters. As $\mathbb{Q}$ is countable, we can test the conditions one by one and we will get one condition satisfying by X, f, L. This gives the lower bound we need.

Previous results. Here we ignore the difficulty from the computability theory as in Example 1.2, and we assume that (X, f) is represented in a reasonable form.

There are plenty of works concerning the computation of dynamical degrees in special cases.

If X is a smooth projective variety and $f : X \to X$ is an endomorphism, we have that

$$\lambda_i(f) = \rho(f^* : N^i(X)_{\mathbb{R}} \to N^i(X)_{\mathbb{R}})$$

where $N^i(X)_{\mathbb{R}}$ is the $\mathbb{R}$-vector space spanned by the numerical classes of i-cocycles of X. As $N^i(X)_{\mathbb{R}}$ is a finite dimensional vector space, the sequence $\deg_{i,L} f^n, n \geq 0$ satisfies a linear recursive equation of order $\leq \dim_{\mathbb{R}} N^i(X)_{\mathbb{R}}$. In particular, $\lambda_i(f)$ is an algebraic integer of degree $\leq \dim_{\mathbb{R}} N^i(X)_{\mathbb{R}}$. In this case, $\lambda_i(f)$ should be computable for a given (X, f). On the other hand, there is a lot of interesting works on constructing examples of endomorphisms (especially automorphisms) having certain properties on the dynamical degrees, e.g. [Can99, McM02, McM07, McM11, McM16, Dol18, Ogu10, Ogu14, CO15, Ogu09, OT15, OY20, Ueh16, Res17, Les21].

When f is merely rational, most of the previous work focus on the first dynamical degree $\lambda_1(f)$. In [Sib99], Sibony introduced the important notion of algebraically stable maps. If (X, f) is algebraically stable, as in the endomorphism case, we still have

$$\lambda_1(f) = \rho(f^* : N^1(X) \to N^1(X)),$$

and one can compute $\lambda_1(f)$ using linear algebra. In most of the works, the strategy to compute $\lambda_1(f)$ is to construct a birational model (X', f') of (X, f) for which (X', f') is algebraicaly stable. For certain classes of maps, such as birational self-maps of surfaces [DF01] or endomorphism of $\mathbb{A}^2$ [FJ07, FJ11], we can find such birational models, after a suitable iterate. On the other hand, it was proved by Favre [Fav03] that algebraically stable model may not exist even for monomial self-maps on $\mathbb{P}^2$. However, the dynamical degrees $\lambda_i(f), i = 0, \ldots, d$ are computed for monomial self-maps on $\mathbb{P}^N$ by Favre-Wulcan and Lin [FW12, Lin12]. Dinh and Sibony [DS] computed the dynamical degrees for automorphisms $f : \mathbb{A}^m \to \mathbb{A}^m$ on complex affine spaces that are regular (i.e. the indeterminacy loci of f and its inverse f^{-1} are disjoint on the hyperplane at infinity in $\mathbb{P}^m$). As far as we know, these are the only non-trivial cases for which higher dynamical degrees are computed. See [BK04, BK08, Ngu06, AdMV06, AAdBM99, BV99, MHV97, DF01, FJ11, BDJ20, DF21, BDJK23] and the reference therein for more related works.

In the above cases, the dynamical degrees are always algebraic. Moreover, as (X, f) can be defined on a finitely generated field, the set of all possible values of dynamical degrees are countable [BFs00, Ure18]. However, in the breakthrough work [BDJ20], Bell, Diller and Jonsson give examples of rational self-maps of $\mathbb{P}^2_{\mathbb{C}}$ whose first dynamical degrees are transcendental numbers. In their examples, $\lambda_1(f)$ is the unique positive real zero of certain transcendental power series. So we do not expect the existence of an algorithm (stoping in finite time) computing the exact value of $\lambda_i(f)$ in general.

The surface case. In [Xie15, Key Lemma], the author proved the following result.[1]

Theorem 1.5. *Let* $\mathbf{k}$ *be a field. Let* X *be a projective surface over* $\mathbf{k}$. *Let* $f : X \dashrightarrow X$ *be a dominant rational self-map. Let* L *be a big and nef line bundle.*

[1] In [Xie15, Key Lemma], the result is stated only for birational self-maps. However, its proof indeed works for any dominant rational self-map, replacing all f^* by $\frac{f^*}{\lambda_2^{1/2}}$.

Then we have

$$\lambda_1(f) \geq \frac{\deg_{1,L} f^2}{2^{\frac{1}{2}} \times 3^{18} \deg_{1,L} f}.$$

We will see, in Section 5.3, that this indeed implies a positive answer of Question 1.1 for rational self-maps of surfaces.

The proof of Theorem 1.5 relies on the theory of hyperbolic geometry and the natural linear action of f on a suitable hyperbolic space of infinite dimension. This space is constructed as a set of cohomology classes in the Riemann-Zariski space of X and was introduced by Cantat [Can11]. Unfortunately, such a space can only be constructed in dimension two. Also the coefficient $2^{\frac{1}{2}} \times 3^{18}$ is quite large. In this paper, we give a new proof of Theorem 1.5 with a better coefficient i.e. from $2^{\frac{1}{2}} \times 3^{18}$ to 4.

Theorem 1.6. *Let* $\mathbf{k}$ *be a field. Let* X *be a projective surface over* $\mathbf{k}$. *Let* $f : X \dashrightarrow X$ *be a dominant rational self-map. Let* L *be a big and nef line bundle. Then we have*

$$\lambda_1(f) \geq \frac{\deg_{1,L} f^2}{4 \deg_{1,L} f}.$$

The proof of Theorem 1.6 does not rely on hyperbolic geometry and is much simpler (c.f. Section 5.3).

1.3. **Lower semi-continuity of dynamical degrees.** Besides the the dynamical degrees of a single map, we also study the behavior of the dynamical degree in families.

Let S be an integral noetherian scheme and $d \in \mathbb{Z}_{\geq 0}$. Initially, a family of self-maps on S should be a collection of dominant rational self-maps $f_p : X_p \dashrightarrow X_p, p \in S$ on varieties $p \in S$. So we introduce the following definition.

Definition 1.7. A *family of* d*-dimensional dominant rational self-maps on* S is a flat and projective scheme $\pi : \mathcal{X} \to S$ satisfying $\dim \mathcal{X}/S = d$ with a dominant rational self-map $f : \mathcal{X} \dashrightarrow \mathcal{X}$ over S such that for every $p \in S$,

(i) the fiber X_p of π at p is geometrically reduced and irreducible;
(ii) $X_p \not\subseteq I(f)$;
(iii) the induced map $f_p : X_p \dashrightarrow X_p$ is dominant.

We prove the following result in Section 6.

Theorem 1.8. *Let* S *be an integral noetherian scheme and* $\pi : \mathcal{X} \to S$ *be a flat and projective scheme over* S *with* $\dim \mathcal{X}/S = d$. *Let* $f : \mathcal{X} \dashrightarrow \mathcal{X}$ *be a family of* d*-dimensional dominant rational self-maps on* S. *Then for every* $i = 0, \ldots, d$, *the function* $p \in S \mapsto \lambda_i(f_p)$ *is lower semi-continuous in the Zariski topology on* S.

A special case of Theorem 6.1 is the following result.

Corollary 1.9. *Let* $f : \mathbb{P}^d_{\mathbb{Z}} \dashrightarrow \mathbb{P}^d_{\mathbb{Z}}$ *be a dominant rational self-map over* $\mathbb{Z}$. *Then for every* $i = 0, \ldots, d$, *we have*

$$\lambda_i(f \otimes_{\mathbb{Z}} \mathbb{Q}) = \lim_{p \text{ prime}, p \to \infty} \lambda_i(f_p).$$

Theorem 1.8 generalizes [Xie15, Theorem 4.3] from dimension two to any dimension. The special case where $i = 1$ and $\mathcal{X} = \mathbb{P}^N_S$ of Theorem 1.8 implies Call-Silverman's conjecture [SC18, Conjecture 1] and its generalized version [BIJ$^+$19, Conjecture 14.13]. Corollary 1.9 gives a positive answer to [BIJ$^+$19, Question 14.10].

In [Xie15, Section 4.3], the author provided the following example showing that Corollary 1.9 cannot be strengthened to the statement that $\lambda_i(f \otimes_{\mathbb{Z}} \mathbb{Q}) = \lambda_i(f_p)$ for infinitely many prime p.

Example 1.10. Let $f : \mathbb{P}^2_{\mathbb{Z}} \dashrightarrow \mathbb{P}^2_{\mathbb{Z}}$ be the rational self-map sending $[x : y : z]$ to $[xy : xy - 2z^2 : yz + 3z^2]$. Then $\lambda_1(f \otimes_{\mathbb{Z}} \mathbb{Q}) = 2$, but $\lambda_1(f_p) < 2$ for all primes p.

Strategy of the proof. There are three steps in the proof. In the first step, we get a simple criterion for lower semi-continuity functions on noetherian schemes (c.f. Lemma 6.2). Next we show that the mixed degrees are lower semi-continuous (c.f. Lemma 6.5). This step can be shown using our criterion Lemma 6.2, Raynaud-Gruson flattening theorem [RG71, Theorem 5.2.2], and the constancy of intersection numbers on flat families [Ful84, Proposition 10.2]. The function $p \in S \mapsto \lambda_i(f_p)$ of the i-th dynamical degree is the point-wise limit of the functions $p \in S \mapsto (\deg_{i,L_p} f_p)^{1/n}$. By the second step, the later function is lower semi-continuous. However, as shown in Remark 6.3, limit of lower semi-continuous functions may not be lower semi-continuous. To complete the proof of Theorem 6.1, we need to show that the dynamical degrees are continuous at the generic point of S. In this step, the main ingredient is the lower bounds of the dynamical degrees obtained in Section 3.

1.4. **Periodic points.**

One of the most basic problem in algebraic dynamics is to determine when the set of periodic points is Zariski dense.

Question 1.11. Under which condition does f admit a Zariski dense set of periodic points?

We give a positive answer to Question 1.11 for cohomologically hyperbolic self-maps in Section 7.

Theorem 1.12. *If f is cohomologically hyperbolic i.e. there is a unique $i \in \{1, \dots, d\}$ such that $\lambda_i(f) = \max_{j=0,\dots,d} \lambda_j(f)$, then the set of periodic closed points is Zariski dense.*

We indeed prove a stronger statement in Theorem 7.1.

When f is not cohomologically hyperbolic, the answer to Question 1.11 can be either positive or negative. Indeed, in Section 7.4, we give examples to show that for cohomologically non-hyperbolic maps, one can not determine whether the set of periodic points are Zariski dense from their dynamical degrees.

Historical notes. The first fundamental result for Question 1.11 is the positive answer for polarized endomorphisms[2], which can be achieved both by analytic and algebraic method.

Suppose that X is smooth projective over $\mathbb{C}$ and f is a polarized endomorphism. Using complex analytic methods, Briend-Duval [BD01] and subsequently Dinh-Sibony [DS10] have proved that the set of periodic points is Zariski dense in X. By the Lefschetz principle, these results hold true whenever $\mathbf{k}$ has characteristic zero. Later, Hrushovski and Fakhruddin [Fak03] gave a purely algebraic proof of the Zariski density of periodic points over any algebraically closed field[3].

The complex analytic methods alluded to above have been used to give a positive answer to Question 1.11 for several other classes of maps. For example, building on the work of Guedj [Gue05b], Dinh, Nguyên and Truong [TCD] proved it when f is $(\dim X)$-cohomologically hyperbolic. See [DS, BD05, Duj06, DDG10, BLS93, JR18] for other cases obtained using complex analytic method. All these cases are cohomologically hyperbolic, hence implied by our Theorem 1.12. However, when the complex analytic method works, usually it not only proves the Zariski density of periodic points, but also show that the periodic points equidistribute to the maximal entropy measure in the complex topology.

In [Xie15, Theorem 1.1], the author classified the birational self-maps on surfaces whose periodic points are not Zariski dense by algebraic method. The essential step of [Xie15, Theorem 1.1], is the case where $\lambda_1(f) > 1$ (hence cohomologically hyperbolic). An advantage of the algebraic method is that it works in arbitrary characteristic.

Strategy of the proof. We follow the original method of Hrushovski and Fakhruddin [Fak03] by reducing our result to the case of finite fields. This method was also used in the proof of [Xie15, Theorem 1.1].

For the sake of simplicity, we shall assume that $X = \mathbb{P}^N$ and $f = [f_0 : \cdots : f_N]$ is a rational self-map having integral coefficients. Assume that f is cohomologically hyperbolic. By Corollary 1.9, we can find a prime $p \geq 2$ such that the reduction f_p modulo p of f is cohomologically hyperbolic. Then the set of periodic closed points of f_p is Zariski dense in $\mathbb{P}^N(\overline{\mathbb{F}_p})$ by an argument of Fakhruddin based on Hrushovski's twisted Lang-Weil estimate c.f. Theorem 7.3. We show that most of the f_p-periodic points are "isolated" in a certain sense (c.f. Corollary 7.6). The main ingredient of this step is a recursive inequality proved in Theorem 3.6 and [MW, Proposition 3.5]. By Lemma 7.4 (which generalizes [Fak03, Theorem 5.1]), we can lift isolated periodic points from the special fiber to the generic fiber. This concludes the proof.

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[2]Recall that an endomorphism f on a projective variety X is said to be *polarized* if there exists an ample line bundle L on X satisfying $f^*L = qL$ for some integer $q \geq 2$. In this case, $\lambda_i(f) = q^i$ for $i = 0, \ldots, d$. Hence polarized endomorphisms are cohomologically hyperbolic.

[3]Their proof indeed works for amplified endomorphisms which are more general than polarized ones.

1.5. **Uniform growth of the degrees.** For two sequences $a_n, b_n \in [1, +\infty)$, we say that a_n and b_n are equivalent if there is $C > 1$ such that

$$C^{-1}a_n \leq b_n \leq Ca_n.$$

Define

$$\rho(a_n, n \geq 0) := \lim_{n\to\infty} a_n^{1/n}$$

if the limit exists. For equivalent sequences, the above exist in the same time and take the same value.

Let $f : X \dashrightarrow X$ be a dominant rational self-map and let L be a big and nef line bundle on X. Then for $i = 0, \dots, \dim X$, the i-th degree sequence $\deg_{i,L} f^n, n \geq 0$ does not depend on the choice of L up to equivalence. It is natural to ask

Question 1.13. Which class of sequences can be realized as an i-th degree sequence for some X, f and $i = 0, \dots, \dim X$?

A well-known fact is that an i-th degree sequence is equivalent to a submultiplicative sequence. Indeed, by Lemma 3.2, for every $m, n \geq 0$, we have

$$\deg_i(f^{m+n}) \leq \frac{\binom{d}{i}}{(L^d)} \deg_i(f^m)\deg_i(f^n). \tag{1.3}$$

Hence the sequence $\frac{\binom{d}{i}}{(L^d)} \deg_i(f^n), n \geq 0$ is submultiplicative. By Fekete's lemma, if a sequence a_n is equivalent to some submultiplicative sequence, $\rho(a_n, n \geq 0)$ is well-defined. This is the reason why the dynamical degrees are well-defined.

In this paper, we give a further obstacle for a submultiplicative sequence to be a degree sequence. We show that, after a large iterate, the degree grows almost at a uniform rate.

Corollary 1.14. *Let $f : X \dashrightarrow X$ be a dominant rational self-map and let L be a big and nef line bundle on X. For every $i = 1, \dots, d$, $\delta \in (0, 1)$, there is $m_\delta \geq 1$ such that for every $m \geq m_\delta$ and $n \geq 0$, we have*

$$\delta^m \lambda_i^m(f) \leq \frac{\deg_{i,L}(f^{m(n+1)})}{\deg_{i,L}(f^{mn})} \leq \delta^{-m}\lambda_i(f)^m.$$

The statement of Corollary 1.14 for a sequence is invariant under the equivalence relation. In particular, it does not depend on the choice of L. In Section 4.1, we give an example to show that this Corollary is not satisfied for general submultiplicative sequences.

We will prove Corollary 1.14 in Section 4. The proof is based on the recursive inequalities for degree sequences constructed in Section 3.

1.6. **Kawaguchi-Silverman conjecture.** Assume that $\mathbf{k} = \overline{\mathbb{Q}}$. Let X be a projective variety over $\overline{\mathbb{Q}}$ and let L be an ample line bundle on X. We denote by $h_L : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ a Weil height associated to L c.f. [HS00, BG06]. It is unique up to adding a bounded function. Set $h_L^+ := \min\{1, h_L\}$.

Let $X_f(\overline{\mathbb{Q}})$ be the set of points $x \in X(\overline{\mathbb{Q}})$ whose orbit is well-defined i.e. $f^n(x) \notin I(f)$ for every $n \geq 0$. In [KS16], Kawaguchi and Silverman introduced the fundamental notion of arithmetic degree to describe the arithmetic complexity of an orbit. For $x \in X(\overline{\mathbb{Q}})$, the *upper/ lower arithmetic degree* for X, f, x are

$$\overline{\alpha}_f(x) := \limsup_{n\to\infty} h_L^+(f^n)^{1/n} \text{ and } \underline{\alpha}_f(x) := \liminf_{n\to\infty} h_L^+(f^n)^{1/n}.$$

If $\overline{\alpha}_f(x) = \underline{\alpha}_f(x)$, we set

$$\alpha_f(x) := \overline{\alpha}_f(x) = \underline{\alpha}_f(x).$$

In this case, we say that $\alpha_f(x)$ is well-defined and call it the *arithmetic degree* of f at x.

The following conjecture was proposed by Kawaguchi and Silverman [Sil14, KS16]. It connects the arithmetic degree with the first dynamical degree.

Conjecture 1.15 (Kawaguchi-Silverman conjecture)**.** Let X be a projective variety over $\overline{\mathbb{Q}}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map. Then for every $x \in X_f(\overline{\mathbb{Q}})$, $\alpha_f(x)$ is well defined. Moreover, if $O_f(x)$ is Zariski dense, then we have $\alpha_f(x) = \lambda_1(f)$.

The general form of the Kawaguchi-Silverman conjecture is wildly open. However many special cases are known especially when f is well-defined everywhere. When f is a polarized endomorphism, the Kawaguchi-Silverman conjecture is implied by the Northcott property. It was completely solved when X is a projective surface and f is a surjective endomorphism by Kawaguchi, Silverman, Matsuzawa, Sano, Shibata, Meng and Zhang [Kaw08, KS14, MSS18, MZ22]. Except Kawaguchi's automorphism case, the proof heavily relies on classification (or minimal model theory) of surfaces. In higher dimension, serval cases are proven by minimal model theory. These results are on surjective endomorphisms on projective varieties (c.f. [MZ23]). Few results are known when f is merely rational. The following are two remarkable cases.

(1) Conjecture 1.15 was proved for regular affine automorphisms on $\mathbb{A}^N$ by Kawaguchi [Kaw06, Kaw13].
(2) Conjecture 1.15 holds by Matsuzawa and Wang [Wan23, MW] when X is a smooth projective variety, f is a 1-cohomologically hyperbolic rational map, and the f-orbit of x is generic i.e. for every proper subvariety Y of X, the set $\{n \geq 0 |\ f^n(x) \in Y\}$ is finite.

The Dynamical Mordell-Lang conjecture proposed by Ghioca and Tucker asserts that for every $x \in X_f(\overline{\mathbb{Q}})$, if $O_f(x)$ is Zariski dense, then the f-orbit of x is *generic* c.f. [GT09] (see also [Xie23a, Conjecture 1.2]). So (2) implies that the Kawaguchi-Silverman conjecture for 1-cohomologically hyperbolic self-maps assuming the Dynamical Mordell-Lang conjecture. For more results, see [Mat23] and the references therein.

In Section 8, we prove the following result (the case $\lambda_2(f) = \lambda_1(f)^2$ was already proved by Wang and Matsuzawa [MW, Theorem 1.17]).

Theorem 1.16. *Let X be a projective surface over $\overline{\mathbb{Q}}$ and $f : X \dashrightarrow X$ be a dominant rational self-map such that $\lambda_1(f) > \lambda_2(f)$ or $\lambda_2(f) = \lambda_1(f)^2$. Let $x \in X_f(\overline{\mathbb{Q}})$. If the orbit $O_f(x)$ of x is Zariski dense, then $\alpha_f(x) = \lambda_1(f)$.*

In particular, Theorem 1.16 implies the Kawaguchi-Silverman conjecture for birational self-maps on projective surfaces. We first explain how to prove Theorem 1.16 assuming the Dynamical Mordell-Lang conjecture (which was done in (2)). In the proof of (2), Matsuzawa and Wang construct some recursive inequality for the heights $h(f^n(x))$ when $f^n(x)$ is not contained in the base locus B of some big line bundle. Applying the Dynamical Mordell-Lang conjecture, one can show that the orbit meets B in at most finitely many times. So we may ignore the base locus and assume that the recursive inequality holds for all n. This implies our result easily. As the Dynamical Mordell-Lang conjecture is wildly open in general, we can not ignore the base locus. This is the main difficulty of our proof. Our idea is to construct a weaker recursive inequality when $f^n(x)$ is contained in B. We then combine this inequality with the one when $f^n(x) \in B$ to get a lower bound of the growth of the height. We apply the Weak dynamical Mordell-Lang [BGT15, Corollary 1.5] (see also [Fav00, Theorem 2.5.8], [Gig14, Theorem D, Theorem E],[Pet15, Theorem 2], [BHS20, Theorem 1.10], [Xie23b, Theorem 1.17] and [Xie23a, Theorem 5.2]) to show that the density of n with $f^n(x) \in B$ is zero. Using this, one can show that we can ignore B asymptotically and conclude the proof.

1.7. **Further problems.** Though our Theorem 1.4 gives an algorithm to compute the dynamical degrees to any given precision, it seems that our algorithm is not so efficient. Either theoretically, or practically, it is interesting to have a more efficient algorithm. In a private communication with Silverman, he asked the following more precise question:

Question 1.17. Is there an algorithm to compute the dynamical degrees $\lambda_i(f)$ to within $1/2^l$ using only $O(l^e)$ storage for some "not too large" e?

Blanc, Cantat and McMullen [McM07, Can11, BC16] showed that there is a gap on the first dynamical degree for surface birational self-maps. More precisely, for every surface birational self-map f, we have $\lambda_1(f) \notin (1, \lambda_L)$, where $\lambda_L \simeq 1.176280$ is the Lehmer number i.e. the unique root > 1 of the irreducible polynomial $x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$. This result relies on the existence of algebraically stable models for surface birational self-maps proved by Diller-Favre [DF01]. It is interesting to ask whether such a gap exists for general rational self-maps and for higher dynamical degrees.

Question 1.18. Is there a $\lambda > 1$ depending on $d \geq 1$ and $i \in \{1, \dots, d\}$ such that for every dominant rational self-maps f on a d-dimensional projective variety X, we have $\lambda_i(f) \notin (1, \lambda)$?

When $i = d$, Question 1.18 has positive answer by taking $\lambda = 2$. Corollary 6.8 gives positive answer to Question 1.18 for self-maps coming from a given family. In particular, Corollary 6.8 shows that for every $d \geq 1$, $D \geq 1$, there is $\lambda > 1$ depending on d and D, such that for every $i = 0, \dots, d$ and every dominant rational self-map f of $\mathbb{P}^d$ with $\deg_{1,O(1)} f \leq D$, we have $\lambda_i \notin (1, \lambda)$.

The same question can be asked for the cohomological Lyapunov exponents.

Question 1.19. Is there a $\mu > 1$ depending on $d \geq 1$ and $i \in 1, \ldots, d$ such that for every dominant rational self-maps f on a d-dimensional projective variety X, we have $\mu_i(f) \notin (\mu^{-1}, 1) \cup (1, \mu)$.

Acknowledgement. The author would like to thank Long Wang for interesting discussion on the Kawaguchi-Silverman conjecture. The author would like to thank Zhiqiang Li and Xianghui Shi for interesting discussion on computability theory. The author would like to thank Tien Cuong Dinh, Keiji Oguiso, Mattias Jonsson, Claude-Michel Viallet, Guolei Zhong and Fei Hu for helpful comments on the first version of this work. The author would like to thank Joseph Silverman for proposing Question 1.17. The author would like to thank Serge Cantat for his careful reading of first version and his helpful comments.

2. Birational models

In this section, we introduce the terminology of cocycles on birational models. This terminology is not completely necessary for our paper, but it naturally fits our setting and it simplifies the notations a lot and makes the presentations clearer.

Let $\mathbf{k}$ be a field. Let X be a projective variety of dimension d over $\mathbf{k}$. A *birational model* of X is a projective variety X_π with a birational morphism $\pi : X_\pi \to X$. For two birational models X_π and $X_{\pi'}$, we say that $X_{\pi'}$ *dominates* X_π and write $X_{\pi'} \geq X_\pi$ if the birational map $\mu := \pi^{-1} \circ \pi' : X_{\pi'} \to X_\pi$ is a morphism.

2.1. Line bundles.

Let $\widetilde{\mathrm{Pic}}(X)$ and $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$ be the inductive limits

$$\widetilde{\mathrm{Pic}}(X) := \varinjlim_{\pi} \mathrm{Pic}(X_\pi)$$

and

$$\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}} := \varinjlim_{\pi} \mathrm{Pic}(X_\pi)_{\mathbb{R}}.$$

with respect to pullback arrows. In particular, $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}} = \widetilde{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. To simplify the notations, for $L \in \widetilde{\mathrm{Pic}}(X)$, we still denote by L its image in $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$.

For every element $L \in \widetilde{\mathrm{Pic}}(X)$ (resp. $L \in \widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$) there is a birational model X_π of X such that L is represented by some $L_\pi \in \mathrm{Pic}(X_\pi)$ (resp. $L_\pi \in \mathrm{Pic}(X_\pi)_{\mathbb{R}}$); we say that L is defined on X_π by $(\pi^{-1} \circ \pi')^* L_\pi$. For every $X_{\pi'} \geq X_\pi$, L is also defined on $X_{\pi'}$. We say that $L \in \widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$ is big (resp. nef, effective, pseudo-effective) if it is represented by $L_\pi \in \mathrm{Pic}(X_\pi)_{\mathbb{R}}$ for some birational model X_π of X such that L_π is big (resp. nef, effective, pseudo-effective). For $L, M \in \widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$, write $L >_{big} M$ if $L - M$ is big and $L \geq M$ if $L - M$ is pseudo-effective.

For $L \in \widetilde{\mathrm{Pic}}(X)$ and every birational model X_π of X, define the *stable base locus* as follows: Pick any model $\pi_0 : X_{\pi_0} \to X$ such that L is defined on X_{π_0}, X_{π_0} is normal and X_{π_0} dominates X_π. Define $\mathbf{B}_{X_{\pi_0}}(L) := \cap_{n \geq 0} Bs_{X_{\pi_0}}(nL)$ where $Bs_{X_{\pi_0}}(\cdot)$ is the usual base locus, and let $\mathbf{B}_{X_\pi}(L)$ be the image of $\mathbf{B}_{X_{\pi_0}}(L)$ in X_π.

It is easy to check that this definition does not depend on the choice of π_0. If L is effective, then $\mathbf{B}_{X_\pi}(L) \neq X_\pi$.

Let C be a curve in X and $L \in \mathrm{Pic}(X)_{\mathbb{R}}$. Assume that L is represented by some $L_\pi \in \mathrm{Pic}(X_\pi)_{\mathbb{R}}$ on some birational model X_π of X such that C is not contained in the indeterminacy locus $I(\pi^{-1})$ of $\pi^{-1} : X \dashrightarrow X_\pi$. Let $C_\pi := \overline{\pi^{-1}(C \setminus I(\pi^{-1}))}$ be the strict transform of C by π. We define $(C \cdot L)$ to be $(C_\pi \cdot L_\pi)$. It is easy to see that this definition does depend on the choice of X_π, L_π.

Let $f : X \dashrightarrow X$ be a rational self-map and C be a curve in X. Assume that C is not contained in $I(f)$. Define $f_*(C)$ as follows: Pick a birational model X_π of X such that $f_\pi := f \circ \pi : X_\pi \to X$ is a morphism and $C \not\subseteq I(\pi^{-1})$. Such a model exists, as we can pick X_π to be the graph of f in $X \times X$ and let π be the projection to the first factor. Define $f_*(C) := (f_\pi)_*(C_\pi)$. This definition does not depend on the choice of X_π. Let $M \in \mathrm{Pic}(X)_{\mathbb{R}}$. We pick X_π as above. Then M is defined on X_π. The previous paragraph shows that the intersection $(f^*(M)\cdot C)$ is well-defined and is equal to $(C_\pi \cdot f_\pi^* M)$. By projection formula, we get

$$(f^*(M) \cdot C) = (M \cdot f_*(C)). \tag{2.1}$$

2.2. **Cocycles.** For $i = 0, \dots, d$, let $\widetilde{\mathrm{CH}}^i(X)_{\mathbb{R}}$ be the inductive limit

$$\widetilde{\mathrm{CH}}^i(X)_{\mathbb{R}} := \varinjlim_{\pi} \mathrm{CH}^i(X_\pi)_{\mathbb{R}}$$

with respect to pullback arrows, where $\mathrm{CH}^i(\cdot)$ is the Chow group of degree i cocycles [Ful84]. In particular, $\widetilde{\mathrm{CH}}^1(X)_{\mathbb{R}} = \widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$.

For $L_1, \dots, L_i \in \widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$ and $Z \in \widetilde{\mathrm{CH}}^j(X)_{\mathbb{R}}$, we define the intersection $L_1 \cdots L_i \cdot Z \in \widetilde{\mathrm{CH}}^{i+j}(X)_{\mathbb{R}}$ as follows: There is a birational model X_π of X such that $L_1, \dots, L_i$ and Z are all defined on X_π. Let $L_{1,\pi}, \dots, L_{i,\pi} \in \mathrm{Pic}(X_\pi)$ and $Z_\pi \in \mathrm{CH}^j(X_\pi)_{\mathbb{R}}$ represent $L_1, \dots, L_i$ and Z. Define $L_1 \cdots L_i \cdot Z$ to be the element in $\widetilde{\mathrm{CH}}^{i+j}(X)_{\mathbb{R}}$ represented by $L_{1,\pi} \cdots L_{i,\pi} \cdot Z_\pi \in \mathrm{CH}^{i+j}(X_\pi)$. This definition does not depend on the choice of the birational model X_π.

For $P \in \widetilde{\mathrm{CH}}^d(X)_{\mathbb{R}}$, define (P) to be the degree of P_π where $P_\pi \in CH^d(X_\pi)$ for some birational model X_π of X which defines P. This does not depend on the choice of birational model X_π.

For $Z, W \in \widetilde{\mathrm{CH}}^i(X)_{\mathbb{R}}$, write $Z \geq_n W$ if for every $(d-i)$-tuple of nef line bundles $H_1, \dots, H_{d-i}$ in $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$, $((Z - W) \cdot H_1 \cdots H_{d-i}) \geq 0$. Write $Z >_n W$ if for some big and nef line bundle $L \in \widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$, $Z \geq_n W + L^i$.

2.3. **Siu's inequalities.** Siu's inequality [Laz04, Theorem 2.2.13] for nef line bundles is useful in our paper. For the convenience of the applications, we write it in the following form for nef line bundles in $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$.

Theorem 2.1. *Let L, M be nef line bundles in $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$. Assume that $(M^d) > 0$, then*

$$L \leq d\frac{(L \cdot M^{d-1})}{(M^d)}M.$$

In particular, for every $\epsilon \in (0,1)$, $\epsilon L <_{big} d\frac{(L\cdot M^{d-1})}{(M^d)}M$.

Applying Siu's inequality inductively, Dang proved a version of Siu's inequality in arbitrary codimension [Dan20, Corollary 3.4.6]. In [JL23, Theorem 3.5], Jiang and Li give another proof using (multipoint) Okounkov bodies and get the optimal coefficient. For the convenience of the applications, we write [JL23, Theorem 3.5] in the following form for nef line bundles in $\widetilde{\mathrm{Pic}}(X)$.

Theorem 2.2. *Let $i = 0, \dots, d$, and $L_1, \dots, L_i, M$ be nef line bundles in $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$. Then we have*

$$(M^d)L_1 \cdots L_i \leq_n \binom{d}{i}(L_1 \cdots L_i \cdot M^{d-i})M^i$$

i.e. for every $(d-i)$-tube of nef line bundles $H_1, \dots, H_{d-i}$ in $\widetilde{\mathrm{Pic}}(X)_{\mathbb{R}}$, we have

$$(L_1 \cdots L_i \cdot H_1 \cdots H_{d-i})(M^d) \leq \binom{d}{i}(L_1 \cdots L_i \cdot M^{d-i})(M^i \cdot H_1 \cdots H_{d-i}).$$

In particular, for every $\epsilon \in (0,1)$,

$$\epsilon(M^d)L_1 \cdots L_i <_n \binom{d}{i}(L_1 \cdots L_i \cdot M^{d-i})M^i.$$

3. Recursive inequalities for degree sequences

Let $\mathbf{k}$ be a field. Let X be a projective variety of dimension d over $\mathbf{k}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map. Let L be a big and nef line bundle in $\widetilde{\mathrm{Pic}}(X)$.

To simplify the notations, we write

$$\lambda_i := \lambda_i(f), \mu_i := \mu_i(f), \deg_i f^n := \deg_{i,L} f^n, \text{ and } L_n := (f^n)^*L$$

for every $n \geq 0$.

3.1. **A lemma on recursive inequalities.** The following simple lemma on recursive inequalities is useful.

Lemma 3.1. *Let $a_n, n \geq 0$ be a sequences of non-negative real numbers. Let α, β, γ be real numbers with $\alpha \geq 0$ and $\gamma \geq \alpha + \beta$. Assume that $a_1 > \beta a_0$ and*

$$a_{n+2} + \alpha\beta a_n \geq \gamma a_{n+1}$$

for every $n \in \{0, \dots, N\}$ where $N \in \mathbb{Z}_{\geq 0} \cup \{+\infty\}$. Then for every $n \in \{0, \dots, N\}$, we have

$$a_{n+2} > \beta a_{n+1} \text{ and } a_{n+2} \geq \alpha^n(a_1 - \beta a_0).$$

In particular, if $N = +\infty$, then $\liminf_{n\to\infty} a_n^{1/n} \geq \alpha$.

Proof. As $\gamma \geq \alpha + \beta$ and $a_{n+1} \geq 0$, we have $a_{n+2} + \alpha\beta a_n \geq (\alpha + \beta)a_{n+1}$. Hence we have $(a_{n+2} - \beta a_{n+1}) \geq \alpha(a_{n+1} - \beta a_n)$, which concludes the proof. □

3.2. **Mixed degrees.** Let $s\geq 1$, consider two sequence of non-negative integers: $m_1>\cdots>m_s\geq 0$ and $r_1,\dots,r_s\geq 0$ with $\sum_{i=1}^s r_i=d$. We will compute the *mixed degree*, which is defined to be

$$(L_{m_1}^{r_1}\cdots L_{m_s}^{r_s}).$$

Our computation is based on a direct application of the higher codimensional Siu's inequality.

Lemma 3.2. *Let $r_1,r_2\geq 0$ with $r_1+r_2\leq d$. Let A be a product of $d-r_1-r_2$ nef line bundles in $\widetilde{\mathrm{Pic}}(X)$. Let $n_1,n_2\geq 0$ and $0\leq t\leq r_1$. If $n_1\geq n_2$, then we have*

$$(L_{n_1}^{r_1}\cdot L_{n_2}^{r_2}\cdot A)\leq \binom{d-r_1-r_2+t}{t}\frac{\deg_{r_1}f^{n_1-n_2}}{\deg_{r_1-t}f^{n_1-n_2}}(L_{n_1}^{r_1-t}\cdot L_{n_2}^{r_2+t}\cdot A);$$

if $n_2\geq n_1$, then we have

$$(L_{n_1}^{r_1}\cdot L_{n_2}^{r_2}\cdot A)\leq \binom{d-r_1-r_2+t}{t}\frac{\deg_{d-r_1}f^{n_2-n_1}}{\deg_{d-r_1+t}f^{n_2-n_1}}(L_{n_1}^{r_1-t}\cdot L_{n_2}^{r_2+t}\cdot A).$$

Proof. Up to some small pertubations of L of the form $L+\epsilon H$ for some H ample and positive rational $\epsilon\to 0$, we can suppose that L is ample on X. After replacing L by some positive multiple, we may further assume that L is very ample.

Replace X by a sufficiently large model, we may assume that L_{n_1} and L_{n_2} are defined over X. Hence they are generated by global sections. Let V be the intersection of (r_1-t) general sections of L_{n_1} and r_2 general sections of L_{n_2}. Then $V=L_{n_1}^{r_1-t}\cdot L_{n_2}^{r_2}$ in $\mathrm{CH}^{r_1+r_2-t}(X)$.

By Theorem 2.2, we have

$$L_{n_1}^t|_V\leq \binom{d-r_1-r_2+t}{t}\frac{((L_{n_1}|_V)^s\cdot(L_{n_2}|_V)^{d-r_1-r_2})}{((L_{n_2}|_V)^{d-r_1-r_2+t})}L_{n_2}^t|_V.$$

Hence

$$L_{n_1}^t\cdot V\leq \binom{d-r_1-r_2+t}{t}\frac{(L_{n_1}^t\cdot L_{n_2}^{d-r_1-r_2}\cdot V)}{(L_{n_2}^{d-r_1-r_2+t}\cdot V)}L_{n_1}^{r_1-t}\cdot L_{n_2}^{r_2+t}.$$

Intersecting with A, we conclude the proof by the projection formula. □

Applying Lemma 3.2, we get upper and lower bounds on the mixed degrees.

Proposition 3.3. *Let $l_i:=r_1+\cdots+r_i$, we have*

$$(L_{m_1}^{r_1}\cdots L_{m_s}^{r_s})\leq (L^d)\prod_{i=1}^s\binom{d-r_{i+1}}{l_i}\prod_{i=1}^s\frac{\deg_{l_i,L}(f^{m_i-m_{i+1}})}{(L^d)}$$

and

$$\deg_d(f^{m_1})\prod_{i=1}^{s-1}\binom{d-l_i}{r_{i+1}}^{-1}\prod_{i=1}^{s-1}\frac{\deg_{l_i}(f^{m_1-m_{i+1}})}{\deg_{l_{i+1}(f^{m_1-m_{i+1}})}}\leq (L_{m_1}^{r_1}\cdots L_{m_s}^{r_s})$$

Proof. Apply the first inequality in Lemma 3.2 for $L_{m_1}^{r_1}$, $L_{m_2}^{r_2}$, $t = r_1$ and $A := L_{m_2}^{r_2} \cdots L_{m_s}^{r_s}$, we have

$$(L_{m_1}^{l_1} \cdots L_{m_s}^{l_s - l_{s-1}}) \le \binom{d-r_2}{l_1} \frac{\deg_{l_1}(f^{m_1-m_2})}{(L^d)} (L_{m_2}^{l_2} \cdots L_{m_s}^{l_s - l_{s-1}}).$$

We get the first inequality by induction.

Apply the first inequality in Lemma 3.2 for $L_{m_1}^{l_2}$, $L_{m_2}^{0}$, $t = r_2$ and $A := L_{m_3}^{l_3-l_2} \cdots L_{m_s}^{l_s-l_{s-1}}$, we have

$$(L_{m_1}^{l_2} \cdot L_{m_3}^{l_3-l_2} \cdots L_{m_s}^{l_s - l_{s-1}}) \le \binom{d-l_1}{r_2} \frac{\deg_{l_2}(f^{m_1-m_2})}{\deg_{l_1}(f^{m_1-m_2})} (L_{m_1}^{l_1} \cdots L_{m_s}^{l_s - l_{s-1}}).$$

Hence we have

$$(L_{m_1}^{l_1} \cdots L_{m_s}^{l_s - l_{s-1}}) \ge \binom{d-l_1}{r_2}^{-1} \frac{\deg_{l_1}(f^{m_1-m_2})}{\deg_{l_2}(f^{m_1-m_2})} (L_{m_1}^{l_2} \cdot L_{m_3}^{l_3-l_2} \cdots L_{m_s}^{l_s - l_{s-1}}).$$

We get the second inequality by induction, which concludes the proof. □

By Proposition 3.3, we get the following corollary directly.

Corollary 3.4. *For every $\delta \in (0,1)$, there is a constant $D_\delta \ge 1$ such that*

$$D_\delta^{-1} \delta^{m_1} \prod_{i=1}^{s} \lambda_{l_i}(f)^{m_i - m_{i+1}} \le (L_{m_1}^{r_1} \cdots L_{m_s}^{r_s}) \le D_\delta \delta^{-m_1} \prod_{i=1}^{s} \lambda_{l_i}(f)^{m_i - m_{i+1}}.$$

where $l_i := r_1 + \cdots + r_i$.

3.3. **Recursive inequalities.** For two functions $\theta_1, \theta_2 : \mathbb{Z}_{\ge 0} \to \mathbb{R}_{>0}$, define $\theta_1 \gtrsim \theta_2$ if

$$\limsup_{n\to\infty} (\theta_2/\theta_1)^{1/n} \le 1.$$

This defines a partial ordering on the space of functions from $\mathbb{Z}_{\ge 0}$ to $\mathbb{R}_{>0}$. Define $\theta_1 \approx \theta_2$ if $\theta_1 \gtrsim \theta_2$ and $\theta_2 \gtrsim \theta_1$. This is an equivalence relation.

Theorem 3.5. *For $r_1, r_2 \ge 0$, set*

$$t := \min\{i \ge 1 | \ \mu_{r_1+1+i}\mu_{r_1+r_2+1+i} < \mu_{r_1+1}\mu_{r_1+r_2+1}\}.$$

Set $\eta_{r_1,r_2} := \max\{\mu_{r_1+t}\mu_{r_1+r_2+1+t}, \mu_{r_1+t+1}\mu_{r_1+r_2+t}\}$. Then for every $\epsilon \in (0,1)$, there is $m_\epsilon > 0$, such that for every $m \ge m_\epsilon$,

$$\frac{(L_{2m} + \eta_{r_1,r_2}^m L)^{d-r_2} \cdot L_m^{r_2}}{\mu_{r_1+1}^m (L_{2m} + \eta_{r_1,r_2}^m L)^{d-r_2-1} \cdot L_m^{r_2+1}} > (d-r_2)\epsilon^m.$$

In particular, we have

$$L_{2m} \cdot L_m^{r_2} + \eta_{r_1,r_2}^m L \cdot L_m^{r_2} >_n \epsilon^m \mu_{r_1+1}^m L_m^{r_2+1}.$$

We note that η_{r_1,r_2} in Theorem 3.6 is strictly less than $\mu_{r_1+1}\mu_{r_1+r_2+1}$.

Proof. Set $\eta := \eta_{r_1,r_2}$. We note that μ_{r_1+1+i} and $\mu_{r_1+r_2+1+i}$ are constant when $i \in [0, t-1]$. In particular $\mu_{r_1+1} = \mu_{r_1+t}$.

The decreasing of μ_i implies that $\eta^{(d-r_2-j)m}\lambda_j^m\lambda_{j+r_2}^m, j = 0, \dots, d-r_2$ takes maximal value when $j = r_1 + t$. By Corollary 3.4, we have

$$\begin{aligned}(L_{2m} + \eta^m L)^{d-r_2} \cdot L_m^{r_2} &\approx \sum_{j=0}^{d-r_2} \eta^{(d-r_2-j)m}(L_{2m}^j \cdot L_m^{r_2} \cdot L^{d-j}) \\ &\approx \max_{j=0}^{d-r_2} \eta^{(d-r_2-j)m}\lambda_j^m\lambda_{j+r_2}^m \\ &\approx \eta^{(d-r_2-r_1-t)m}\lambda_{r_1+t}^m\lambda_{r_1+r_2+t}^m \qquad (3.1)\end{aligned}$$

By Corollary 3.4, we have

$$\begin{aligned}\mu_{r_1+1}^m(L_{2m} + \eta^m L)^{d-r_2-1} \cdot L_m^{r_2+1} &\approx \mu_{r_1+1}^m \sum_{j=0}^{d-r_2-1} \eta^{(d-r_2-1-j)m}(L_{2m}^j \cdot L_m^{r_2+1} \cdot L^{d-j}) \\ &\approx \mu_{r_1+1}^m \sum_{j=0}^{d-r_2-1} \eta^{(d-r_2-1-j)m}\lambda_j^m\lambda_{j+r_2+1}^m.\end{aligned}$$

The maximal taken when $j = r_1 + t - 1$. Then we have

$$\begin{aligned}\mu_{r_1+1}^m(L_{2m} + \eta^m L)^{d-r_2-1} \cdot L_m^{r_2+1} &\approx \mu_{r_1+1}^m\eta^{(d-r_2-r_1-t)m}\lambda_{r_1+t-1}^m\lambda_{r_1+t+r_2}^m \\ &= \mu_{r_1+t}^m\eta^{(d-r_2-r_1-t)m}\lambda_{r_1+t-1}^m\lambda_{r_1+t+r_2}^m \\ &= \eta^{(d-r_2-r_1-t)m}\lambda_{r_1+t}^m\lambda_{r_1+t+r_2}^m \qquad (3.2)\end{aligned}$$

By (3.1) and (3.2), we conclude the proof by Theorem 2.1. □

For every $i = 1, \dots, d$, define $U(i) := \max\{j = 0, \dots, d | \ \mu_j = \mu_i\}$. If we take $r_1 = i - 1$ and $r_2 = 0$ in Theorem 3.5, we have $\eta_{r_1,r_2} = \mu_i\mu_{U(i)+1}$ and we get the following special case for line bundles.

Theorem 3.6. *For $i = 1, \dots, d$ and every $\epsilon \in (0, 1)$, there is $m_\epsilon > 0$, such that for every $m \geq m_\epsilon$,*

$$L_{2m} + \mu_i^m\mu_{U(i)+1}^m L - \epsilon^m\mu_i^m L_m$$

is big.

A weaker version of Theorem 3.6 was proved in [MW, Proposition 3.5], when $\lambda_i = \max_{j=0}^d \lambda_j$.

Remark 3.7. As $\mu_i = \mu_{U(i)}$ and $\mu_{i+1} \geq \mu_{U(i)+1}$, Theorem 3.6 can be reformulated in the following form: For $i = 1, \dots, d$ and every $\epsilon \in (0, 1)$, there is $m_\epsilon > 0$, such that for every $m \geq m_\epsilon$,

$$L_{2m} + \mu_i^m\mu_{i+1}^m L - \epsilon^m\mu_i^m L_m$$

is big.

We define two families of conditions on X, f, L which depend only on some intersection numbers. Let $\alpha_1, \ldots, \alpha_d \in \mathbb{R}_{>0}$ be a sequence of decreasing numbers. Let $\gamma, \epsilon \in (0,1)$, and $m \in \mathbb{Z}_{\geq 1}$. Set $\beta_i := \prod_{i=1}^{i} \alpha_i$.

Definition 3.8. For $i = 1, \ldots, d$, we say that X, f, L has condition $I_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m)$ if for every $j = 0, \ldots, i-1$, we have

$$\frac{(L_{2m} + \alpha_{j+1}^m \alpha_i^m \gamma^m L)^{d-i+j+1} \cdot L_m^{i-j-1}}{\mu_{r_1+1}^m (L_{2m} + \alpha_{j+1}^m \alpha_i^m \gamma^m L)^{d-i+j} \cdot L_m^{i-j}} > (d-i+j+1)\epsilon^m.$$

Definition 3.9. For $i = 1, \ldots, d$, define

$$B(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m) := \sum_{j=0}^{i-1} \binom{d}{j} \binom{d}{i-1} \frac{\alpha_i^m \gamma^m}{\epsilon^{m(j+1)} \beta_j^m} \frac{\deg_j(f^m) \deg_{i-1}(f^m)}{(L^d)^2}.$$

We say that X, f, L has condition $J_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m)$ if

$$B(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m) < \epsilon^{2mi} \beta_i^m (1 - \epsilon^{mi}).$$

By Theorem 2.1, If (X, f, L) has condition $I_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m)$, then

$$L_{2m} \cdot L_m^{i-j-1} + \mu_{j+1}^m \mu_i^m \gamma^m L_m^{i-j-1} \cdot L >_n \epsilon^m \mu_{j+1}^m L_m^{i-j}.$$

Definition 3.10. For $i = 1, \ldots, d$ and $N \geq 0$, we say that X, f, L has condition $K_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m; N)$ if

$$\deg_i(f^{m(N+1)}) > B(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m) \epsilon^{-mi} \deg_i(f^{mN}).$$

Condition $I_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m)$ and $J_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m)$ only depend on the top intersection numbers using L_{2m}, L_m, L. Condition $K_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m; N)$ only depends on the top intersection numbers of $L_{Nm}, L_{(N+1)m}, L_{2m}, L_m, L$.

Remark 3.11. If we fix X, f, L, i, m and n, $I_i(\alpha_1, \ldots, \alpha_d; \gamma, \epsilon, m)$, $J_i(\alpha_1, \ldots, \alpha_d; \gamma, \epsilon, m)$ and $K_i(\alpha_1, \ldots, \alpha_d; \gamma, \epsilon, m; n)$ are open condition on $(\alpha_1, \ldots, \alpha_d, \gamma, \epsilon)$.

Lemma 3.12. *If (X, f, L) has conditions $I_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m)$. Then for every $n \geq 0$, we have*

$$\deg_i(f^{m(n+2)}) + B\epsilon^{mi} \beta_i^m \deg_i(f^{mn}) \geq \epsilon^{mi} \beta_i^m \deg_i(f^{m(n+1)}),$$

where $B := B(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m)$. Assume further that the conditions $J_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m)$ and $K_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m; N)$ are satisfied for some $N \geq 0$. Then for every $n \geq 1$, we have

$$\deg_i(f^{m(N+n)}) - \epsilon^{-mi} B \deg_i(f^{m(N+n-1)}) \geq (\epsilon^{2mi} \beta_i^m)^{n-1} (\deg_i f^{m(N+1)} - \epsilon^{-mi} B \deg_i f^{mN}).$$

In particular, $K_i(\alpha_1, \ldots, \alpha_d; \gamma; \epsilon; m; N+n)$ is satisfied for every $n \geq 0$ and we have

$$\lambda_i \geq \epsilon^{2i} \beta_i.$$

Proof. As $I_i(\alpha_1,\dots,\alpha_d;\gamma;\epsilon;m)$ is satisfied, for every $j=0,\dots,i-1$, we have

$$L_{2m}^{j+1}\cdot L_m^{i-j-1}+\alpha_{j+1}^m\alpha_i^m\gamma^m L_{2m}^j\cdot L_m^{i-j-1}\cdot L>_{big}\epsilon^m\alpha_{j+1}^m L_{2m}^j L_m^{i-j}. \tag{3.3}$$

To simplify the notations, for $u,v,w\geq 0$ with $u+v+w\leq d$, write

$$D^{u,v,w}(n):=(L_{(2+n)m}^u\cdot L_{m(n+1)}^v\cdot L_{mn}^w\cdot L^{d-u-v-w}).$$

Apply $(f^n)^*$ to (3.3) and intersect them with L^{d-i}, we get

$$D^{j+1,i-j-1,0}(n)+\alpha_{j+1}^m\alpha_i^m\gamma^m D^{j,i-j-1,1}(n)>\epsilon^m\alpha_{j+1}^m D^{j,i-j,0}(n).$$

Dividing both side by $\epsilon^{(j+1)m}\beta_{j+1}^m$, we get

$$\frac{D^{j+1,i-j-1,0}(n)}{\epsilon^{(j+1)m}\beta_{j+1}^m}+\frac{\alpha_i^m\gamma^m D^{j,i-j-1,1}(n)}{\epsilon^{(j+1)m}\beta_j^m}>\frac{D^{j,i-j,0}(n)}{\epsilon^{jm}\beta_j^m}.$$

Then we get

$$\frac{D^{i,0,0}(n)}{\epsilon^{im}\beta_i^m}+\sum_{j=0}^{i-1}\frac{\alpha_i^m\gamma^m D^{j,i-j-1,1}(n)}{\epsilon^{(j+1)m}\beta_j^m}>D^{0,i,0}(n).$$

We note that

$$D^{i,0,0}(n)=\deg_i(f^{(n+2)m})\text{ and }D^{0,i,0}(n)=\deg_i(f^{(n+1)m}).$$

By Proposition 3.3, we have

$$D^{j,i-j-1,1}(n)\leq\binom{d}{j}\binom{d}{i-1}\frac{\deg_j(f^m)\deg_{i-1}(f^m)}{(L^d)^2}.$$

Then we get

$$\deg_i(f^{m(n+2)})+B(\alpha_1,\dots,\alpha_d;\gamma;\epsilon;m)\epsilon^{mi}\beta_i^m\deg_i(f^{mn})\geq\epsilon^{mi}\beta_i^m\deg_i(f^{m(n+1)}).$$

Now assume that conditions $J_i(\alpha_1,\dots,\alpha_d;\gamma;\epsilon;m),K_i(\alpha_1,\dots,\alpha_d;\gamma;\epsilon;m;N)$ are satisfied. Write

$$\phi:=\epsilon^{2mi}\beta_i^m\text{ and }\psi:=B(\alpha_1,\dots,\alpha_d;\gamma;\epsilon;m)\epsilon^{-mi}.$$

Condition $J_i(\alpha_1,\dots,\alpha_d;\gamma;\epsilon;m)$ implies that

$$\epsilon^{mi}\beta_i^m>\phi+\psi.$$

Condition $K_i(\alpha_1,\dots,\alpha_d;\gamma;\epsilon;m;N)$ implies that

$$\deg_i(f^{m(N+1)})>\psi\deg_i(f^{mN}).$$

We then conclude the proof by Lemma 3.1. □

Lemma 3.13. *Fix $i=1,\dots,d$. For every $j=0,\dots,i-1$, pick $\gamma\in[\max_{j=0}^{i-1}\frac{\eta_{j,i-j-1}}{\mu_{j+1}\mu_i},1)$. Then for every $\epsilon\in(\gamma^{\frac{1}{3d}},1)$ there is $m_\epsilon>0$, such that for every $m\geq m_\epsilon$, conditions $I_i(\mu_1,\dots,\mu_d;\gamma;\epsilon;m)$, $J_i(\mu_1,\dots,\mu_d;\gamma;\epsilon;m)$ hold for (X,f,L) and moreover, for every $N\geq 0$, $K_i(\alpha_1,\dots,\alpha_d;\gamma;\epsilon;m;N)$ is satisfied.*

Proof. The case $r_1 = j, r_2 = i - j - 1$ of Theorem 3.5 implies the condition $I_i(\mu_1, \dots, \mu_d; \gamma; \epsilon; m)$ holds for $m \gg 0$.

As

$$\begin{aligned}\binom{d}{j}\binom{d}{j-1}\frac{\mu_i^m\gamma^m}{\epsilon^{m(j+1)}\lambda_j^m}\frac{\deg_j(f^m)\deg_{i-1}(f^m)}{(L^d)^2} &\approx \frac{\mu_i^m\gamma^m}{\epsilon^{m(j+1)}\lambda_j^m}\lambda_j^m\lambda_{i-1}^m \\ &\lesssim \frac{\gamma^m}{\epsilon^{md}}\lambda_i^m,\end{aligned}$$

we get

$$B(\mu_1, \dots, \mu_d; \gamma; \epsilon; m) \lesssim \frac{\gamma^m}{\epsilon^{md}}\lambda_i^m. \tag{3.4}$$

Then we have

$$\frac{B(\mu_1, \dots, \mu_d; \gamma; \epsilon; m)}{\epsilon^{2mi}\lambda_i^m(1-\epsilon^{mi})} \lesssim (\frac{\gamma}{\epsilon^{3d}})^m.$$

Since $\frac{\gamma}{\epsilon^{3d}} < 1$, $J_i(\mu_1, \dots, \mu_d; \gamma; \epsilon; m)$ holds for $m \gg 0$.

By (3.4),

$$\frac{\deg_i(f^m)}{B(\alpha_1, \dots, \alpha_d; \gamma; \epsilon; m)\epsilon^{-mi}} \gtrsim \frac{\lambda_i^m\epsilon^{md}}{\frac{\gamma^m}{\epsilon^{md}}\lambda_i^m} = (\frac{\epsilon^{2d}}{\gamma})^m.$$

$K_i(\alpha_1, \dots, \alpha_d; \gamma; \epsilon; m; 0)$ is holds for $m \gg 0$. Hence there is $m_\epsilon > 0$, such that for every $m \geq m_\epsilon$, conditions $I_i(\mu_1, \dots, \mu_d; \gamma; \epsilon; m)$, $J_i(\mu_1, \dots, \mu_d; \gamma; \epsilon; m)$ and $K_i(\alpha_1, \dots, \alpha_d; \gamma; \epsilon; m; 0)$ hold for (X, f, L). By 3.12, for every $m \geq m_\epsilon$ and $N \geq 0$, $K_i(\alpha_1, \dots, \alpha_d; \gamma; \epsilon; m; N)$ holds. □

4. Uniform growth of the degrees

The following Fekete's lemma is well-known.

Fekete's Lemma. Let $A_n, n \geq 0$ be a submultiplicative sequence of positive real numbers, i.e. for every $m, n \geq 0$,

$$A_{n+m} \leq A_n A_m,$$

Then $\lim_{n\to\infty} A_n^{1/n}$ exists.

By [JL23, Corollary 1.3] (or Theorem 2.2), for every $m, n \geq 0$, we have

$$\deg_i(f^{m+n}) \leq \frac{\binom{d}{i}}{(L^d)} \deg_i(f^m)\deg_i(f^n). \tag{4.1}$$

In particular, the sequence $\frac{\binom{d}{i}}{(L^d)}\deg_i(f^n), n \geq 0$ is submultiplicative. This is the reason that the dynamical degrees are well-defined.

The aim of this section is to prove the following result, which shows that, after a large iterate, the degrees grows almost at a uniform rate. We will give an example to show that such a property is not satisfied for general submultiplicative sequences. We keep the notations of Section 3.

Corollary 4.1 (=Corollary 1.14). *For every $i = 1, \dots, d$, $\delta \in (0,1)$, there is $m_\delta \geq 1$ such that for every $m \geq m_\delta$ and $n \geq 0$, we have*

$$\delta^m \lambda_i^m \leq \frac{\deg_i(f^{m(n+1)})}{\deg_i(f^{mn})} \leq \delta^{-m}\lambda_i^m.$$

Proof. By Lemma 3.2, we have

$$\deg_i(f^{m(n+1)}) \leq \frac{\binom{d}{i}}{(L^d)} \deg_i(f^{mn}) \deg_i f^m.$$

As

$$\lambda_i = \lim_{m\to\infty} (\deg_i f^m)^{1/m},$$

we get the upper bound. We now prove the lower bound. The case $n = 0$ is trivial, we only need to show the case where $n \geq 1$.

Pick $\gamma \in [\max_{j=0}^{i-1} \frac{\eta_{j,i-j-1}}{\mu_{j+1}\mu_i}, 1)$ and $\epsilon_1, \epsilon_2 \in (\gamma^{\frac{1}{3d}}, 1)$ satisfying $\epsilon_1 < \epsilon_2$. For every $\epsilon \in (0,1)$, define

$$\phi(\epsilon, m) := \epsilon^{2mi}\lambda_i^m \text{ and } \psi(\epsilon, m) := B(\lambda_1, \dots, \lambda_d; \gamma; \epsilon; m)\epsilon^{-mi}.$$

The definition of $B(\lambda_1, \dots, \lambda_d; \gamma; \epsilon; m)$ shows that

$$\lim_{m\to\infty} \frac{\psi(\epsilon_2, m)}{\psi(\epsilon_1, m)} < 1. \tag{4.2}$$

There is $N_0 > 0$ such that for every $m \geq N_0$, $\frac{\psi(\epsilon_2,m)}{\psi(\epsilon_1,m)} < 1/2$. By Lemma 3.13, there is $N_1 \geq N_0$, such that for every $m \geq N_1$, conditions $I_i(\mu_1, \dots, \mu_d; \gamma; \epsilon_j; m)$, $J_i(\mu_1, \dots, \mu_d; \gamma; \epsilon_j; m)$ and $K_i(\alpha_1, \dots, \alpha_d; \gamma; \epsilon_j; m; 0)$, $j = 1, 2$ hold for (X, f, L). By Lemma 3.12, we have

$$\deg_i(f^{m(n+1)}) - \psi(\epsilon_1, m)\deg_i(f^{mn}) > 0$$

for every $n \geq 0$. Hence we have

$$\deg_i(f^{m(n+1)}) - \psi(\epsilon_2, m)\deg_i(f^{mn}) \geq \deg_i(f^{m(n+1)})(1 - \frac{\psi(\epsilon_2, m)}{\psi(\epsilon_1, m)}) > \deg_i(f^{m(n+1)})/2.$$

By Lemma 3.12, for every $n \geq 0$, we have

$$\begin{aligned}\deg_i(f^{m(n+2)}) \geq& \deg_i(f^{m(n+2)}) - \psi(\epsilon_2, m)\deg_i(f^{m(n+1)})\\ \geq& \psi_2(\epsilon, m)(\deg_i(f^{m(n+1)}) - \psi(\epsilon_2, m)\deg_i(f^{mn}))\\ \geq& \frac{\psi_2(\epsilon, m)}{2}\deg_i(f^{m(n+1)}).\end{aligned}$$

As $\epsilon^{2i} > \delta$, $\frac{\psi_2(\epsilon,m)}{2} > \delta^m\lambda_i^m$ for $m \gg 0$. This concludes the proof. □

In the rest of this section, we give an example to show that Corollary 4.1 does not hold for general submultiplicative sequences.

4.1. **An example of non-uniform growth.** We first define a sequence $a_n, n \geq 1$ inductively as follows: Define

$$a_1 = \cdots = a_6 = 1 \text{ and } a_7 = a_8 = a_9 = 0.$$

Note that $9 = 3! + 3$. Assume that a_n is already defined for $n \in [1, (r+2)! + r + 2]$ for some $r \geq 1$. Define

$$a_n := (1 - \frac{1}{(r+1)!})a_{(r+2)!}$$

for $n \in [(r+2)! + r + 3, (r+3)!]$ and

$$a_n = 0$$

for $n \in [(r+3)! + 1, (r+3)! + r + 3]$. It is easy to check that for every $n \geq 1$,

$$a_{n+1} \leq \frac{\sum_{j=1}^{n} a_j}{n}. \tag{4.3}$$

Hence the sequence $\frac{\sum_{j=1}^{n} a_j}{n}, n \geq 1$ is decreasing.

We claim that for every $r, s \in \mathbb{Z}_{\geq 1}$, we have

$$\sum_{j=1}^{s} a_j \geq \sum_{l=r}^{r+s-1} a_l. \tag{4.4}$$

We only need to check the case where $r \geq s+1$. By (4.3), for every $l \in [r, r+s-1]$, $a_l \leq (\sum_{j=1}^{s} a_j)/s$, which implies the claim.

For $n \geq 0$, set

$$A_n := 2^{\sum_{j=1}^{n} a_j}.$$

It is submultiplicative by (4.4) and

$$\lim_{n\to\infty} A_n^{\frac{1}{n}} = 2^{\prod_{r\geq 1}(1 - \frac{1}{(r+1)!})} > 1.$$

However, for every $m \geq 1$, there are infinitely many $n \geq 0$ such that

$$\frac{A_{m(n+1)}}{A_{mn}} = 1.$$

Indeed for every $r \geq 0$, we may take n above to be $\frac{(m+r)!}{m}$.

5. Algorithm to compute the dynamical degrees

We keep the notations of Section 3. Let $\mathbf{k}$ be a field. Let X be a projective variety of dimension d over $\mathbf{k}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map. Let L be a big and nef line bundle in $\widetilde{\mathrm{Pic}}(X)$.

The aim of this section is to give an algorithm to compute the dynamical degrees to arbitrary precision. In other words, we will give an algorithm, such that for any given number $l \in \mathbb{Z}_{>0}$, the algorithm gives a rational number $\tilde{\lambda}$ such that $\lambda_i \in (\tilde{\lambda}, \tilde{\lambda} + \frac{1}{2^l})$. For a given precision l, the algorithm will stop in finitely many steps and it only uses certain intersection numbers between $L_n, n \geq 0$.

5.1. **Upper and lower bounds.** By Lemma 3.2, we have the following well-known fact:

Fact 5.1. The sequence

$$(\frac{\binom{d}{i}}{(L^d)}\deg_i(f^n))^{1/n}, n\geq 0$$

tends to λ_i from above.

This controls the dynamical degrees from above. By Lemma 3.13, Lemma 3.12 and Remark 3.11, we get the following result, which controls the dynamical degrees from below.

Theorem 5.2. *For $i=1,\dots,d$, every $\beta\geq 1$, the following statements are equivalents:*

(i) $\lambda_i>\beta$;

(ii) *there are decreasing numbers $\alpha_1,\dots,\alpha_d\in\mathbb{R}_{>0}$, with $\beta_i:=\prod_{j=1}^i\alpha_j>\beta$, $\gamma\in(0,1)$, $\epsilon\in((\frac{\beta}{\beta_i})^{\frac{1}{2i}},1)$ and $m\in\mathbb{Z}_{\geq 1}$, such that the conditions $I_i(\alpha_1,\dots,\alpha_d;\gamma,\epsilon,m)$, $J_i(\alpha_1,\dots,\alpha_d;\gamma,\epsilon,m)$ and $K_i(\alpha_1,\dots,\alpha_d;\gamma,\epsilon,m;0)$ are satisfied for (X,f,L);*

(iii) *there are decreasing numbers $\alpha_1,\dots,\alpha_d\in\mathbb{Q}_{>0}$, with $\beta_i:=\prod_{j=1}^i\alpha_j>\beta$, $\gamma\in(0,1)\cap\mathbb{Q}$, $\epsilon\in((\frac{\beta}{\beta_i})^{\frac{1}{2i}},1)\cap\mathbb{Q}$ and $m\in\mathbb{Z}_{\geq 1}$; such that the conditions $I_i(\alpha_1,\dots,\alpha_d;\gamma,\epsilon,m)$, $J_i(\alpha_1,\dots,\alpha_d;\gamma,\epsilon,m)$ and $K_i(\alpha_1,\dots,\alpha_d;\gamma,\epsilon,m;0)$ are satisfied for (X,f,L);*

Combining Fact 5.1 with Theorem 5.2, we can control λ_i both from above and form below. This give us an algorithm to compute λ_i which only use certain intersection numbers. Now we explain the algorithm in more details.

5.2. **Algorithm.**

Proof of Theorem 1.4. Let Ω be the set of $(\alpha_1,\dots,\alpha_d,\gamma,\epsilon,m,n)\in\mathbb{Q}_{>0}^d\times((0,1)\cap\mathbb{Q})^2\times\mathbb{Z}_{>0}^2$, such that with $\beta_i:=\prod_{j=1}^i\alpha_j>\beta$ and $\epsilon^{2i}>\frac{\beta}{\beta_i}$. This is a countable set. We may fix a (computable) arrangement to write $\Omega=\{\omega_j,j\geq 0\}$. Define a function $\theta:\Omega\to\{0,1\}$ as follows. For $\omega=(\alpha_1,\dots,\alpha_d,\gamma,\epsilon,m,n)\in\Omega$, $\theta(\omega)=1$ if and only if the followings hold:

(1) the conditions I_i,J_i hold for $(\alpha_1,\dots,\alpha_d;\gamma,\epsilon,m)$ and the condition K_i holds for $(\alpha_1,\dots,\alpha_d;\gamma,\epsilon,m;0)$;

(2) $(\frac{\binom{d}{i}}{(L^d)}\deg_i(f^n))^{1/n}-\epsilon^{2i}\beta_i<\frac{1}{2^l}$.

To compute $\theta(\omega)$, we only need to compute $\deg_i f^n$ and finitely many intersection numbers among L,L_m,L_{2m}. Now we compute $\theta(\omega_j), j\geq 0$ one by one. Fact 5.1 and Theorem 5.2 shows that there is a smallest j such that $\theta(\omega_j)=1$. Write $\omega_j=(\alpha_1,\dots,\alpha_d,\gamma,\epsilon,m,n)$. We output $\tilde{\lambda}:=\epsilon^{2i}\beta_i$. By Fact 5.1 and Theorem 5.2, we have

$$\tilde{\lambda}<\lambda_i\leq\frac{\binom{d}{i}}{(L^d)}\deg_i(f^n))^{1/n}<\tilde{\lambda}+\frac{1}{2^l}.$$

So the output $\tilde{\lambda}$ is what we need. □

5.3. **Lower bounds in dimension two.** By Fact 5.1, we have a direct upper bound of dynamical degrees, but to get lower bounds we need to try many possible parameters to see whether θ equal to 1. This makes the algorithm in Section 5.2 far from being efficient. I suspect that a direct way to get the lower bounds should make the algorithm more efficient. In the surface case, a such lower bound was proved by the author [Xie15, Key Lemma].

Theorem 5.3 (=Theorem 1.5)**.** *Let* $\mathbf{k}$ *be a field. Let* X *be a projective surface over* $\mathbf{k}$*. Let* $f : X \dashrightarrow X$ *be a dominant rational self-map. Let* L *be a big and nef line bundle in* $\widetilde{\mathrm{Pic}}(X)$*. Then we have*

$$\lambda_1 \geq \frac{\deg_1 f^2}{2^{\frac{1}{2}} \times 3^{18} \deg_1 f}.$$

The proof relies on the theory of hyperbolic geometry and the natural linear action of f on a suitable hyperbolic space of infinite dimension. This space is constructed as a set of cohomology classes in the Riemann-Zariski space of X and was introduced by Cantat [Can11]. Unfortunately, such space can be only constructed in dimension two. Also the coefficient $2^{\frac{1}{2}} \times 3^{18}$ is quite large.

In this section, we use the idea of constructing recursive inequalities to get a better lower bound for the first dynamical degree in dimension two. This result has the same form as Theorem 5.3, but it improves the coefficient a lot i.e. from $2^{\frac{1}{2}} \times 3^{18}$ to 4. Moreover, the proof become much simpler.

Theorem 5.4 (=Theorem 1.6)**.** *Let* $\mathbf{k}$ *be a field. Let* X *be a projective surface over* $\mathbf{k}$*. Let* $f : X \dashrightarrow X$ *be a dominant rational self-map. Let* L *be a big and nef line bundle in* $\widetilde{\mathrm{Pic}}(X)$*. Then we have*

$$\lambda_1 \geq \frac{\deg_1 f^2}{4 \deg_1 f}.$$

Proof. Set $Q := \frac{\deg_1 f^2}{\deg_1 f}$. As $\lambda_1 \geq \lambda_2^{1/2}$, we may assume that $Q/4 \geq \lambda_2^{1/2}$. We claim that the line bundle

$$L_2 + \frac{Q^2}{16} L - \frac{Q}{2} L_1$$

is big. For this, by Theorem 2.1, we only need to show that

$$\frac{(L_2 + \frac{Q^2}{16} L)^2}{2(L_2 + \frac{Q^2}{16} L)L_1} \geq \frac{Q}{2}. \tag{5.1}$$

Indeed,

$$\begin{aligned}\frac{(L_2+\frac{Q^2}{16}L)^2}{2(L_2+\frac{Q^2}{16}L)L_1} \geq & \frac{2\frac{Q^2}{16}\deg_1 f^2}{2(L_2+\frac{Q^2}{16}L)L_1}\\ \geq & \frac{2\frac{Q^2}{16}\deg_1 f^2}{2(\lambda_2\deg_1 f+\frac{Q^2}{16}\deg_1 f)}\\ \geq & \frac{2\frac{Q^2}{16}\deg_1 f^2}{4\frac{Q^2}{16}\deg_1 f}\\ \geq & \frac{Q}{2}\end{aligned}$$

Then we get (5.1). Apply $(f^n)^*$ to $L_2+\frac{Q^2}{16}L-\frac{Q}{2}L_1$ and multiply it by L, we get that for every $n\geq 0$,

$$\deg_1(f^{n+2})+\frac{Q^2}{16}\deg_1(f^n)\geq\frac{Q}{2}\deg_1(f^{n+1}).$$

By (4.1), $\deg_1 f^2\leq\frac{2}{(L^2)}(\deg_1 f)^2$. Then we have

$$\deg_1 f-\frac{Q}{4}(L^2)\geq\frac{Q}{2}(L^2)-\frac{Q}{4}(L^2)>0.$$

We concludes the proof by Lemma 3.1. □

6. Lower semi-continuity of dynamical degrees

Let S be an integral noetherian scheme. Recall that A *family of d-dimensional dominant rational self-maps on S* is a flat and projective scheme $\pi:\mathcal{X}\to S$ satisfying $d:=\dim\mathcal{X}/S$ with a dominant rational self-map $f:\mathcal{X}\dashrightarrow\mathcal{X}$ over S such that the following hold:

(i) For every $p\in S$, the fiber X_p of π at p is geometrically reduced and irreducible;
(ii) For every point $p\in S$, $X_p\not\subseteq I(f)$.
(iii) The induced map $f_p:X_p\dashrightarrow X_p$ is dominant.

The aim of this section is to prove the lower semi-continuity of dynamical degrees for a family of dominant rational maps.

Theorem 6.1 (=Theorem 1.8)**.** *Let S be an integral noetherian scheme and $\pi:\mathcal{X}\to S$ be a flat and projective scheme over S. Let $f:\mathcal{X}\dashrightarrow\mathcal{X}$ be a family of d-dimensional dominant rational self-maps on S. Then for every $i=0,\dots,d$, the function $p\in S\mapsto\lambda_i(f_p)$ is lower semi-continuous.*

6.1. **Lower semi-continuity functions on noetherian schemes.** The following lemma gives a criterion for the lower semi-continuity.

Lemma 6.2. *Let S be a noetherian scheme. Then a function $\theta:S\to\mathbb{R}$ is lower semi-continuous if and only if the following hold:*

(i) *for points $x,y\in S$ with $y\in\overline{\{x\}}$, we have $\theta(x)\geq\theta(y)$;*

(ii) *for every* $x \in S$ *and* $a < \theta(x)$, *there is an open subset* V *of* $\overline{\{x\}}$ *containing* x *such that* $V \subseteq \theta^{-1}((a, +\infty))$.

Proof. First assume that θ is lower semi-continuous. Then (ii) is obvious. Let $x, y \in S$ with $y \in \overline{\{x\}}$. Note that $\theta^{-1}((-\infty, \theta(x)])$ is closed and it contains x. Then we have $y \in \overline{\{x\}} \subseteq \theta^{-1}((-\infty, \theta(x)])$. So (i) holds.

Now assume that (i) and (ii) hold. Let $a \in \mathbb{R}$. Let $Z := \overline{\theta^{-1}((-\infty, a])}$. We only need to show that $Z = \theta^{-1}((-\infty, a])$. Otherwise, there is an irreducible component Z' of Z such that $Z' \not\subseteq \theta^{-1}((-\infty, a])$. Let η be the generic point of Z'. By (i), $\eta \notin \theta^{-1}((-\infty, a])$. By (ii), there is open subset V of Z' containing η such that $V \subseteq \theta^{-1}((a, +\infty))$. So $V \subseteq \theta^{-1}((-\infty, a])$. So $\theta^{-1}((-\infty, a])$ is not dense in Z', which is a contradiction. □

Remark 6.3. The following example shows that the limit of lower semi-continuous functions may not be lower semi-continuous: Let $S = \operatorname{Spec} \mathbb{Z}$. Let η be the generic point of S. For $n \geq 1$, let $D_n : S \to \mathbb{R}$ be the function as follows: Define $D_n(\eta) := 1$. for every prime number p, $D_n(p) := 0$ if $p < n$; and $D_n(p) := 1$ if $p \geq n$. Easy to check that D_n are lower semi-continuous. Easy to see that D_n pointwisely converges to the function $D : S \to \mathbb{R}$ satisfying $D(\eta) = 1$ and $D|_{S \setminus \{\eta\}} = 0$, which is not lower semi-continuous.

Constructible topology. Let S be a noetherian scheme. Denote by $|S|$ the underling set of S with the constructible topology; i.e. the topology on a S generated by the constructible subsets (see [Gro64, Section (1.9) and in particular (1.9.13)]). In particular every constructible subset is open and closed. This topology is finer than the Zariski topology on S. Moreover $|S|$ is (Hausdorff) compact.

Lemma 6.4. *Let* S *be a noetherian scheme. Let* $\theta : S \to \mathbb{R}$ *be a lower semi-continuous function. Then* θ *is continuous in the constructible topology. Assume further that* $\theta(S)$ *is discrete. Then* $\theta(S)$ *is finite and for every* $a \in \mathbb{R}$, $\theta^{-1}(a)$ *is a constructible subset of* S.

Proof. For every $a \in \mathbb{R}$, $\theta^{-1}((a, +\infty))$ is Zariski open in S, hence open in the constructible topology. For every $a \in \mathbb{R}$, $\theta^{-1}((-\infty, a)) = \cup_{n \geq 1} \theta^{-1}((-\infty, a - \frac{1}{n}])$. As $\theta^{-1}((-\infty, a - \frac{1}{n}])$ is Zariski closed, it is open in the constructible topology. So $\theta^{-1}((-\infty, a))$ is open in the constructible topology. So θ is continuous in the constructible topology.

Now assume that $\theta(S)$ discrete. As θ is continuous on $|S|$ and $|S|$ is compact, $\theta(S)$ is compact and discrete, hence finite. If $a \notin \theta(S)$, then $\theta^{-1}(a) = \emptyset$. Assume that $a \in \theta(S)$. There is $b < a$ such that $(b, a) \cap \theta(S) = \emptyset$. Then

$$\theta^{-1}(a) = \theta^{-1}((-\infty, a]) \setminus \theta^{-1}((-\infty, b]),$$

which is a constructible set. □

6.2. **Lower semi-continuity of mixed degrees.** Let $\mathcal{L}$ be a π-ample line bundle on $\mathcal{X}$. For every $p \in S$, denote by L_p the restriction of $\mathcal{L}$ to the fiber X_p.

Lemma 6.5. *Let $(\mathcal{X}, f)$ be a family of d-dimensional dominant rational self-maps on S. Let $\mathcal{L}$ be a π-ample line bundle on $\mathcal{X}$. Let $m_1, \dots, m_d \in \mathbb{Z}_{\geq 0}$. Then the function*

$$p \mapsto ((f_p^{m_1})^* L_p \cdots (f_p^{m_d})^* L_p)$$

is lower semi-continuous on S. In particular, for every $i = 0, \dots, d$, the function

$$p \mapsto \deg_{i, L_p} f_p$$

is lower semi-continuous on S.

Proof of Lemma 6.5. Denote by κ the generic point of S. By Lemma 6.2, we only need to show that for any $(\pi : \mathcal{X} \to S, f, \mathcal{L})$ satisfying our assumption, we have

$$((f_p^{m_1})^* L_p \cdots (f_p^{m_d})^* L_p) \leq ((f_\kappa^{m_1})^* L_\kappa \cdots (f_\kappa^{m_d})^* L_\kappa)$$

on S with equality on a Zariski open subset of S.

Let Γ_κ be the closure of the image of the map $X_\kappa \dashrightarrow X_\kappa^d$ sending x to $(f_\kappa^{m_1}(x), \dots, f_\kappa^{m_d}(x))$. Let Γ be its closure in $\mathcal{X}^d_{/S}$.

By [RG71, Theorem 5.2.2], there is a blowup $\phi : S' \to S$ such that the strict transformation $\Gamma' \to S'$ of $\Gamma \to S$ by ϕ is flat. Set $\mathcal{X}' := \mathcal{X} \times_S S'$ with structure morphism $\pi' : \mathcal{X}' \to S'$ and $f' := f \times_S \mathrm{id}$. Set $\psi := \mathrm{id} \times_S \phi : \mathcal{X}' \to \mathcal{X}$ and $\mathcal{L}' := \psi^* \mathcal{L}$. Then $(\pi' : \mathcal{X}' \to S', f', \mathcal{L}')$ has the same property as $(\pi : \mathcal{X} \to S, f, \mathcal{L})$. Let κ' be the generic point of S' and Γ'_κ be the closure of the image of the map $X'_{\kappa'} \dashrightarrow X'^d_{\kappa'}$ sending x to $((f'_{\kappa'})^{m_1}(x), \dots, (f'_{\kappa'})^{m_d}(x))$. Then Γ' is its closure in $\mathcal{X}'^d_{/S}$. For every $p' \in S'$, $(X'_{p'}, f'_{p'}, L'_{p'})$ is a bass change of (X_p, f_p, L_p). Moreover ψ is an isomorphism over a Zariski dense open subset of S. So we may replace $(\pi : \mathcal{X} \to S, f, \mathcal{L})$ by $(\pi' : \mathcal{X}' \to S', f', \mathcal{L}')$ and assume further that the structure morphism $\pi_\Gamma : \Gamma \to S$ is flat.

For every $p \in S$, let Γ''_p be the closure of the image of the map $X_p \dashrightarrow X_p^d$ sending x to $(f^{m_1}(x), \dots, f^{m_d}(x))$. Then Γ''_p is an irreducible component of Γ_p. There is a Zariski dense open subset U of S such that for every $p \in U$, $\Gamma_p = \Gamma''_p$. Let $F_i : \Gamma \to \mathcal{X}$ be the i-th projection. Then we have

$$((f_p^{m_1})^* L_p \cdots (f_p^{m_d})^* L_p) = (F_1^* \mathcal{L}|_{\Gamma_{p''}} \cdots F_d^* \mathcal{L}|_{\Gamma_{p''}}) \leq (F_1^* \mathcal{L}|_{\Gamma_p} \cdots F_d^* \mathcal{L}|_{\Gamma_p}), \tag{6.1}$$

and the equality holds for $p \in U$. By [Ful84, Proposition 10.2], we have

$$(F_1^* \mathcal{L}|_{\Gamma_p} \cdots F_d^* \mathcal{L}|_{\Gamma_p}) = (F_1^* \mathcal{L}|_{\Gamma_\kappa} \cdots F_d^* \mathcal{L}|_{\Gamma_\kappa}) = ((f_\kappa^{m_1})^* L_\kappa \cdots (f_\kappa^{m_d})^* L_\kappa). \tag{6.2}$$

Combine (6.2) with (6.1), we concludes the proof. □

Remark 6.6. For every $p \in S$, we have $((f_p^{m_1})^* L_p \cdots (f_p^{m_d})^* L_p) \in \mathbb{Z}$. By Lemma 6.4, the set $\{((f_p^{m_1})^* L_p \cdots (f_p^{m_d})^* L_p) |\ p \in S\}$ is finite and for every subset $F \subseteq \mathbb{R}$, $\{p \in S |\ ((f_p^{m_1})^* L_p \cdots (f_p^{m_d})^* L_p) \in F\}$ is a constructible subset of S.

The following example shows that the map $p \mapsto \deg_{L_p, i}(f_p)$ is not continuous in general.

Example 6.7. [Xie15, Example 4.2] Consider the birational transformation

$$f[x : y : z] = [xz : yz + 2xy : z^2]$$

of $\mathbb{P}^2$ over $\operatorname{Spec}\mathbb{Z}$. Denote by L the hyperplane line bundle on $\mathbb{P}^2_{\mathbb{Z}}$. Then f_p is birational for every prime $p \in \operatorname{Spec}\mathbb{Z}$. We have that $\deg_{L_p}(f_p) = 1$ for $p = 2$ and $\deg_{L_p}(f_p) = 2$ for any odd prime.

The function $p \in S \mapsto \lambda_i(f_p)$ of the i-th dynamical degree is the point-wise limit of the functions $p \in S \mapsto (\deg_{i,L_p} f_p)^{1/n}$. By Lemma 6.5, the later function is lower semi-continuous. However, as shown in Remark 6.3, this does not directly imply the lower semi-continuity of the i-th dynamical degree. To complete the proof of Theorem 6.1, we need to apply the lower bounds of the dynamical degrees obtained in Section 3.

6.3. **Lower semi-continuity of dynamical degrees.**

Proof of Theorem 6.1. Let κ be the generic point of S. By Lemma 6.2, we only need to show that for any $(\pi : \mathcal{X} \to S, f)$ satisfying our assumption, the followings hold:

(i) $\lambda_i(f_p) \leq \lambda_i(f_\kappa)$ for all $p \in S$
(ii) for any $\beta < \lambda_i(f_\kappa)$, there is a nonempty open set U of S, such that for every point $p \in U$, $\lambda_i(f_p) > \beta$.

Let $\mathcal{L}$ be a π-ample line bundle on $\mathcal{X}$. For every $p \in S$, denote by L_p the restriction of $\mathcal{L}$ to the fiber X_p. By Lemma 6.5, for every integer $n > 0$, we have

$$\deg_{i,L_p}(f_p^n) \leq \deg_{i,L_\kappa}(f_\kappa^n)$$

hence

$$\lambda_i(f_p) \leq \lambda_i(f_\kappa).$$

This implies (i).

Let $\beta < \lambda_i(f_\kappa)$. By Theorem 5.2, there are decreasing numbers $\alpha_1, \dots, \alpha_d \in \mathbb{R}_{>0}$, with $\beta_i := \prod_{j=1}^i \alpha_j > \beta$, $\gamma \in (0,1)$, $\epsilon \in ((\frac{\beta}{\beta_i})^{\frac{1}{2i}}, 1)$ and $m \in \mathbb{Z}_{\geq 1}$, such that the conditions $I_i(\alpha_1, \dots, \alpha_d; \gamma, \epsilon, m)$, $J_i(\alpha_1, \dots, \alpha_d; \gamma, \epsilon, m)$ and $K_i(\alpha_1, \dots, \alpha_d; \gamma, \epsilon, m; 0)$ are satisfied for $(X_\kappa, f_\kappa, L_\kappa)$. Note that the conditions $I_i(\alpha_1, \dots, \alpha_d; \gamma; \epsilon; m)$, $J_i(\alpha_1, \dots, \alpha_d; \gamma; \epsilon; m)$ and $K_i(\alpha_1, \dots, \alpha_d; \gamma; \epsilon; m; 0)$ for (X_p, f_p, L_p) only depend on the top intersection numbers of $(f_p^{2m})^* L_p, (f_p^m)^* L_p, L_p$. By Lemma 6.5, there is a Zariski dense open subset U of S, such that for every $p \in U$, all top intersection numbers of $(f_p^{2m})^* L_p, (f_p^m)^* L_p, L$ are constant. Hence for every $p \in U$, the conditions $I_i(\alpha_1, \dots, \alpha_d; \gamma, \epsilon, m)$, $J_i(\alpha_1, \dots, \alpha_d; \gamma, \epsilon, m)$ and $K_i(\alpha_1, \dots, \alpha_d; \gamma, \epsilon, m; 0)$ are satisfied for (X_p, f_p, L_p). By Theorem 5.2, we get $\lambda_i(f_p) > \beta$. This implies (ii). This concludes the proof. □

Theorem 6.1 implies that for every family of dominant rational self-maps over S, $\lambda_i(f_p)$ can not be arbitrarily closed to 1 if it is not equal to 1.

Corollary 6.8. *Let $(\mathcal{X}, f)$ be a family of d-dimensional dominant rational maps on S. For every $i = 0, \dots, d$, the set $\Lambda_i((\mathcal{X}, f)) := \{\lambda_i(f_p) |\ p \in S\}$ is well-ordered i.e. every subset of $\Lambda_i((\mathcal{X}, f))$ has a minimal element. In particular, there is $\lambda \in (1, +\infty)$ such that for every $p \in S$, if $\lambda_i(f_p) > 1$, then $\lambda_i(f_p) \geq \lambda$.*

Proof. Let $F \subseteq \Lambda_i((\mathcal{X}, f))$. For every $\beta \in \mathbb{R}$, define $Z_\beta := \lambda_i^{-1}((-\infty, \beta])$ which is Zariski closed. For $\beta \in \Lambda_i((\mathcal{X}, f))$, we have

$$\beta = \sup_{p \in Z_\beta} \lambda_i(f_p). \tag{6.3}$$

Set $b := \inf F$. There is a decreasing sequence $\beta_n \in F$ such that $\lim_{n\to\infty} \beta_n = b$. The noetherianity of S shows that there is $N \geq 0$ such that $Z_{\beta_n} = Z_{\beta_N}$ for all $n \geq N$. By (6.3), we get $\beta_n = \beta_N$ for every $n \geq N$. Hence $b = \beta_N$. This implies that $\Lambda_i((\mathcal{X}, f))$ is well-ordered. As $\Lambda_i((\mathcal{X}, f)) \cap (1, +\infty)$ is well-ordered, the last statement is true. □

6.4. **Decidability.** Theorem 6.1 implies that for a family of dominant rational self-maps over S. For every $\beta > 0$, $i = 1, \ldots, d$, and $p \in S$, the question whether $\lambda_i(f_p) > \beta$ is decidable.

Let $(\mathcal{X}, f)$ be a family of d-dimensional dominant rational maps on S. Let $\mathcal{L}$ be a π-ample line bundle on $\mathcal{X}$. As in Section 5.2, let Ω be the set of

$$(\alpha_1, \ldots, \alpha_d, \gamma, \epsilon, m, n) \in \mathbb{Q}^d_{>0} \times ((0,1) \cap \mathbb{Q})^2 \times \mathbb{Z}^2_{>0},$$

with

$$\beta_i := \prod_{j=1}^{i} \alpha_j > \beta \text{ and } \epsilon^{2i} > \frac{\beta}{\beta_i}.$$

This is a countable set. We fix an arrangement to write $\Omega = \{\omega_j, j \geq 0\}$. Define $\Theta : S \times \Omega \to \{0,1\}$ as follows. For $\omega = (\alpha_1, \ldots, \alpha_d, \gamma, \epsilon, m, n) \in \Omega$ and $p \in S$, $\Theta(p, \omega) = 1$ if and only if the followings hold:

(1) for (X_p, f_p, L_p), the three conditions I_i, J_i hold for $(\alpha_1, \ldots, \alpha_d; \gamma, \epsilon, m)$ and K_i holds for $(\alpha_1, \ldots, \alpha_d; \gamma, \epsilon, m; 0)$;
(2) $\epsilon^{2i}\beta_i > \beta$.

To compute $\Theta(p, \omega)$, we only need to compute finitely many intersection numbers between $(f_p^{2m})^* L_p, (f_p^m)^* L_p, L_p$.

Corollary 6.9. *Let $(\mathcal{X}, f)$ be a family of d-dimensional dominant rational maps on S. Let $\mathcal{L}$ be a π-ample line bundle on $\mathcal{X}$. Then for every $\beta \in \mathbb{R}$ and $i = 0, \ldots, d$, there is a finite set $F \subset \Omega$ such that for every $p \in S$, $\lambda_i(f_p) > \beta$ if and only if there is $\omega \in F$ such that $\Theta(p, \omega) = 1$.*

Proof. Set $Z := \{p \in S |\ \lambda_i(f_p) \leq \beta\}$. For every $n \geq 1$, write $V_n := \{p \in S |\ \Theta(p, \omega_n) = 1\}$. By Theorem 5.2, we have $Z = \cup_{n \geq 0} V_n$. By Remark 6.6, V_n is constructible in S. As $|Z|$ is compact and $|V_n|, n \geq 0$ are open in the constructible topology, there is $N \geq 1$ such that $Z = \cup_{n=0}^N V_n$. We concludes the proof by letting $F := \{\omega_1, \ldots, \omega_N\}$. □

7. Periodic points of cohomologically hyperbolic maps

Let X be a variety over a field $\mathbf{k}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map. For $i = 1, \ldots, d$, we say that f is *i-cohomologically hyperbolic* if $\lambda_i(f)$ is strictly larger than the other dynamical degrees i.e.

$$\mu_i(f) > 1 \text{ and } \mu_{i+1}(f) < 1.$$

We say that f is *cohomologically hyperbolic* if it is i-cohomologically hyperbolic for some $i = 1, \dots, d$ i.e. $\mu_j(f) \neq 1$ for every $j = 1, \dots, d$.

Let X_f be the set of (scheme-theoretic) points x whose orbit is well-defined i.e. for every $n \geq 0$, $f^n \notin I(f)$. More generally, for every Zariski open subset V of X, let V_f be the set of points $x \in X_f$ whose orbit $O_f(x)$ is contained in V. Let $X_f(\overline{\mathbf{k}}) := X_f \cap X(\overline{\mathbf{k}})$. For every $n \geq 0$, let $\mathrm{Per}_n(f)(\overline{\mathbf{k}})$ be the set of n-periodic closed points in $X_f(\overline{\mathbf{k}})$. Set $\mathrm{Per}(f)(\overline{\mathbf{k}}) := \cup_{n\geq 1}\mathrm{Per}_n(f)(\overline{\mathbf{k}})$. For every Zariski open subset V of X, let $\mathrm{Per}_V(f)(\overline{\mathbf{k}})$ be the set of $x \in \mathrm{Per}(f)(\overline{\mathbf{k}})$ whose orbit $O_f(x)$ is contained in V.

The aim of this section is to prove the following result, which implies Theorem 1.12.

Theorem 7.1. *If f is cohomologically hyperbolic, then for every Zariski dense open subset V of X, $\mathrm{Per}_V(f)(\overline{\mathbf{k}})$ is Zariski dense in X.*

7.1. **Rational self-maps over finite fields.** The following result shows that, for dominant rational self-maps over finite fields, the periodic points are always Zariski dense. It was originally proved by Fakhruddin and Poonen [Fak03, Proposition 5.5] for endomorphisms. Their proof indeed works for arbitrary dominant rational self-maps with minor modifications. For the convenience of the readers, we provide a proof here in the general case. Our proof is based on the proof of [Xie15, Proposition 5.2].

Proposition 7.2. *Let $p > 0$ be a prime number. Let X be a variety over $\overline{\mathbb{F}_p}$. Let $f : X \dashrightarrow X$ be a dominant rational self-maps. Then for every Zariski dense open subset W of X, $\mathrm{Per}_W(f)(\overline{\mathbb{F}_p})$ is Zariski dense in X.*

The key ingredient to prove Proposition 7.2 is Hrushovski's twisted Lang-Weil estimate.

Theorem 7.3 ([Hru, SV22])**.** *Let $g : X \to \mathrm{Spec}\, k$ be an irreducible affine variety of dimension r over an algebraically closed field k of characteristic p, and let q be a power of p. We denote by ϕ_q the q-Frobenius map of k, and by X^{ϕ_q} the same scheme as X with g replaced by $g \circ \phi_q^{-1}$. Let $V \subseteq X \times X^{\phi_q}$ be an irreducible subvariety of dimension r such that both projections*

$$\pi_1 : V \to X \text{ and } \pi_2 : V \to X^{\phi_q}$$

are dominant and the second one is quasi-finite. Let $\Phi_q \subseteq X \times X^{\phi_q}$ be the graph of the q-Frobenius map ϕ_q. Set

$$u = \frac{\deg \pi_1}{\deg_{insep} \pi_2},$$

where $\deg \pi_1$ denotes the degree of field extension $K(V)/K(X)$ and $\deg_{insep} \pi_2$ is the purely inseparable degree of the field extension $K(V)/K(X)$.

Then there is a constant C that does not depend on q, such that

$$|\#(V \bigcap \Phi_q) - uq^r| \leq Cq^{r-1/2}.$$

Hrushovski's original proof of Theorem 7.3 relies on model theory. See [SV22] for an algebro-geometric proof.

Proof of Proposition 7.2. After replacing X, f by $W, f|_W$, we may assume that $W = X$. Let $Z := \overline{\mathrm{Per}\,(f)(\overline{\mathbb{F}_p})}$ and assume by contradiction that $Z \neq X$. Set $Y := Z \bigcup I(f)$. Then Y is a proper closed subset of X. Let $q = p^n$ be such that X and f are defined over the subfield $\mathbb{F}_q$ of $\overline{\mathbb{F}_p}$ having exactly q elements. Let ϕ_q denote the Frobenius morphism acting on X and let Γ_f(resp. Γ_m) denote the graph of f (resp. ϕ_q^m) in $X \times X$. Let U be an irreducible affine open subset of $X \setminus Y$ that is also defined over $\mathbb{F}_q$ and such that f is an open embedding from U to X. Set $V = \Gamma_f \bigcap (U \times U)$. By Theorem 7.3, there exists an integer $m > 0$ such that $(\mathrm{V}\bigcap\Gamma_m)(\overline{\mathbb{F}_p})\neq\emptyset$ i.e. there exists $u \in U(\overline{\mathbb{F}_p})$ such that $f(u) = \phi_q^m(u) \in U$. Since f is defined over $\mathbb{F}_q$, it follows that $f^l(u) = \phi_q^{lm}(u) \in U$ for all $l \geq 0$. This contradicts the definition of Y and U. □

7.2. **Isolated periodic points.** A periodic point $x \in \mathrm{Per}\,(f)(\overline{\mathbf{k}})$ is called *isolated* if it is isolated in $\mathrm{Per}_r(f)$ for some period $r \geq 1$ of x. The following result shows we can lift isolated periodic points from the special fiber to the generic fiber. This result was originally proved by Fakhruddin and Poonen [Fak03, Theorem 5.1] for endomorphisms. However, its proof works for arbitrary dominant rational self-maps with minor modifications. For the convenience of the reader, we provide a proof here in the general case. Our proof is based on the proof of [Xie15, Proposition 5.4].

Lemma 7.4. *Let $\mathcal{X}$ be a quasi-projective scheme, flat over a discrete valuation ring R with fraction field K and residue field k_p. Let f_R be a dominant rational self-map $\mathcal{X} \dashrightarrow \mathcal{X}$ over R. Let X_p be the special fiber of $\mathcal{X}$ and X be the generic fiber of $\mathcal{X}$. Assume that X_p is reduced and $X_p \not\subseteq I(f)$. Let f be the restriction of f_R to X, and f_p be the restriction of f_R to X_p. Let U_p be a Zariski dense open subset of X_p such that $U_p \cap I(f_R) = \emptyset$. Let $r \geq 1$ and $x_p \in U_p$ be a closed point in $\mathrm{Per}_{U_p}(f_p)$ of period r. Assume that X is regular at x_p and x_p is isolated in $\mathrm{Per}_r(f_p)$. Them there is a closed isolated periodic point $x \in \mathrm{Per}_r(X)$ such that $x_p \in \overline{\{x\}}$.*

Moreover, if isolated closed periodic points in $\mathrm{Per}_{U_p}(X_p)$ are Zariski dense in X_p, then the set of isolated f-periodic points is Zariski dense in X.

Proof. The set of periodic $\overline{k_p}$-points of f_p of period n can be viewed as the set of $\overline{k_p}$-points in $\Delta_{X_p} \bigcap \Gamma_{f_p^n}$, where Δ_{X_p} is the diagonal and $\Gamma_{f_p^n}$ is the graph of f_p^n in $X_p \times X_p$.

For any positive integer $r \geq 1$, consider the subscheme $\Delta_{\mathcal{X}} \bigcap \Gamma_{f_R^r}$ of $\mathcal{X} \times_R \mathcal{X}$, where $\Delta_{\mathcal{X}}$ is the diagonal and $\Gamma_{f_R^r}$ is the graph of f_R^r in $\mathcal{X} \times_R \mathcal{X}$. Note that $(x_p, x_p) \subset \Delta_{\mathcal{X}} \bigcap \Gamma_{f_R^r}$. As $\mathcal{X} \times_R \mathcal{X}$ is regular at (x_p, x_p), $\dim_{(x_p,x_p)} \Delta_{\mathcal{X}} \bigcap \Gamma_{f_R^r} \geq 1$. As $U_p \cap I(f_R) = \emptyset$, and $x_p \in \mathrm{Per}_{U_p}(f_p)$, $x_p \notin I(f_R^r)$, $\Delta_{X_p} \bigcap \Gamma_{f_p^n}$ and the special fiber of $\Delta_{\mathcal{X}} \bigcap \Gamma_{f_R^r}$ are locally the same at (x_p, x_p). As x_p is isolated in $\Delta_{X_p} \bigcap \Gamma_{f_p^n}$, we have $\dim_{(x_p,x_p)} \Delta_{\mathcal{X}} \bigcap \Gamma_{f_R^r} = 1$ and every irreducible component of $\Delta_{\mathcal{X}} \bigcap \Gamma_{f_R^r}$ passing through (x_p, x_p) dominates $\mathrm{Spec}\, R$. Pick an irreducible component $\mathcal{V}$ of

$\Delta_{\mathcal{X}} \bigcap \Gamma_{f^r_R}$ passing through (x_p, x_p). Let x' be the generic point of $\mathcal{V}$. Then $x' \subseteq \Delta_X \bigcap \Gamma_{f^n}$, where Δ_X is the diagonal and Γ_{f^n} is the graph of f^n in $X \times X$. Identify Δ_X with X, we get a closed isolated periodic point $x \in \mathrm{Per}_r(X)$ such that $x_p \in \overline{\{x\}}$.

Now assume that isolated closed periodic points in $\mathrm{Per}_{U_p}(X_p)$ are Zariski dense in X_p. We identify $\mathcal{X}$ with $\Delta_{\mathcal{X}}$. For any open subset U' of X, let Z be a Cartier divisor of X containing $X \backslash U'$. Let $\mathcal{Z}$ be the closure of Z in $\mathcal{X}$, then $\mathrm{codim}(\mathcal{Z}) = 1$ and every component of $\mathcal{Z}$ meets Z. Every irreducible component of X_p is of codimension 1. If $X_p \subseteq \mathcal{Z}$, every irreducible component of X_p is a component of $\mathcal{Z}$. Since $X \bigcap X_p = \emptyset$, we get $X_p \not\subseteq \mathcal{Z}$. Let $\mathcal{V} = \mathcal{X} \setminus \mathcal{Z}$ and $V_p = \mathcal{V} \bigcap X_p$, then $\mathcal{V} \bigcap X = U'$ and $V_p \neq \emptyset$. Hence $V_p \cap U_p \neq \emptyset$. As X_p is reduced, by our assumption, there is a closed isolated periodic point x_p of period $r \geq 1$ in $\mathrm{Per}_{U_p}(X_p) \cap V_p$ such that $\mathcal{X}$ is regular at x_p. The previous paragraph shows that there is a closed isolated periodic point $x \in \mathrm{Per}_r(X)$ such that $x_p \in \overline{\{x\}}$. Then $x \in U'$, this concludes the proof. □

Next we show that for cohomologically hyperbolic self-maps, periodic points under mild conditions are isolated.

Lemma 7.5. *Assume that X is projective. Let L be an ample line bundle on X. If f is i-cohomologically hyperbolic, then for every $\beta < \mu_i$ there is an affine Zariski open subset U of X such that for every irreducible curve C, if $C \cap U_f \neq \emptyset$ and* $\dim f^n(C) = 1$ *for all $n \geq 0$, we have*

$$\liminf_{n\to\infty} (L_n \cdot C)^{1/n} \geq \beta.$$

Recall that the above intersections $(L_n \cdot C)$ are well-defined as in the last paragraph of Section 2.1.

Proof. To simplify the notations, we write

$$\lambda_j := \lambda_j(f), \mu_j := \mu_j(f), \text{ and } L_n := (f^n)^* L$$

for every $n \geq 0$.

We may assume that $\beta > 1$. Pick $\epsilon \in (0,1)$, such that $\mu_i \epsilon^2 > \beta$ and $\mu_{i+1}\epsilon^{-2} < 1$. There is $m_0 \geq 1$ such that for every $m \geq m_0$, we have

$$\mu_i^m \epsilon^{2m} + \mu_{i+1}^m \epsilon^{-2m} \leq \mu_i^m \epsilon^m. \tag{7.1}$$

By Theorem 3.6 or [MW, Proposition 3.5], there is $m_1 > m_0$, such that for every $m \geq m_1$,

$$M_m := L_{2m} + \mu_i^m \mu_{i+1}^m L - \epsilon^m \mu_i^m L_m$$

is big. Fix $m \geq m_1$. Set $U := X \setminus \mathbf{B}_X(M_m)$. Let C be an irreducible curve satisfying $C \cap U_f \neq \emptyset$. Then for every $n \geq 0$, $f^n(C) \cap U \neq \emptyset$. Then $(M_m \cdot f^n_*(C))$ is well-defined and non-negative. So we get

$$(L_{(2+n)m} \cdot C) + \mu_i^m \mu_{i+1}^m (L_n \cdot C) \geq \epsilon^m \mu_i^m (L_{(n+1)m} \cdot C).$$

As $\dim f^n(C) = 1$ for all $n \geq 0$, $(L_n \cdot C) = (L \cdot (f^n)_* C) \geq 1$ (c.f. (2.1) of Section 2.1). As $\mu_{i+1}^m \epsilon^{-2m} < 1$, there is $N \geq 0$ such that

$$(L_{m(N+1)} \cdot C) > \mu_{i+1}^m \epsilon^{-2m} (L_{mN} \cdot C).$$

By (7.1) and Lemma 3.1, we have

$$\liminf_{n\to\infty}(L_n\cdot C)^{1/n}\geq \beta.$$

This concludes the proof. □

Corollary 7.6. *Let X be a variety over $\mathbf{k}$. Let $f : X \dashrightarrow X$ be a dominant rational self-map which is cohomologically hyperbolic. Then there is a Zariski dense open subset U of X such that for every $x \in \mathrm{Per}_U(f)$, x is isolate in $\mathrm{Per}_r(f)$, where $r \geq 1$ is a period of x.*

Proof. If Corollary 7.6 holds for one Zariski dense open subset U, it holds for any Zariski dense open subset U' of U.

After replace X by a Zariski dense affine open subset X' and f by $f|_{X'}$, we may assume that X is quasi-projective. Pick a projective compactification X'' of X. Then f extends to an dominant rational self-map f'' on X''. After replace X, f by X'', f'', we may assume that X is projective. Let L be an ample line bundle on X.

Let U as in Lemma 7.5. Let $x \in \mathrm{Per}_U(f)$ of period $r \geq 1$. If it is not isolated in $\mathrm{Per}_r(f)$, then there is an irreducible curve C containing x such that $f^r|_C = \mathrm{id}$. It follows that for every $n \geq 0$,

$$((f^{rn})^*L\cdot C) = (L\cdot C),$$

which is a contradiction. □

7.3. **Periodic points of cohomologically hyperbolic self-maps.**

Proof of Theorem 7.1. As we may replace X, f by $V, f|_V$, we only need to prove the case where $X = V$.

Assume that f is i-cohomologically hyperbolic for some $i \geq 0$. After base change by $\overline{\mathbf{k}}$, we may assume that $\mathbf{k}$ is algebraically closed. We may assume that the transcendence degree of $\mathbf{k}$ over its prime field F is finite, since we can find a subfield of $\mathbf{k}$ which is finitely generated over F such that X and f are all defined over this subfield. We complete the proof by induction on the transcendence degree of $\mathbf{k}$ over F.

If $\mathbf{k}$ is the closure of a finite field, we conclude the proof by Proposition 7.2.

If $\mathbf{k} = \overline{\mathbb{Q}}$, there is a regular subring R of $\overline{\mathbb{Q}}$ which is finitely generated over $\mathbb{Z}$, such that X and f are defined over R i.e. there is a flat projective R-scheme $\pi : \mathcal{X} \to \mathrm{Spec}\, R$, a dominant rational self-map $f_R : \mathcal{X} \dashrightarrow \mathcal{X}$ such that X, f are the generic fiber of $\mathcal{X}, f_R$. After shrinking $\mathrm{Spec}\, R$, we may assume that for every point $p \in \mathrm{Spec}\, R$, the fiber X_p is reduced and irreducible, $X_p \not\subseteq I(f_R)$ and $f_p := f_R|_{X_p}$ is a dominant rational map. By Theorem 6.1, there is a closed point $p \in \mathrm{Spec}\, R$ such that f_p is i-cohomologically hyperbolic. Since R is regular and finitely generated over $\mathbb{Z}$, the localization R_p of R at p is a discrete valuation ring such that $\overline{\mathrm{Frac}\,(R_p)} = \overline{\mathbb{Q}}$. So $R_p/pR_p = R/p$ is a finite field. Let U_p be a Zariski dense open subset of X_p with $I(f_R) \cap U_p = \emptyset$. By Corollary 7.6, after shrinking U_p, we may assume that every periodic point in $\mathrm{Per}_{U_p}(f_p)$ are isolated.

By Proposition 7.2, $\mathrm{Per}_{U_p}(f_p)$ is Zariski dense. We then conclude the proof by Lemma 7.4.

If the transcendence degree of $\mathbf{k}$ over F is greater than 1, we pick an algebraically closed subfield K of $\mathbf{k}$ such that the transcendence degree of K over F equals the transcendence degree of $\mathbf{k}$ over F minus 1. Then we pick a subring R of $\mathbf{k}$ which is finitely generated over K, such that X and f are all defined over R. Since $\mathrm{Spec}\, R$ is regular on an open set, we may assume that R is regular by adding finitely many inverses of elements in R. We may repeat the same arguments as in the case $\mathbf{k} = \overline{\mathbb{Q}}$ to conclude the proof. □

In the end, we give examples to show that for cohomologically non-hyperbolic maps, one can not determine whether the set of periodic points are Zariski dense from the dynamical degrees.

7.4. **Examples of cohomologically non-hyperbolic maps.** Let X be a projective variety over $\mathbf{k}$ of dimension $d \geq 1$. Let $i = 1, \dots, d$ and $f : X \dashrightarrow X$ be a i-cohomologically hyperbolic maps. Let $g : \mathbb{P}^N \to \mathbb{P}^N$ be an automorphism over $\mathbf{k}$. Consider the rational self-map $F := f \times g : X \times \mathbb{P}^N \dashrightarrow X \times \mathbb{P}^N$.

The product formula for relative dynamical degrees (c.f. [DN11], [Dan20] and [Tru20, Theorem 1.3]) shows the following lemma.

Lemma 7.7. *We have*

$$\lambda_j(F) = \lambda_j(f) \text{ for } j \leq i;$$
$$\lambda_j(F) = \lambda_i(f) \text{ for } i \leq j \leq i + N;$$
$$\lambda_j(F) = \lambda_{j-N}(f) \text{ for } j \geq i + N.$$

In particular, the dynamical degrees of F does not depend on g.

We note that $\mathrm{Per}\,(g)$ is Zariski dense if and only if g is of finite order i.e. $g^m = \mathrm{id}$ for some $m \geq 1$. Combining this fact with Theorem 7.1, we get the following statement.

Lemma 7.8. *The set of periodic points of F is Zariski dense if and only if g is of finite order i.e. $g^m = \mathrm{id}$ for some $m \geq 1$.*

8. Applications to the Kawaguchi-Silverman conjecture

The aim of this section is to prove the Kawaguchi-Silverman conjecture for certain rational self-maps on projective surfaces. In particular, our result implies the Kawaguchi-Silverman conjecture for birational self-maps on projective surfaces. We first recall the arithmetic degree and the Kawaguchi-Silverman conjecture.

8.1. **Arithmetic degree.** The arithmetic degree was first defined in [KS16] over a number field or a function field of characteristic zero. As in [Xie23b, Xie23a] and [Mat20, Remark 1.14], this definition can be extended to characteristic positive. Here we only recall the definition in the number fields cases.

Let X be a projective variety over $\overline{\mathbb{Q}}$. For every $L \in \mathrm{Pic}(X)$, we denote by $h_L : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ a Weil height associated to L. It is unique up to adding a bounded function.

Let $f\colon X \dashrightarrow X$ is a dominant rational self-map and $x \in X_f(\mathbf{k})$. As in [JSXZ21, Xie23b, Xie23a], we will associate to (X, f, x) a subset

$$A_f(x) \subseteq [1, \infty]$$

as follows: Let L be an ample divisor on X, define

$$A_f(x) := \cap_{m\geq 0}\overline{\{(h_L^+(f^n(x)))^{1/n}|\ n \geq m\}} \subseteq [1, \infty]$$

to be the limit set of the sequence $(h_L^+(f^n(x)))^{1/n}$, $n \geq 0$, where $h_L^+(\cdot) := \max\{h_L(\cdot), 1\}$. Indeed we have $A_f(x) \subseteq [1, \lambda_1(f)]$ by [KS16, Mat20, JSXZ21, Xie23b, Son23, Xie23a]. The following lemma shows that the set $A_f(x)$ does not depend on the choice of L.

Lemma 8.1. [Xie23b, Lemma 2.7] *Let $\pi\colon X \dashrightarrow Y$ be a dominant rational map between projective varieties. Let U be a Zariski dense open subset of X such that $\pi|_U\colon U \to Y$ is well-defined. Let L be an ample divisor on X and M an ample divisor on Y. Then there are constants $C \geq 1$ and $D > 0$ such that for every $x \in U$, we have*

$$h_M(\pi(x)) \leq Ch_L(x) + D. \tag{8.1}$$

Moreover if $V := \pi(U)$ is open in Y and $\pi|_U\colon U \to V$ is an isomorphism, then there are constants $C \geq 1$ and $D > 0$ such that for every $x \in U$, we have

$$C^{-1}h_L(x) - D \leq h_M(\pi(x)) \leq Ch_L(x) + D. \tag{8.2}$$

As in [KS16], define

$$\overline{\alpha}_f(x) := \sup A_f(x), \qquad \underline{\alpha}_f(x) := \inf A_f(x),$$

and call them *upper/lower arithmetic degree*. By Lemma 8.1, we have the following basic properties:

Proposition 8.2. [Xie23a, Proposition 6.4] *We have:*

(1) $A_f(x) = A_f(f^\ell(x))$, *for any* $\ell \geq 0$.
(2) $A_f(x) = \bigcup_{i=0}^{\ell-1}(A_{f^\ell}(f^i(x)))^{1/\ell}$. *In particular,* $\overline{\alpha}_{f^\ell}(x) = \overline{\alpha}_f(x)^\ell$, $\underline{\alpha}_{f^\ell}(x) = \underline{\alpha}_f(x)^\ell$.

The following result is the Kawaguchi-Silverman-Matsuzawa's upper bound. See [KS16, Mat20, JSXZ21, Xie23b, Son23, Xie23a] for its proof.

Theorem 8.3. *Let h be any Weil height on X associated to some ample line bundle. Then for any $x \in X_f(\mathbf{k})$, we have*

$$\overline{\alpha}_f(x) \leq \lambda_1(f).$$

If $\overline{\alpha}_f(x) = \underline{\alpha}_f(x)$, we set

$$\alpha_f(x) := \overline{\alpha}_f(x) = \underline{\alpha}_f(x).$$

In this case, we say that $\alpha_f(x)$ is well-defined and call it the *arithmetic degree* of f at x.

8.2. **Kawaguchi-Silverman conjecture.** The following conjecture was proposed by Kawaguchi and Silverman [Sil14, KS16].

Conjecture 8.4 (Kawaguchi-Silverman conjecture)**.** Let X be a projective variety over $\overline{\mathbb{Q}}$. Let $f : X \dashrightarrow X$ be a dominant rational map. Then for every $x \in X_f(\overline{\mathbb{Q}})$, $\alpha_f(x)$ is well defined. Moreover, if $O_f(x)$ is Zariski dense, then we have

$$\alpha_f(x) = \lambda_1(f).$$

8.3. **Our result.** The following is the main result of this section.

Theorem 8.5 (=Theorem 1.16)**.** *Let X be a projective surface over $\overline{\mathbb{Q}}$ and $f : X \dashrightarrow X$ be a dominant rational self-map such that $\lambda_1(f) > \lambda_2(f)$ or $\lambda_2(f) = \lambda_1(f)^2$. Let $x \in X_f(\overline{\mathbb{Q}})$. If the orbit $O_f(x)$ of x is Zariski dense, then $\alpha_f(x) = \lambda_1(f)$.*

In particular, Theorem 8.5 implies the Kawaguchi-Silverman conjecture for birational self-maps on projective surfaces.

Proof. Let L be an ample line bundle on X. To simplify the notations, we write

$$\lambda_j := \lambda_j(f), \mu_j := \mu_j(f), \text{ and } L_n := (f^n)^*L$$

for every $n \geq 0$. Let $i := 1$ if $\lambda_1 > \lambda_2$ and $i := 2$ if $\lambda_1^2 = \lambda_2$. Then we have $\mu_i = \lambda_1 > 1$ and $\mu_{i+1} < 1$. If $\lambda_1 = 1$, Theorem 8.5 trivially holds. So we assume that $\lambda_1 > 1$.

We denote by $h : X(\mathbf{k}) \to \mathbb{R}$ a Weil height associated to L. It is unique up to adding a bounded function. We may assume that $h(y) \geq 1$ for every $y \in X(\overline{\mathbb{Q}})$.

Let x be a point in $X_f(\overline{\mathbb{Q}})$ whose orbit is Zariski dense. By Theorem 8.3, $\overline{\alpha}_f(x) \leq \lambda_1$. We only need to show that for every $\beta \in (0, \lambda_1)$, $\underline{\alpha}_f(x) \geq \beta$. Pick $\epsilon \in (0, 1)$ such that

$$\epsilon^2\mu_i > \beta \text{ and } \epsilon^{-2}\mu_{i+1} < 1. \tag{8.3}$$

There is $m_0 \geq 1$ such that for every $m \geq m_0$, we have

$$(\epsilon^2\mu_i)^m + (\epsilon^{-2}\mu_{i+1})^m < \epsilon^m\mu_i^m - 1. \tag{8.4}$$

Set

$$\beta_1 := (\epsilon^2\mu_i)^m \text{ and } \beta_2 := (\epsilon^{-2}\mu_{i+1})^m.$$

By Theorem 3.6, there is $m \geq m_0$ such that

$$M_m := L_{2m} + \mu_i^m\mu_{i+1}^m L - \epsilon^m\mu_i^m L_m$$

is big. There is a constant $C > 0$ such that for every $y \in X_f(\mathbf{k}) \setminus \mathbf{B}_X(M_m)$, we have

$$h(f^{2m}(y)) + \mu_i^m\mu_{i+1}^m h(y) \geq \epsilon^m\mu_i^m h(f^m(y)) - C.$$

By the Northcott property, after replacing x by some $f^l(x)$ for some $l \geq 0$, we may assume that $h(f^n(x)) \geq C$ for every $n \geq 0$. Set $h_n := h(f^{mn}(x))$. We only need to show that

$$\liminf_{n\to\infty} h_n^{1/n} \geq \beta^m. \tag{8.5}$$

By Lemma 8.1, there is a constant $D > 1$ such that for every $n \geq 0$,

$$h_{n+1} \leq D h_n. \tag{8.6}$$

If $f^{mn}(x) \notin \mathbf{B}_X(M_m)$, we have

$$h_{n+2} + \mu_i^m \mu_{i+1}^m h_n \geq \epsilon^m \mu_i^m h_{n+1} - C \geq (\epsilon^m \mu_i^m - 1) h_{n+1}. \tag{8.7}$$

Hence we have

$$h_{n+2} - \beta_2 h_{n+1} \geq \beta_1 (h_{n+1} - \beta_2 h_n). \tag{8.8}$$

Let B_0 be the union of irreducible components of $\mathbf{B}_X(M_m)$ of dimension 0 and B_1 be the union of irreducible components of $\mathbf{B}_X(M_m)$ of dimension 1. We have $\mathbf{B}_X(M_m) = B_0 \sqcup B_1$. After replacing x by some $f^l(x)$ for some $l \geq 0$, we may assume that $f^n(x) \notin B_0$ for every $n \geq 0$.

Let $C_j, j = 1, \ldots, s'$ be the irreducible components of B_1 such that $C_j \cap X_f \neq \emptyset$. As $f^n(x) \in X_f(\overline{\mathbb{Q}})$ for every $n \geq 0$, $f^n(x) \in \mathbf{B}_X(M_m)$ if and only if $f^n(x) \in C_j$ for some $j = 1, \ldots, s'$. We may assume that $C_j, j = 1, \ldots, s''$ are exactly the C_j such that for every $n \geq 0$, $\dim(f^n(C_j)) = 1$. After replacing x by some $f^l(x)$ for some $l \geq 0$, we may assume that $f^n(x) \notin \cup_{j=s''+1}^{s'} C_j$ for every $n \geq 0$. We may assume that $C_j, j = 1, \ldots, s$ are exactly the $C_j, j = 1, \ldots, s''$ which are not preperiodic. As $O_f(x)$ is Zariski dense, $f^n(x) \in \mathbf{B}_X(M_m)$ if and only if $f^n(x) \in C_j$ for some $j = 1, \ldots, s$.

Lemma 8.6. *Let C be an irreducible curve in X with $C \cap X_f \neq \emptyset$. Assume that $\dim f^n(C) = 1$ for every $n \geq 0$ and C is not preperiodic. Recall that the above intersections $(L_n \cdot C)$ are well-defined as in the last paragraph of Section 2.1. Then the sequence $(L_n \cdot C), n \geq 0$ is not bounded.*

Applying Lemma 8.6 for f^m and $f(C_j), j = 1, \ldots, s$, for every $j = 1, \ldots, s$, there is $N_j \geq 1$ such that $(L_{N_j m} \cdot f(C_j)) > 3D^3 (L \cdot f(C_j))$. As $f(C_j)$ is one dimensional, there is $C' > 0$ such that for every $j = 1, \ldots, s$, for every $y \in X_f(\mathbf{k}) \cap f(C_j)$ we have $h(f^{N_j m}(y)) > 3D^3 h(y) - C'$. By the Northcott property, after replacing x by $f^l(x)$ for some $l \geq 0$, we may assume that $h(f^n(x)) > C'$ for every $n \geq 0$. Moreover, we may assume that $h(f^m(x)) > h(x)$. If $f^{mn}(x) \in f(C_j), j = 1, \ldots, s$, we have

$$h(f^{(N_j+n)m}(x)) > 3D^3 h(f^{mn}(x)) - C' > 2D^3 h(f^{mn}(x)). \tag{8.9}$$

By (8.6), there is $t_n \in \{0, \ldots, N_j - 1\}$ such that

$$h(f^{(t_n+1+n)m}(x)) > h(f^{(t_n+n)m}(x)) > h(f^{nm}(x)).$$

Set $N := \max\{N_j |\ j = 1, \ldots, s\}$. The above discussion shows the following: If $f^{m(n-1)}(x) \in \mathbf{B}_X(M_m)$, there is $t_n \in \{0, \ldots, N\}$, such that

$$h_{n+t_n+1} > h_{n+t_n} > h_n. \tag{8.10}$$

Set $W := \{n \geq 1 |\ f^{m(n-1)}(x) \in \mathbf{B}_X(M_m)\}$. By the Weak dynamical Mordell-Lang [BGT15, Corollary 1.5] (see also [Fav00, Theorem 2.5.8], [Gig14, Theorem

D, Theorem E],[Pet15, Theorem 2], [BHS20, Theorem 1.10], [Xie23b, Theorem 1.17] and [Xie23a, Theorem 5.2]), we have

$$\lim_{n\to\infty}\frac{w_n}{n}=0 \tag{8.11}$$

where $w_n := \#(\{1,\dots,n\}\cap W)$. We define a sequence $p(n)$ by induction. Define $p(0)=0$. Assume that $p(n)$ is defined for $n\leq n_1$. Define $p(n_1+1)=p(n_1)+1$ if $p(n_1)\notin W$; otherwise if $n_1\notin W$, define $p(n_1+1)=p(n_1)+t_{p(n_1)}+1$. It is clear that $p(n)$ is strictly increasing. As $t_{p(n')}\leq N$ for every $n'\geq 0$, for every $n\geq 0$, we have

$$p(n)\geq n \text{ and } p(n+1)\leq p(n)+N. \tag{8.12}$$

For every $n\geq 0$, there is a minimal $r(n)\geq 0$ such that $n\leq p(r(n))$. It is clear that

$$r(n)\leq n \text{ and } p(r(n))\leq n+N. \tag{8.13}$$

Lemma 8.7. *We have*

$$\lim_{n\to\infty}\frac{r(n)}{n}=1.$$

We note that $r(p(n))=n$. By Lemma 8.7, we have

$$\lim_{n\to\infty}\frac{p(n)}{n}=1. \tag{8.14}$$

The definition of $p(n)$ shows that for $n\geq 1$, if $p(n-1)\in W$, then $h_{p(n)}>h_{p(n)-1}$ and $h_{p(n)}>h_{p(n-1)}$. Moreover, we have $h_1>h_0$. For a set I of consecutive integers, we say that I is of type 0 if $p(n)\notin W$ for every $n\in I$ and say that I is of type 1 if $p(n)\in W$ for every $n\in I$. Let $n\geq 1$, write $\{1,\dots,n\}$ as $I_1\sqcup\cdots\sqcup I_s$ where I_i are set of consecutive integers such that

- for $i=1,\dots,s-1$, $\max\{I_i\}+1=\min\{I_{i+1}\}$;
- for every $i=1,\dots,s$, I_i are either of type 0 or of type 1;
- for $i=1,\dots,s-1$, the type of I_i and I_{i+1} are different.

Write $I_i=\{a_i,a_i+1\dots,b_i\}$. As $0\notin W$, we have $a_1=1$ and $p(a_1)=1$. If $i\leq s-1$, then $a_{i+1}=b_i+1$. If further that I_i is of type 0, then $p(a_{i+1})=p(b_i)+1$.

By (8.8), if I_i is of type 0, then for every $j\in p(a_i),\dots,p(b_i)+1$, we have

$$h_j\geq \beta_1^{j-a_i}(h_{p(a_i)}-\beta_2 h_{p(a_i)-1}).$$

If $i=1$, we have $h_{p(a_i)}=h_1>h_0=h_{p(a_i)-1}$. If $i\geq 2$, we have $a_i-1=b_{i-1}$ and I_{i-1} is of type 1. As $p(a_i-1)\in W$, we have $h_{p(a_i)}>h_{p(a_i)-1}$. Hence we have

$$h_{p(b_i)+1}\geq \beta_1^{b_i+1-a_i}(1-\beta_2)h_{p(a_i)}=\beta_1^{\#I_i}(1-\beta_2)h_{p(a_i)} \tag{8.15}$$

and

$$h_{p(b_i)}\geq \beta_1^{b_i-a_i}(1-\beta_2)h_{p(a_i)}=\beta_1^{\#I_i-1}(1-\beta_2)h_{p(a_i)} \tag{8.16}$$

When $i\leq s-1$, $p(b_i)+1=p(a_{i+1})$.

If I_i is of type 1, then for every $n\in I_i$, we have $h_{p(n+1)}>h_{p(n)}$. So we have

$$h_{p(b_i+1)}\geq h_{p(b_i+1)}\geq h_{p(a_i)} \tag{8.17}$$

When $i \leq s-1$, $p(b_i+1) = p(a_{i+1})$.

Set $O_n := \{i = 1, \dots, s |\ I_i \text{ is of type } 0\}$. Set $e := 0$ if $s \notin O$ and $e = 1$ if $s \in O_n$. Set $l_n := \sum_{i \in O_n} \#I_i$. Combining (8.15), (8.16) and (8.17), we get

$$h_{p(n)} \geq \beta_1^{(\sum_{i\in O_n} \#I_i)-e}(1-\beta_2)^{\#O_n} h_1 \geq \beta_1^{l_n-1}(1-\beta_2)^{\#O_n} h_1. \tag{8.18}$$

Since

$$\#O_n \leq n - l_n = \sum_{i\in\{1,\dots,s\}\setminus O_n} \#I_i \leq w_{p(n)},$$

by (8.11) and (8.14), we get that

$$\lim_{n\to\infty} \frac{l_n}{n} = 1 \text{ and } \frac{\#O_n}{n} = 0. \tag{8.19}$$

Then by (8.18), we have

$$\liminf_{n\to\infty} h_{p(n)}^{1/n} \geq \beta_1. \tag{8.20}$$

By (8.13) and (8.6), for every $n \geq 1$, we have $h_n \geq D^{-N} h_{p(r(n))}$. By (8.20) and Lemma 8.7, we get that

$$\liminf_{n\to\infty} h_n^{1/n} \geq \liminf_{n\to\infty} ((D^{-N} h_{p(r(n))})^{1/r(n)})^{n/r(n)} \geq \beta_1 \geq \beta^m.$$

Hence (8.5) holds. This concludes the proof. □

Proof of Lemma 8.6. If $\lambda_1 > \lambda_2$, then f is cohomologically 1-hyperbolic. If $\lambda_1^2 = \lambda_2$, as $\lambda_1 > 1$, f is cohomologically 2-hyperbolic. By Lemma 7.5, there is a non-empty Zariski open subset V of X such that for every irreducible curve C'' of X, if $\dim f^n(C'') = 1$ for every $n \geq 0$ and $C'' \cap V_f \neq \emptyset$. Then the sequence $(L_n \cdot C''), n \geq 0$ is not bounded.

As C is not preperiodic and $\dim X \setminus V \leq 1$, after replacing C by $f^l(C)$ for some $l \geq 0$, we may assume that $f^n(C) \cap V \neq \emptyset$. Hence the generic point of C is contained in V_f. We conclude the proof by the previous paragraph. □

Proof of Lemma 8.7. For $j = 0, \dots, r(n)-1$, we have

$$p(j+1) - p(j) = 1$$

if $j \notin W$; and

$$p(j+1) - p(j) \leq N$$

if $j \in W$. Hence we have

$$r(n) \leq n \leq p(r(n)) \leq r(n) + (N-1) w_{p(r(n))} \leq r(n) + (N-1) w_{n+N}.$$

So we have

$$\limsup_{n\to\infty} \frac{r_n}{n} \leq 1 \leq \liminf_{n\to\infty} \frac{r_n}{n} + (N-1) \lim_{n\to\infty} \frac{w_{n+N}}{n} = \liminf_{n\to\infty} \frac{r_n}{n}.$$

This concludes the proof. □

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