THE GEOMETRIC BOGOMOLOV CONJECTURE

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ABSTRACT. We prove the geometric Bogomolov conjecture over a function field of characteristic zero.

1. INTRODUCTION

1.1. The geometric Bogomolov conjecture.

1.1.1. *Abelian varieties and heights.* Let $\mathbf{k}$ be an algebraically closed field. Let B be an irreducible normal projective variety over $\mathbf{k}$ of dimension $d_B \geq 1$. Let $K := \mathbf{k}(B)$ be the function field of B. Let A be an abelian variety defined over K of dimension g. Fix an ample line bundle M on B, and a symmetric ample line bundle L on A.

Denote by $\hat{h} : A(\overline{K}) \to [0, +\infty)$ the canonical height on A with respect to L and M where $\overline{K}$ is an algebraic closure of K (see Section 3.1). For any irreducible subvariety X of $A_{\overline{K}}$ and any $\varepsilon > 0$, we set

$$X_\varepsilon := \{x \in X(\overline{K}) |\, \hat{h}(x) < \varepsilon\}. \tag{1.1}$$

Set $A_{\overline{K}} = A \otimes_K \overline{K}$, and denote by $(A^{\overline{K}/\mathbf{k}}, \mathsf{tr})$ the $\overline{K}/\mathbf{k}$-trace of $A_{\overline{K}}$: it is the final object of the category of pairs (C, f), where C is an abelian variety over $\mathbf{k}$ and f is a morphism from $C \otimes_{\mathbf{k}} \overline{K}$ to $A_{\overline{K}}$ (see [12]). If $\operatorname{char} \mathbf{k} = 0$, tr is a closed immersion and $A^{\overline{K}/\mathbf{k}} \otimes_{\mathbf{k}} \overline{K}$ can be naturally viewed as an abelian subvariety of $A_{\overline{K}}$. By definition, a **torsion coset** of A is a translate $a + C$ of an abelian subvariety $C \subset A$ by a torsion point a. An irreducible subvariety X of $A_{\overline{K}}$ is said to be **special** if

$$X = \mathsf{tr}(Y \otimes_{\mathbf{k}} \overline{K}) + T \tag{1.2}$$

for some torsion coset T of $A_{\overline{K}}$ and some subvariety Y of $A^{\overline{K}/\mathbf{k}}$. When X is special, X_ε is Zariski dense in X for all $\varepsilon > 0$ ([12, Theorem 5.4, Chapter 6]).

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1.1.2. *Bogomolov conjecture.* The following conjecture was proposed by Yamaki [19, Conjecture 0.3], but particular instances of it were studied earlier by Gubler in [9]. It is an analog over function fields of the Bogomolov conjecture which was proved by Ullmo [15] and Zhang [25].

Geometric Bogomolov Conjecture.– *Let X be an irreducible subvariety of $A_{\overline{K}}$. If X is not special there exists $\varepsilon > 0$ such that X_ε is not Zariski dense in X.*

The aim of this paper is to prove the geometric Bogomolov conjecture over a function field of characteristic zero.

Theorem A. *Assume that $\mathbf{k}$ is an algebraically closed field of characteristic 0. Let X be an irreducible subvariety of $A_{\overline{K}}$. If X is not special then there exists $\varepsilon > 0$ such that X_ε is not Zariski dense in X.*

1.1.3. *Historical note.* Gubler proved the geometric Bogomolov conjecture in [9] when A is totally degenerate at some place of K. When $\dim B = 1$ and $X \subset A$ is a curve in its Jacobian, Yamaki proved it for nonhyperelliptic curves of genus 3 in [17] and for any hyperelliptic curve in [18]. If moreover $\operatorname{char} \mathbf{k} = 0$, Faber [5] proved it if X is a curve of genus at most 4 and Cinkir [2] covered the case of arbitrary genus. Later on Yamaki proved the cases $(\mathrm{co})\dim X = 1$ [23] and $\dim(A^{\overline{K}/\mathbf{k}}) \geq \dim(A) - 5$ [22]; in [21], he reduced the conjecture to the case of abelian varieties with trivial $\overline{K}/k$-trace and good reduction everywhere. In [11], the third-named author gave a new proof of this conjecture in characteristic 0 when A is the power of an elliptic curve and $\dim B = 1$, introducing the original idea of considering the Betti map and its monodromy. Recently, the second and the third-named authors [6] proved the conjecture in the case $\operatorname{char} \mathbf{k} = 0$ and $\dim B = 1$.

1.2. An overview of the proof of Theorem A.

1.2.1. *Notation.* From now on, the algebraically closed field $\mathbf{k}$ has characteristic 0. There exists an algebraically closed subfield $\mathbf{k}'$ of $\mathbf{k}$ such that B, A, X, M and L are defined over $\mathbf{k}'$ and the transcendental degree of $\mathbf{k}'$ over $\overline{\mathbf{Q}}$ is finite. In particular, $\mathbf{k}'$ can be embedded in the complex field $\mathbf{C}$. Thus, *in the rest of the paper, we assume* $\mathbf{k} = \mathbf{C}$ and we denote by K the function field $\mathbf{C}(B)$.

Let $\pi : \mathcal{A} \to B$ be an irreducible projective scheme over B whose generic fiber is isomorphic to A. We may assume that $\mathcal{A}$ is normal, and we fix an ample line bundle $\mathcal{L}$ on $\mathcal{A}$ such that $\mathcal{L}|_A = L$. For $b \in B$, we set $\mathcal{A}_b := \pi^{-1}(b)$. We denote by $e : B \dashrightarrow \mathcal{A}$ the zero section and by $[n]$ the multiplication by n on A; it defines a rational mapping $\mathcal{A} \dashrightarrow \mathcal{A}$.

We may assume that M is very ample, and we fix an embedding of B in a projective space such that the restriction of $O(1)$ to B coincides with M. The restriction of the Fubini-Study form to B is a Kähler form ν.

Fix a Zariski dense open subset B^o of B such that B^o is smooth and $\pi|_{\pi^{-1}(B^o)}$ is smooth; then, set $\mathcal{A}^o := \pi^{-1}(B^o)$.

Let X be a geometrically irreducible subvariety of A such that X_ε is Zariski dense in X for every $\varepsilon > 0$. We denote by $\mathcal{X}$ its Zariski closure in $\mathcal{A}$, by $\mathcal{X}^o$ its Zariski closure in $\mathcal{A}^o$, and by $\mathcal{X}^{o,reg}$ the regular locus of $\mathcal{X}^o$. Our goal is to show that X is special.

1.2.2. *The main ingredients.* One of the main ideas of this paper is to consider the Betti foliation (see Section 2.1). It is a smooth foliation of $\mathcal{A}^o$ by holomorphic leaves, which is transverse to π. Every torsion point of A gives local sections of $\pi|_{\pi^{-1}(B^o)}$: these sections are local leaves of the Betti foliation, and this property characterizes it.

To prove Theorem A, the **first step** is to show that $\mathcal{X}^o$ is invariant under the foliation when small points are dense in X. In other words, at every smooth point $x \in \mathcal{X}^o$, the tangent space to the Betti foliation is contained in $T_x\mathcal{X}^o$. For this, we introduce a semi-positive closed $(1,1)$-form ω on $\mathcal{A}^o$ which is canonically associated to L and vanishes along the foliation. An inequality of Gubler implies that the canonical height $\hat{h}(X)$ of X is 0 when small points are dense in X; Theorem B asserts that the condition $\hat{h}(X) = 0$ translates into

$$\int_{\mathcal{X}^o} \omega^{\dim X+1} \wedge (\pi^*\nu)^{m-1} = 0 \tag{1.3}$$

where ν is any Kähler form on the base B^o. From the construction of ω, we deduce that X is invariant under the Betti foliation.

The first step implies that the fibers of $\pi|_{\mathcal{X}^o}$ are invariant under the action of the holonomy of the Betti foliation; the **second step** shows that a subvariety of a fiber $\mathcal{A}_b$ which is invariant under the holonomy is the sum of a torsion coset and a subset of $A^{\overline{K}/\mathbf{k}}$. The conclusion easily follows from these two main steps. The second step already appeared in [11] and [6], but here, we make use of a more efficient dynamical argument which may be derived from a result of Muchnik and is independent of the Pila-Zannier's counting strategy.

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2. The Betti form

In this section, we define a foliation, and a closed $(1,1)$-form on $\mathcal{A}^o$ which is naturally associated to the line bundle L.

2.1. **The local Betti maps.** Let b be a point of B^o, and $U \subseteq B^o(\mathbf{C})$ be a connected and simply connected open neighbourhood of b in the euclidean topology. Fix a basis of $H_1(\mathcal{A}_b;\mathbf{Z})$ and extend it by continuity to all fibers above U.

There is a natural real analytic diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ such that

(1) $\pi_1 \circ \phi_U = \pi$ where $\pi_1 : U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to U$ is the projection to the first factor;
(2) for every $b \in U$, the map $\phi_U|_{\mathcal{A}_b} : \mathcal{A}_b \to \pi_1^{-1}(b)$ is an isomorphism of real Lie groups that maps the basis of $H_1(\mathcal{A}_b;\mathbf{Z})$ onto the canonical basis of $\mathbf{Z}^{2g}$.

For b in U, denote by $i_b : \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ the inclusion $y \mapsto (b,y)$. The **Betti map** is the C^∞-projection $\beta_U^b : \pi^{-1}(U) \to \mathcal{A}_b$ defined by

$$\beta_U^b := (\phi_U|_{\mathcal{A}_b})^{-1} \circ i_b \circ \pi_2 \circ \phi_U \tag{2.1}$$

where $\pi_2 : U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ is the projection to the second factor. Changing the basis of $H_1(\mathcal{A}_b;\mathbf{Z})$, we obtain another trivialization ϕ'_U that is given by post-composing ϕ_U with a constant linear transformation

$$(b,z) \in U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g} \mapsto (b, h(z)) \tag{2.2}$$

for some element h of the group $\mathsf{GL}_{2g}(\mathbf{Z})$; thus, β_U^b does not depend on ϕ_U.

Note that β_U^b is the identity on $\mathcal{A}_b$. In general, β_U^b is not holomorphic. However, for every $p \in \mathcal{A}_b$, $(\beta_U^b)^{-1}(p)$ is a complex submanifold of $\mathcal{A}^o$. (For instance, every section of $\pi|_{\pi^{-1}U}$ which is given by a torsion point provides a fiber of β_U^b, and continous limits of holomorphic sections are holomorphic.)

2.2. **The Betti foliation.** The local Betti maps determine a natural foliation $\mathcal{F}$ on $\mathcal{A}^o$: for every point p, the local leaf $\mathcal{F}_{U,p}$ through p is the fiber $(\beta_U^{\pi(p)})^{-1}(p)$. We call $\mathcal{F}$ the **Betti foliation**. The leaves of $\mathcal{F}$ are holomorphic, in the following sense: for every $p \in \mathcal{A}^o$, the local leaf $\mathcal{F}_{U,p}$ is a complex submanifold of $\pi^{-1}(U) \subset \mathcal{A}^o$. But a global leaf $\mathcal{F}_p$ can be dense in $\mathcal{A}^o$ for the euclidean topology. Moreover, $\mathcal{F}$ is everywhere transverse to the fibers of π, and $\pi|_{\mathcal{F}_p} \colon \mathcal{F}_p \to B^o$ is a regular holomorphic covering for every point p (it may have finite or infinite degree, and this may depend on p).

Remark 2.1. The foliation $\mathcal{F}$ is characterized as follows. Let q be a torsion point of $\mathcal{A}_b$; it determines a multisection of the fibration π, obtained by analytic continuation of q as a torsion point in nearby fibers of π. This multisection coincides with the leaf $\mathcal{F}_q$. There is a unique foliation of $\mathcal{A}^o$ which is everywhere transverse to π and whose set of leaves contains all those multisections.

Remark 2.2. One can also think about $\mathcal{F}$ dynamically. The endomorphism $[n]$ determines a rational transformation of the model $\mathcal{A}$ and induces a regular transformation of $\mathcal{A}^o$. It preserves $\mathcal{F}$, mapping leaves to leaves. Preperiodic leaves correspond to preperiodic points of $[n]$ in the fiber $\mathcal{A}_b$; they are exactly the leaves given by the torsion points of A.

Remark 2.3. Assume that the family $\pi : \mathcal{A}^o \to B^o$ is trivial, i.e. $\mathcal{A}^o = B^o \times A_{\mathbf{C}}$ where $A_{\mathbf{C}}$ is an abelian variety over $\mathbf{C}$ and π is the first projection. Then, the leaves of $\mathcal{F}$ are exactly the fibers of the second projection.

2.3. **The Betti form.** The Betti form is introduced by Mok in [13, pp. 374] to study the Mordell-Weil group over function fields. We hereby sketch the construction of this $(1,1)$-form. For $b \in B^o$, there exists a unique smooth $(1,1)$-form $\omega_b \in c_1(\mathcal{L}|_{\mathcal{A}_b})$ on $\mathcal{A}_b$ which is invariant under translations. If we write $\mathcal{A}_b = \mathbf{C}^g/\Lambda$ and denote by $z_1, \ldots, z_g$ the standard coordinates of $\mathbf{C}^g$, then

$$\omega_b = \sum_{1 \le i,j \le g} a_{i,j} dz_i \wedge d\bar{z}_j \tag{2.3}$$

for some complex numbers $a_{i,j}$. This form ω_b is positive, because $\mathcal{L}|_{\mathcal{A}_b}$ is ample.

Now, we define a smooth 2-form ω on $\mathcal{A}^o$. Let p be a point of $\mathcal{A}^o$. First, define $P_p : T_p\mathcal{A}^o \to T_p\mathcal{A}_{\pi(p)}$ to be the projection onto the first factor in

$$T_p\mathcal{A}^o = T_p\mathcal{A}_{\pi(p)} \oplus T_p\mathcal{F}. \tag{2.4}$$

Since the tangent spaces $T_p\mathcal{F}$ and $T_p\mathcal{A}_{\pi(p)}$ are complex subspaces of $T_p\mathcal{A}^o$, the map P_p is a complex linear map. Then, for v_1 and $v_2 \in T_p\mathcal{A}^o$ we set

$$\omega(v_1, v_2) := \omega_{\pi(p)}(P_p(v_1), P_p(v_2)). \tag{2.5}$$

We call ω the **Betti form**. By construction, $\omega|_{\mathcal{A}_b} = \omega_b$ for every b. Since ω_b is of type $(1,1)$ and P_p is $\mathbf{C}$-linear, ω is an antisymmetric form of type $(1,1)$. Since ω_b is positive, ω is semi-positive.

Let U and ϕ_U be as in Section 2.1. Let y_i, $i = 1, \ldots, 2g$, denote the standard coordinates of $\mathbf{R}^{2g}$. Then there are real numbers $b_{i,j}$ such that

$$(\phi_U^{-1})^*\omega = \sum_{1 \le i < j \le 2g} b_{i,j} dy_i \wedge dy_j. \tag{2.6}$$

It follows that $d((\phi_U^{-1})^*\omega) = 0$ and that ω is closed. Moreover, $[n]^*\omega = n^2\omega$. Thus, we get the following lemma.

Lemma 2.4. *The Betti form ω is a real analytic, closed, semi-positive $(1,1)$-form on $\mathcal{A}^o$ such that $\omega|_{\mathcal{A}_b} = \omega_b$ for every point $b \in B^o$. In particular, the cohomology class of $\omega|_{\mathcal{A}_b}$ coincides with $c_1(\mathcal{L}|_{\mathcal{A}_b})$ for every $b \in B^o$.*

Since the monodromy of the foliation preserves the polarization $\mathcal{L}_{\mathcal{A}_b}$, it preserves ω_b and is contained in a symplectic group.

3. THE CANONICAL HEIGHT AND THE BETTI FORM

3.1. **The canonical height.** Recall that $K = \mathbf{C}(B)$. Let X be any subvariety of $A_{\overline{K}}$. There exists a finite field extension K' over K such that X is defined over

K'; in other words, there exists a subvariety X' of $A_{K'}$ such that $X = X' \otimes_{K'} \overline{K}$. Let $\rho' : B' \to B$ be the normalization of B in K'. Set $\mathcal{A}' := \mathcal{A} \times_B B'$ and denote by $\rho : \mathcal{A}' \to \mathcal{A}$ the projection to the first factor; then, denote by $\mathcal{X}'$ the Zariski closure of X' in $\mathcal{A}'$. The **naive height** of X associated to the model $\pi \colon \mathcal{A} \to B$ and the line bundles $\mathcal{L}$ and M is defined by the intersection number

$$h(X) = \frac{1}{[K' : K]} \left(\mathcal{X}' \cdot c_1(\rho^* \mathcal{L})^{d_X+1} \cdot \rho^* \pi^* (c_1(M))^{d_B-1} \right) \tag{3.1}$$

where $d_X = \dim X$ and $d_B = \dim B$. It depends on the model $\mathcal{A}$ and the extension $\mathcal{L}$ of L to $\mathcal{A}$ but it does not depend on the choice of K'.

The **canonical height** is the limit

$$\hat{h}(X) = \lim_{n \to +\infty} \frac{h([n]_* X)}{n^{2(d_X+1)}} = \lim_{n \to +\infty} \frac{\deg([n]|_X) h([n]X)}{n^{2(d_X+1)}}. \tag{3.2}$$

It depends on L but not on the model $(\mathcal{A}, \mathcal{L})$; we refer to Gubler's work [8] for more details. By [12, Theorem 5.4, Chapter 6], the condition $\hat{h}(X) = 0$ does not depend on L. In particular, we may modify $\mathcal{L}$ on special fibers to assume that $\mathcal{L}$ *is ample*. See also [9, Section 3].

Now we reformulate the canonical height in differential geometric terms. For simplicity, assume that X is already defined over K. Set $\mathcal{A}_1 := \mathcal{A}$, $\pi_1 := \pi$ and $\mathcal{L}_1 := \mathcal{L}$. Pick a Kähler form α_1 in $c_1(\mathcal{L})$ (such a form exists because we choose $\mathcal{L}$ ample). For every $n \geq 1$, there exists an irreducible smooth projective scheme $\pi_n : \mathcal{A}_n \to B$ over B, extending $\pi|_{\mathcal{A}^o} : \mathcal{A}^o \to B^o$, such that the rational map $[n] : \mathcal{A}^o \to \mathcal{A}^o$ lifts to a morphism $f_n : \mathcal{A}_n \to \mathcal{A}$ over B. Write $\mathcal{L}_n := f_n^* \mathcal{L}$ and $\alpha_n := f_n^* \alpha_1$. Denote by $\mathcal{X}_n$ the Zariski closure of $\mathcal{X}^o$ in $\mathcal{A}_n$. Since the Kähler form ν introduced in Section 1.2.1 represents the class $c_1(M)$, the projection formula gives

$$\begin{aligned} \hat{h}(X) &= \lim_{n\to\infty} n^{-2(d_X+1)} (\mathcal{X}_n \cdot \mathcal{L}_n^{d_X+1} \cdot (\pi_n^* M)^{d_B-1}) && (3.3)\\ &= \lim_{n\to\infty} n^{-2(d_X+1)} \int_{\mathcal{X}_n} \alpha_n^{d_X+1} \wedge (\pi_n^* \nu)^{d_B-1} && (3.4)\\ &= \lim_{n\to\infty} n^{-2(d_X+1)} \int_{\mathcal{X}^o} ([n]^* \alpha)^{d_X+1} \wedge (\pi^* \nu)^{d_B-1} && (3.5) \end{aligned}$$

because the integral on $\mathcal{X}_n$ is equal to the integral on the dense Zariski open subset $\mathcal{X}^o$ (and even on the regular locus $\mathcal{X}^{o,reg}$).

3.2. **Gubler-Zhang inequality.** By definition, the **essential height** $\mathrm{ess}(X)$ of a subvariety $X \subset A$ is the real number

$$\mathrm{ess}(X) = \sup_Y \inf_{x \in X(\overline{K}) \setminus Y} \hat{h}(x), \tag{3.6}$$

where Y runs through all proper Zariski closed subsets of X. The following inequality is due to Gubler in [9, Lemma 4.1]; it is an analogue of Zhang's inequality [24, Theorem 1.10] over number fields.

$$0 \leq \frac{\hat{h}(X)}{(d_X+1)\deg_L(X)} \leq \mathrm{ess}(X). \tag{3.7}$$

The converse inequality $\mathrm{ess}(X) \leq \hat{h}(X)/\deg_L(X)$ also holds, but we shall not use it in this article.

Definition 3.1. We say that X is **small**, if X_ε is Zariski dense in X for all $\varepsilon > 0$.

The above inequalities comparing $\hat{h}(X)$ to $\mathrm{ess}(X)$ show that X is small if, and only if $\hat{h}(X) = 0$.

Proposition 3.2. *Let $g : A \to A'$ be a morphism of abelian varieties over K, and let $a \in A(K)$ be a torsion point. Let X be an absolutely irreducible subvariety of A over K.*

(1) *If X is small, then $g(X)$ is small.*
(2) *If g is an isogeny and $g(X)$ is small, then X is small.*
(3) *X is small if and only if $a+X$ is small.*

Proof. Assertions (1) and (2) follow from [20, Proposition 2.6.]. To prove the third one fix an integer $n \geq 1$ such that $na = 0$. By assertions (1) and (2), $a+X$ is small if and only if $[n](a+X) = [n](X)$ is small, if and only if X is small. □

3.3. **Smallness and the Betti form.** Here is the key relationship between the density of small points and the Betti form.

Theorem B. *Let X be an absolutely irreducible subvariety of A over $\mathbf{C}(B)$. If X is small, then*

$$\int_{X^o} \omega^{d_X+1} \wedge (\pi^*\nu)^{d_B-1} = 0,$$

with ω the Betti form associated to L and ν the Kähler form on B representing the class $c_1(M)$.

Proof. Since X is small, $\hat{h}(X) = 0$ and equation (3.5) shows that

$$0 = \hat{h}(X) = \lim_{n\to\infty} n^{-2(d_X+1)} \int_{X^o} ([n]^*\alpha)^{d_X+1} \wedge (\pi^*\nu)^{d_B-1}. \tag{3.8}$$

Let $U \subset B^o$ be any relatively compact open subset of B^o in the euclidean topology. There exists a constant $C_U > 0$ such that $C_U\alpha - \omega$ is semi-positive on $\pi^{-1}(U)$. Since $[n] : \mathcal{A}^o \to \mathcal{A}^o$ is regular, the $(1,1)$-form $n^{-2}[n]^*(C_U\alpha - \omega) = C_U n^{-2}[n^*]\alpha - \omega$ is semi-positive. Since ω and ν are semi-positive, we get

$$0 \leq \int_{\pi^{-1}(U)\cap X^o} \omega^{d_X+1} \wedge (\pi^*\nu)^{d_B-1} \leq \left(\frac{C_U}{n^2}\right)^{d_X+1} \int_{X^o} ([n]^*\alpha)^{d_X+1} \wedge (\pi^*\nu)^{d_B-1}$$

for all $n \geq 1$. Letting n go to $+\infty$, equation (3.8) gives

$$\int_{\pi^{-1}(U)\cap \mathcal{X}^o} \omega^{d_X+1} \wedge (\pi^*\nu)^{d_B-1} = 0. \tag{3.9}$$

Since this holds for all relatively compact subsets U of B^o, the theorem is proved. □

Corollary 3.3. *Assume that X is small. Let U and V be open subsets of B^o and $\mathcal{X}^o$ with respect to the euclidean topology such that U contains the closure of $\pi(V)$. Let μ be any smooth real semi-positive $(1,1)$-form on U. We have*

$$\int_V \omega^{d_X+1} \wedge (\pi^*\mu)^{d_B-1} = 0.$$

Proof of the Corollary. Since ω and μ are semi-positive, the integral is non-negative. Since ν is strictly positive on U, there is a constant $C > 0$ such that $C\nu - \mu$ is semi-positive. From Theorem B we get

$$0 \leq \int_V \omega^{d_X+1} \wedge (\pi^*\mu)^{d_B-1} \leq C^{d_B-1} \int_V \omega^{d_X+1} \wedge (\pi^*\nu)^{d_B-1} = 0, \tag{3.10}$$

and the conclusion follows. □

Theorem B'. *Assume that X is small. Then at every point $p \in \mathcal{X}^o$, we have $T_p\mathcal{F} \subseteq T_p\mathcal{X}^o$. In other words, $\mathcal{X}^o$ is invariant under the Betti foliation: for every $p \in \mathcal{X}^o$, the leaf $\mathcal{F}_p$ is contained in $\mathcal{X}^o$.*

Proof. We start with a simple remark. Let $P\colon \mathbf{C}^{N+1} \to \mathbf{C}^N$ be a complex linear map of rank N. Let ω_0 be a positive $(1,1)$-form on $\mathbf{C}^N$. If V is a complex linear subspace of $\mathbf{C}^{N+1}$ of dimension N, then $\ker(P) \subset V$ if and only if $P|V$ is not onto, if and only if $(P^*\omega_0^N)|V = 0$. Now, assume that B has dimension 1. Then, the integral of ω^{d_X+1} on $\mathcal{X}^o$ vanishes; since the form ω is non-negative, the remark implies that the kernel of P_p from Section 2.3 is contained in $T_p\mathcal{X}^o$ at every smooth point p of $\mathcal{X}^o$. This proves the proposition when $d_B = 1$.

The general case reduces to $d_B = 1$ as follows. Let U and U' be open subsets of $B^o(\mathbf{C})$ such that: (i) $\overline{U} \subset U'$ in the euclidean topology and (ii) there are complex coordinates (z_j) on U' such that $U = \{|z_j| < 1, j = 1, \ldots, d_B\}$. Set

$$\mu := i(dz_2 \wedge d\overline{z_2} + \ldots + dz_{d_B} \wedge d\overline{z_{d_B}}). \tag{3.11}$$

It is a smooth real non-negative $(1,1)$-form on U'. By Corollary 3.3, we have

$$\int_{\pi^{-1}(U)\cap \mathcal{X}} \omega^{d_X+1} \wedge (\pi^*\mu)^{d_B-1} = 0. \tag{3.12}$$

For $(w_2, \ldots, w_{d_B})$ in $\mathbf{C}^{d_B-1}$ with norm $|w_i| < 1$ for all i, consider the slice

$$\mathcal{X}(w_2, \ldots, w_{d_B}) = \mathcal{X} \cap \pi^{-1}(U \cap \{z_2 = w_2, \ldots, z_{d_B} = w_{d_B}\}); \tag{3.13}$$

this slice provides a family of subsets of $\mathcal{A}$ over the one-dimensional disk $\{(z_1, w_2, \ldots, w_{d_B})\ ;\ |z_1| < 1\}$. Then, the integral of ω^{d_X+1} over $\mathcal{X}(w_2, \ldots, w_{d_B})$ vanishes for almost every point $(w_2, \ldots, w_{d_B})$; from the case $d_B = 1$, we deduce that, at every point p of $\mathcal{X}^o \cap \pi^{-1}U$, the tangent $T_p\mathcal{X}^o$ intersects $T_p\mathcal{F}$ on a line whose projection in $T_{\pi(p)}B$ is the line $\{z_2 = \cdots = z_{d_B} = 0\}$. Doing the same for all coordinates z_i, we see that $T_p\mathcal{F}$ is contained in $T_p\mathcal{X}^o$. □

As a direct application of Theorem B' and Remark 2.3, we prove Theorem A in the isotrivival case.

Corollary 3.4. *If $A_{\overline{K}} = A^{\overline{K}/\mathbf{C}} \otimes_{\mathbf{C}} \overline{K}$ and X is small, then there exists a subvariety $Y \subseteq A^{\overline{K}/\mathbf{C}}$ such that $X \otimes_K \overline{K} = Y \otimes_{\mathbf{C}} \overline{K}$.*

Proof. Replacing K by a suitable finite extension K' and then B by its normalization in K', we may assume that $\mathcal{A}^o = B^o \times A^{\overline{K}/\mathbf{C}}$ and that $\pi \colon \mathcal{A}^o \to B$ is the projection to the first factor. By Remark 2.3, the leaves of the Betti foliation are exactly the fibers of the projection π_2 onto the second factor. Since X is small, Theorem B' shows that $\mathcal{X} = \pi_2^{-1}(Y)$, with $Y := \pi_2(\mathcal{X})$. □

4. Invariant analytic subsets of real and complex tori

Let m be a positive integer. Let $M = \mathbf{R}^m/\mathbf{Z}^m$ be the torus of dimension m and $\pi \colon \mathbf{R}^m \to M$ be the natural projection. The group $\mathsf{GL}_m(\mathbf{Z})$ acts by real analytic homomorphisms on M. In this section, we study analytic subsets of M which are invariant under the action of a subgroup $\Gamma \subset \mathsf{SL}_m(\mathbf{Z})$. The main ingredient is a result of Muchnik and of Guivarc'h and Starkov.

4.1. **Zariski closure of Γ.** We denote by

$$G = \mathrm{Zar}(\Gamma)^{irr} \tag{4.1}$$

the neutral component, for the Zariski topology, of the Zariski closure of Γ in $\mathsf{GL}_m(\mathbf{R})$. We shall assume that G is semi-simple. The real points $G(\mathbf{R})$ form a real Lie group, and the neutral component in the euclidean topology is denoted $G(\mathbf{R})^+$. Let Γ_0 be the intersection of Γ with $G(\mathbf{R})^+$; then Γ_0 is both contained in $\mathsf{GL}_m(\mathbf{Z})$ and Zariski dense in G: every polynomial equation that vanishes identically on Γ_0 vanishes also on G. But the Zariski closure of Γ_0 in $\mathsf{GL}_m(\mathbf{R})$ may be larger than $G(\mathbf{R})^+$ (it may include other connected components).

We shall denote by V the vector space $\mathbf{R}^m$; the lattice $\mathbf{Z}^m$ determines an integral, hence a rational structure on V. The Zariski closures $\mathrm{Zar}(\Gamma)$ and $\mathrm{Zar}(\Gamma_0)$ are $\mathbf{Q}$-algebraic subgroups of SL_m for this rational structure.

We shall say that Γ (or G) has **no trivial factor** if every G-invariant vector $u \in V$ is equal to 0. Note that this notion depends only on G, not on Γ.

4.2. **Results of Muchnik and Guivarc'h and Starkov.** Assume that V is an irreducible representation of G over $\mathbf{Q}$; this means that every proper $\mathbf{Q}$-subspace of V which is G-invariant is the trivial subspace $\{0\}$. We decompose V into irreducible subrepresentations of G over $\mathbf{R}$,

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_s. \tag{4.2}$$

To each W_i corresponds a subgroup G_i of $\mathsf{GL}(W_i)$ given by the restriction of the action of G to W_i. Some of the groups $G_i(\mathbf{R})$ may be compact, and we denote by V_c the sum of the corresponding subspaces: V_c is the maximal G-invariant subspace of V on which $G(\mathbf{R})$ acts by a compact factor. It is a proper subspace of V; indeed, if V_c were equal to V then $G(\mathbf{R})$ would be compact, Γ would be finite, and G would be trivial (contradicting the non-existence of trivial factor).

Theorem 4.1 (Muchnik [14]; Guivarc'h and Starkov [10])**.** *Assume that G is semi-simple, and its representation on $\mathbf{Q}^m$ is irreducible. Let x be an element of M. Then, one of the following two exclusive properties occur*

(1) *the Γ-orbit of x is dense in M;*
(2) *there exists a torsion point $a \in M$ such that $x \in a + \pi(V_c)$.*

In the second assertion, the torsion point a is uniquely determined by x, because otherwise V_c would contain a non-zero rational vector and the representation V would not be irreducible over $\mathbf{Q}$. As a corollary, if $F \subset M$ is a closed, proper, connected and Γ-invariant subset, then F is contained in a translate of $\pi(V_c)$ by a (unique) torsion point. Also, if x is a point of M with a finite orbit under the action of Γ, then x is a torsion point.

Remark 4.2. Theorem 4.1 will be used to describe Γ-invariant real analytic subsets $Z \subset M$. If it is infinite, such a set contains the image of a non-constant real analytic curve. The existence of such a curve in Z is the main difficulty in Muchnik's argument, but in our situation it is given for free.

Remark 4.3. Assume that $m = 2g$ for some $g \geq 1$ and M is in fact a complex torus $\mathbf{C}^g/\Lambda$, with $\Lambda \simeq \mathbf{Z}^{2g}$. Suppose that F is a complex analytic subset of M. The inclusion $F \to M$ factors through the Albanese torus $F \to A_F$ of F, via a morphism $A_F \to M$, and the image of A_F is the quotient of a subspace W in $\mathbf{C}^g$ by a lattice $W \cap \Lambda$. So, if $F \subset a + \pi(V_c)$, the subspace V_c contains a subspace $W \subset \mathbf{R}^m$ which is defined over $\mathbf{Q}$, contradicting the irreducibility assumption. To separate clearly the arguments of complex geometry from the arguments of dynamical systems, we shall not use this type of idea before Section 4.4.

Remark 4.4. Theorem 2 of [10] should assume that the group G has no compact factor (this is implicitely assumed in [10, Proposition 1.3]).

4.3. **Invariant real analytic subsets.** Let F be an analytic subset of M. We say that F does not **fully generate** M if there is a proper subspace W of V and a non-empty open subset $\mathcal{U}$ of F such that $T_xF \subset W$ for every regular point x of F in $\mathcal{U}$. Otherwise, we say that F fully generates M.

Proposition 4.5. *Let Γ be a subgroup of $\mathsf{GL}_m(\mathbf{Z})$. Assume that the neutral component $\mathrm{Zar}(\Gamma)^{irr} \subset \mathsf{GL}_m(\mathbf{R})$ is semi-simple, and has no trivial factor. Let F be a real analytic and Γ-invariant subset of M. If F fully generates M, it is equal to M.*

To prove this result, we decompose the linear representation of $G = \mathrm{Zar}(\Gamma)^{irr}$ on V into a direct sum of irreducible representations over $\mathbf{Q}$:

$$V = V_1 \oplus \cdots \oplus V_s. \tag{4.3}$$

Since there is no trivial factor, non of the V_i is the trivial representation. For each index i, we denote by $V_{i,c}$ the compact factor of V_i. The projection π is a diffeomorphism from $V_{i,c}$ onto its image in M_i, because otherwise $V_{i,c}$ would contain a non-zero vector in $\mathbf{Z}^m$ and V_i would not be an irreducible representation over $\mathbf{Q}$. Set

$$M_i = V_i/(\mathbf{Z}^m \cap V_i). \tag{4.4}$$

Then, each M_i is a compact torus of dimension $\dim(V_i)$, and M is isogenous to the product of the M_i. We may, and we shall assume that M is in fact equal to this product:

$$M = M_1 \times \cdots \times M_s; \tag{4.5}$$

this assumption simplifies the exposition without any loss of generality, because the image and the pre-image of a real analytic set by an isogeny is analytic too. We also assume, with no loss of generality, that Γ is contained in G. For every index $1 \leq i \leq s$, we denote by π_i the projection on the i-th factor M_i.

Lemma 4.6. *If F fully generates M, the projection $F_i := \pi_i(F)$ is equal to M_i for every $1 \leq i \leq s$.*

Proof. By construction, F_i is a closed, Γ-invariant subset of M_i. Fix a connected component F_i^0 of F_i. If it were contained in a translate of $\pi(V_{i,c})$, then F would not fully generate M. Thus, Theorem 4.1 implies $F_i^0 = M_i$. □

We do an induction on the number s of irreducible factors. For just one factor, this is the previous lemma. Assuming that the proposition has been proven for $s-1$ irreducible factors, we now want to prove it for s factors. To simplify the exposition, we suppose that $s = 2$, which means that M is the product of just two factors $M_1 \times M_2$. The proof will only use that $\pi_1(f) = M_1$ and F fully generates M; thus, changing M_1 into $M_1 \times \ldots \times M_{s-1}$, this proof also establishes the induction in full generality.

There is a closed subanalytic subset Z_1 of M_1 with empty interior such that π_1 restricts to a locally trivial analytic fibration from $F \setminus \pi_1^{-1}(Z_1)$ to $M_1 \setminus Z_1$. If F does not coincide with M, the fiber F_x is a proper, non-empty analytic subset of $\{x\} \times M_2$ for every x in $M_1 \setminus Z_1$. We shall derive a contradiction from the fact that F fully generates M.

Theorem 4.1 tells us that, for every torsion point x in $M_1 \setminus Z_1$, there is a finite set of points $a_j(x)$ in M_2 such that

$$F_x \subset \bigcup_{j=1}^{J} a_j(x) + \pi(V_{2,c}); \tag{4.6}$$

the number of such points $a_j(x)$ is bounded from above by the number of connected components of F_x. Since torsion points are dense in M_1, this property holds for every point x in $M_1 \setminus Z_1$ (the $a_j(x)$ are not torsion points a priori). Since there are points with a dense Γ-orbit in M_1, we can assume that the number J of points $a_j(x)$ does not depend on x.

Assume temporarily that $J = 1$, so that F_x is contained in $a(x) + \pi(V_{2,c})$ for some point $a(x)$ of M_2. The point $a(x)$ is not uniquely defined by this property (one can replace it by $a(x) + \pi(v)$ for any $v \in V_{2,c}$), but there is a way to choose $a(x)$ canonically. First, the action of $G(\mathbf{R})$ on $V_{2,c}$ factors through a compact subgroup of $\mathsf{GL}(V_{2,c})$, so we can fix a $G(\mathbf{R})$-invariant euclidean metric dist_2 on $V_{2,c}$. Then, any compact subset K of $V_{2,c}$ is contained in a unique ball of smallest radius for the metric dist_2; we denote by $c(K)$ and $r(K)$ the center and radius of this ball. Since the projection π is a diffeomorphism from $V_{2,c}$ onto its image in M_2, the center of F_x inside the translate of $\pi(V_{2,c})$ containing F_x is a well defined point

$$c(x) := c(F_x) \tag{4.7}$$

of M_2 such that F_x is contained in $c(x) + \pi(V_{2,c})$. When $J > 1$, this procedure gives a finite set of centers $\{c_j(x)\}_{1 \leq j \leq J}$.

The centers $c_j(x)$ and the radii $r_j(x)$ are (restricted) sub-analytic functions of x. Thus, there is a proper, closed analytic subset D_1 of M_1, containing Z_1, such that all $r_j(x)$ and $c_j(x)$ are smooth and analytic on its complement (see [1, 3, 16]). Let $\mathcal{G}$ be the subset of $\pi_1^{-1}(M_1 \setminus D_1)$ given by the union of the graphs of the centers: $\mathcal{G} = \{(x,y) \in M_1 \times M_2;\, x \in M_1 \setminus D_1,\ y = c_j(x) \text{ for some } j\}$.

Lemma 4.7. *The set $\mathcal{G}$ is contained in finitely many translates of subtori of $M_1 \times M_2$, each of dimension* $\dim M_1$.

This lemma concludes the proof of Proposition 4.5, because if $\mathcal{G}$ is locally contained in $a + \pi(W)$ for some proper subset W of V of dimension $\dim M_1$, then F is locally contained in $a + \pi(W + V_{2,c})$, and F does not fully generate M because $\dim(W + V_{2,c}) < \dim V$.

Proof. By construction, $\mathcal{G}$ is a smooth analytic subset of $\pi_1^{-1}(M_1 \setminus D_1)$ and it is invariant by Γ. For x in $M_1 \setminus D_1$, we denote by $\mathcal{G}_x$ the finite fiber $\pi_1^{-1}(x) \cap \mathcal{G}$.

For every torsion point $x \in M_1 \setminus D_1$, the stabilizer Γ_x of x is a finite index subgroup of Γ that preserves the finite set $\mathcal{G}_x$. Hence, $\mathcal{G}_x$ is a finite set of torsion points of M, and a finite index subgroup Γ'_x of Γ_x fixes individually each of the points $z \in \mathcal{G}_x$. In particular, torsion points are dense in $\mathcal{G}$. Fix one of these torsion points $z = (x, y)$ with x in $M_1 \setminus D_1$, and consider the tangent subspace $T_z\mathcal{G}$. It is the graph of a linear morphism $\varphi_z \colon T_xM_1 \to T_yM_2$. Identifying the tangent spaces T_xM_1 and T_yM_2 with V_1 and V_2 respectively, φ_z becomes a morphism that interlaces the representations ρ_1 and ρ_2 of Γ'_x on V_1 and V_2; since Γ'_x is Zariski dense in G, we get

$$\rho_2(g) \circ \varphi_z = \varphi_z \circ \rho_1(g) \tag{4.8}$$

for every g in G. In other words, $\varphi_z \in \mathsf{End}(V_1; V_2)$ is a morphism of G-spaces. This holds for every torsion point z of $\mathcal{G}$; by continuity of tangent spaces and density of torsion points, this holds everywhere on $\mathcal{G}$.

Since $\mathcal{G}$ is Γ-invariant, we also have

$$\varphi_{g(z)} \circ \rho_1(g) = \rho_2(g) \circ \varphi_z \tag{4.9}$$

for all $g \in \Gamma$ and $z \in \mathcal{G}$. Then equation (4.8) shows that $\varphi_{g(z)} = \varphi_z$, which means that the tangent space $T_z\mathcal{G}$ is constant along the orbits of Γ. Taking a point z in $\mathcal{G}$ whose first projection has a dense Γ-orbit in M_1, we see that the tangent space $w \in \mathcal{G} \mapsto T_w\mathcal{G}$ takes only finitely many values, at most $|\mathcal{G}_{\pi_1(z)}|$.

Let $(W_j)_{1 \leq j \leq k}$ be the list of possible tangent spaces $T_z\mathcal{G}$. Locally, near any point $z \in \mathcal{G}$, $\mathcal{G}$ coincides with $z + \pi(W_j)$ for some j. By analytic continuation $\mathcal{G}$ contains the intersection of $z + \pi(W_j)$ with $\pi_1^{-1}(M_1 \setminus D_1)$; thus, W_j is a rational subspace of V and $\pi(W_j)$ is a subtorus of M. Then $\mathcal{G}$ is contained in a finite union of translates of the tori $\pi(W_j)$. □

4.4. **Complex analytic invariant subsets.** Let J be a complex structure on $V = \mathbf{R}^m$, so that M is now endowed with a structure of complex torus. Then, $m = 2g$ for some integer g, $\mathbf{R}^m$ can be identified to $\mathbf{C}^g$, and $M = \mathbf{C}^g/\Lambda$ where Λ is the lattice $\mathbf{Z}^m$; to simplify the exposition, we denote by A the complex torus $\mathbf{C}^g/\Lambda$ and by M the real torus $\mathbf{R}^m/\mathbf{Z}^m$. Thus, A is just M, together with the complex structure J. Let X be an irreducible complex analytic subset of A, and let X^{reg} be its smooth locus.

Lemma 4.8. *Let W be the real subspace of V generated by the tangent spaces T_xX, for $x \in X^{reg}$. Then W is both complex and rational, and X is contained in a translate of the complex torus $\pi(W)$.*

Proof. Since X is complex, its tangent space is invariant under the complex structure: $\mathrm{J}T_xX = T_xX$ for all $x \in X^{reg}$. So, the sum $W := \sum_x T_xX$ of the T_xX over all points $x \in X^{reg}$ is invariant by J and W is a complex subspace of $V \simeq \mathbf{C}^g$. Observe that if V' is any real subspace of V such that $\pi(V')$ contains some translate of X^{reg}, then $W \subseteq V'$.

Let a be a point of X^{reg}, and Y be the translate $X - a$ of X. It is an irreducible complex analytic subset of A that contains the origin 0 of A and satisfies $T_yY \subset W$ for every $y \in Y^{reg}$. Thus, Y^{reg} is contained in the projection $\pi(W) \subset A$. Set $Y^{(1)} = Y$, $Y_o^{(1)} = Y^{reg}$ and then

$$Y^{(\ell+1)} = Y^{(\ell)} - Y^{(\ell)}, \quad Y_o^{(\ell+1)} = Y_o^{(\ell)} - Y_o^{(\ell)} \tag{4.10}$$

for every integer $\ell \geq 1$. Since $Y^{(1)}$ is irreducible, and $Y^{(2)}$ is the image of $Y^{(1)} \times Y^{(1)}$ by the complex analytic map $(y_1, y_2) \mapsto y_1 - y_2$, we see that $Y^{(2)}$ is an irreducible complex analytic subset of A. Moreover $Y_o^{(2)}$ is a connected, dense, and open subset of $Y^{(2),reg}$. Observe that $Y_o^{(2)}$ is contained in $\pi(W)$ and contains $Y_o^{(1)}$ because $0 \in Y_o^{(1)}$. By a simple induction, the sets $Y^{(\ell)}$ form an increasing sequence of irreducible complex analytic subsets of A, and $Y_o^{(\ell)}$ is a connected, dense and open subset of $Y^{(\ell),reg}$ that is contained in $\pi(W)$. By the Noether property, there is an index $\ell_0 \geq 1$ such that $Y^{(\ell)} = Y^{(\ell_0)}$ for every $\ell \geq \ell_0$. This complex analytic set is a subgroup of A, hence it is a complex subtorus. Write $Y^{(\ell_0)} = \pi(V')$ for some rational subspace V' of V. Since $Y \subset \pi(V')$, we get $W \subseteq V'$. Since $Y_o^{(\ell_0)} \subseteq \pi(W)$, we derive $V' = T_xY_o^{(\ell_0)} \subseteq W$ for every $x \in Y_o^{(\ell_0)}$. This implies $W = V'$, and shows that W is rational.

Thus, $\pi(W)$ is a complex subtorus of A. Since T_xX is contained in W for every regular point, X is locally contained in a translate of $\pi(W)$. Being irreducible, X is connected, and it is contained in a unique translate $a + \pi(W)$. □

Lemma 4.9. *Let X be an irreducible complex analytic subset of A. The following properties are equivalent:*

(i) *X is contained in a translate of a proper complex subtorus $B \subset A$;*
(ii) *X does not fully generate M;*
(iii) *there is a proper real subspace V' of V that contains T_xX for every $x \in X^{reg}$.*

Proof. Obviously (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). We now prove that (ii) implies (i). If X does not fully generate M, then (iii) is satisfied on some non-empty open subset $\mathcal{U}$ of X^{reg}. Since X^{reg} is connected and locally analytic, we deduce from analytic continuation that $T_xX \subset V'$ for every regular point of X. From Lemma 4.8, X is contained in a complex subtorus $B = \pi(W) \subset A$ for some complex subspace W of V'. □

Theorem 4.10. *Let Γ be a subgroup of $\mathsf{SL}_m(\mathbf{Z})$. Assume that the neutral component for the Zariski topology of the Zariski closure of Γ in $\mathsf{SL}_m(\mathbf{R})$ is semi-simple and has no trivial factor. Let J be a complex structure on $M = \mathbf{R}^m/\mathbf{Z}^m$ and let X be an irreducible complex analytic subset of the complex torus $A = (M, \mathrm{J})$. If X is Γ-invariant, it is equal to a translate of a complex subtorus $B \subset A$ by a torsion point.*

Proof. Set $W := \sum_{x \in X^{reg}} T_x X$. Lemma 4.8 shows that W is complex and rational. Since X is Γ-invariant, so is W. Its projection $B = \pi(W)$ is a complex subtorus of A such that

(1) B is Γ-invariant;
(2) B contains a translate $Y = X - a$ of X;
(3) Y fully generates B.

The group Γ acts on the quotient torus A/B and preserves the image of X, *i.e.* the image $\overline{a}$ of a. Since V has no trivial factor, $\overline{a}$ is a torsion point of A/B. Then there exists a torsion point a' in A such that $X \subseteq a' + B$. Replacing a by a' and Γ by a finite index subgroup Γ' which fixes a', we may assume that a is torsion and $Y = X - a$ is invariant by Γ. We apply Proposition 4.5 to B, the restriction Γ_B of Γ to B, and the complex analytic subset Y: we conclude via Lemma 4.9 that Y coincides with B. Thus, $X = a + B$. □

5. Proof of Theorem A

By base change, we may suppose that X is an absolutely irreducible subvariety of A. We assume that X is small (X_ε is dense in X for all $\varepsilon > 0$), and prove that X is a torsion coset of A.

5.1. **Monodromy and invariance.** Let $b \in B^o$ be any point. The monodromy $\rho : \pi_1(B^o) \to \mathsf{GL}_{2g}(\mathbf{Z})$ of the Betti foliation maps the fundamental group of $\pi_1(B^o)$ onto a subgroup $\Gamma := Im(\rho)$ of $\mathsf{GL}_{2g}(\mathbf{Z})$ that acts by linear diffeomorphisms on the torus $\mathcal{A}_b \simeq \mathbf{R}^{2g}/\mathbf{Z}^{2g}$. As in Section 4.1, we denote by G the neutral component $Zar(\Gamma)^{irr}$. We let V^G denote the subspace of elements $v \in \mathbf{R}^{2g}$ which are fixed by G. By Deligne's semi-simplicity theorem, the group G is semi-simple (see [4, Corollary 4.2.9]). Theorem B' implies that X is invariant under the Betti foliation, so that X_b is invariant under the action of Γ.

5.2. **Trivial trace.** We first treat the case when $A^{\overline{K}/\mathbf{C}}$ is trivial. According to [21, Theorem 1.5], this is the only case we need to treat. However we shall also treat the case of a non-trivial trace below for completeness.

By [4, Corollary 4.1.2] and [7] (see also [4, 4.1.3.2]), we have $V^G = \{0\}$ and Theorem 4.10 implies that $\mathcal{X}_b$ is a translation of an abelian subvariety of $\mathcal{A}_b$ by some torsion point $y_b \in \mathcal{A}_b$. Observe that the leaf $\mathcal{F}_{y_b}$ is an algebraic muti-section of $\mathcal{A}^o$ (see Remark 2.1). By base change, we may assume that $\mathcal{F}_{y_b}$ is a section and is the Zariski closure of a torsion point $y \in A(K)$ in $\mathcal{A}^o$. Theorem B' shows that $y \in X$, and replacing X by $X - y$ we may suppose that $0 \in X$; then $\mathcal{X}_b$ is an abelian subvariety of $\mathcal{A}_b$ for all $b \in B^o$. It follows that $\mathcal{X}^o$ is a subscheme of the abelian scheme $\mathcal{A}^o$ over B^o which is stable under the group laws. So X is an abelian subvariety of A.

5.3. **The general case.** We do not assume anymore that $A^{\overline{K}/\mathbf{C}}$ is trivial. Set $A^t = A^{\overline{K}/\mathbf{C}} \otimes_{\mathbf{C}} K$. Replacing K by a finite extension and A by a finite cover, we assume that $A = A^t \times A^{nt}$ where A^{nt} is an abelian variety over K with trivial trace. We also choose the model $\mathcal{A}$ so that $\mathcal{A}^o = (\mathcal{A}^t)^o \times_{B^o} (\mathcal{A}^{nt})^o$ where $(\mathcal{A}^t)^o$ and $(\mathcal{A}^{nt})^o$ are the Zariski closures of A^t and A^{nt} in $\mathcal{A}^o$ respectively. Denote by $\pi^t : \mathcal{A}^o \to (\mathcal{A}^t)^o$ the projection to the first factor and $\pi^{nt} : \mathcal{A}^o \to (\mathcal{A}^{nt})^o$ the projection to the second factor. After replacing K by a further finite extension and B by its normalization, we may assume that $(\mathcal{A}^t)^o = A^{\overline{K}/\mathbf{C}} \times B^o$. Note that $\pi^t|_{\mathcal{A}^t_b} : \mathcal{A}^t_b \to A^{\overline{K}/\mathbf{C}}$ is an isomorphism for every fiber $\mathcal{A}^t_b$ with $b \in B^o$.

By Proposition 3.2-(i), the generic fibers of $\pi^t(\mathcal{X}^o)$ and $\pi^{nt}(\mathcal{X}^o)$ are small. Corollary 3.4 shows that $\pi^t(\mathcal{X}^o) = Y \times B^o$ for some subvariety Y of $A^{\overline{K}/\mathbf{C}}$. Section 5.2 shows that the geometric generic fiber of $\pi^{nt}(\mathcal{X}^o)$ is a torsion coset $a + \mathcal{A}'$ for some torsion point $a \in A^{nt}_{\overline{K}}(\overline{K})$ and some abelian subvariety A'. Replacing K by a finite extension, we may assume that a and A' are defined over K. We have that $\mathcal{X}^o \subseteq \pi^t(\mathcal{X}) \times_{B^o} \pi^{nt}(\mathcal{X}) = \pi^t(\mathcal{X}) + \pi^{nt}(\mathcal{X})$ and we only need to show that $\mathcal{X}^o = \pi^t(\mathcal{X}) \times_{B^o} \pi^{nt}(\mathcal{X})$.

For every $b \in B^o$, $\mathcal{A}_b = \mathcal{A}^t_b \times \mathcal{A}^{nt}_b$. The monodromy on $\mathcal{A}_b$ is the diagonal product of the monodromies on each factor. It is trivial on the first one so, for every $x \in \mathcal{A}^t_b$, the fiber $\pi^t|^{-1}_{\mathcal{A}_b}(x) \simeq \mathcal{A}^{nt}_b$ is invariant under Γ. It follows that $\pi^t|^{-1}_{\mathcal{A}_b}(x) \cap \mathcal{X}_b$ is also Γ-invariant. By Theorem 4.10, $\pi^{nt}(\pi^t|^{-1}_{\mathcal{A}_b}(x) \cap \mathcal{X}_b) \subseteq \pi^{nt}(\mathcal{X}_b)$ is a torsion coset of the abelian variety $\mathcal{A}^{nt}_b$. Since the set of all torsion cosets of $\pi^{nt}(\mathcal{X}_b)$ is countable, $\pi^{nt}(\pi^t|^{-1}_{\mathcal{A}_b}(x) \cap \mathcal{X}_b)$ does not depend on $x \in \pi^t(\mathcal{X}_b)$. Hence, $\mathcal{X}_b = \pi^t(\mathcal{X}_b) \times \pi^{nt}(\mathcal{X}_b)$ for all $b \in B^o$. Then $\mathcal{X}^o = \pi^t(\mathcal{X}) \times_{B^o} \pi^{nt}(\mathcal{X})$ which concludes the proof.

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