FREE ACTIONS OF LARGE GROUPS ON COMPLEX THREEFOLDS

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ABSTRACT. We classify compact Kähler threefolds X with a free group of automorphisms acting freely on X.

Recall that an action of a group Γ on a set X is **free** if the stabilizer of any point $x \in X$ is reduced to the trivial subgroup $\{\mathsf{Id}\} \subset \Gamma$. If M is a compact complex manifold, we denote by $\mathsf{Aut}(M)$ its group of automorphisms, i.e. of holomorphic diffeomorphisms.

Theorem A.– *Let M be a compact Kähler manifold of dimension ≤ 3. If $\mathsf{Aut}(M)$ contains a non-amenable subgroup Γ acting freely on M, then* $\dim(M) = 3$ *and M is a compact torus $\mathbf{C}^3/\Lambda$.*

Moreover, there are examples of non-abelian free groups acting freely on some tori of dimension 3. First, we look at low dimensional tori, then we construct such examples, and we conclude with a proof of Theorem A. In Theorem A, we assume M to be smooth: we refer to Remark 3.5 for an extension to singular varieties. Section 5 gives counter-examples when M is not Kähler.

1. COMPACT TORI

Let $A = \mathbf{C}^n/\Lambda$ be a compact torus of dimension n. Every automorphism $f\colon A \to A$ comes from an affine transformation

$$\hat{f}(z) = L(f)(z) + T(f) \tag{1.1}$$

of $\mathbf{C}^n$, where the translation part is a vector $T(f) \in \mathbf{C}^n$ and the linear part $L(f) \in \mathsf{GL}_n(\mathbf{C})$ preserves the lattice Λ. This defines a homomorphism $\mathsf{Aut}(A) \to \mathsf{GL}_n(\mathbf{C})$, $f \mapsto L(f)$. The following assertions are equivalent:

(i) f has no fixed point;
(ii) the image of the linear transformation $(L - \mathsf{Id})$ does not intersect the set $T(f) + \Lambda$.

In particular, if Γ acts freely on A, then $\det(L(f) - \mathsf{Id}) = 0$ for every f in Γ.

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Lemma 1.1. *Let A be a complex torus of dimension ≤ 2. If $\Gamma \subset \mathsf{Aut}(A)$ acts freely on A, then Γ is solvable. In particular, every free subgroup of $\mathsf{Aut}(A)$ acting freely on A is cyclic.*

Proof. If $\dim(A) = 1$, then $\mathsf{Aut}(A)$ is solvable. Assume $\dim(A) = 2$, and write $A = \mathbf{C}^2/\Lambda$. A subgroup G of $\mathsf{GL}_2(\mathbf{C})$ such that $\det(L - \mathsf{Id}) = 0$ for every $L \in G$ is solvable; hence, the equivalence of (i) and (ii) implies: if Γ acts freely on A, the groups $L(\Gamma)$ and Γ are solvable. □

2. Examples

2.1. **Closed, real analytic manifolds.** The group $\mathsf{SO}_3(\mathbf{R})$ contains a non-abelian free group $\Gamma \subset \mathsf{SO}_3(\mathbf{R})$. This is well known, since the existence of such a group is at the basis of the Banach-Tarsky paradox (see [4]). Now, the action of Γ on $\mathsf{SO}_3(\mathbf{R})$ by left translations is free, and going to the universal cover SU_2 of $\mathsf{SO}_3(\mathbf{R})$, we obtain a free action of a non-abelian free group on a simply connected manifold. These actions being real analytic, the first assertion of the following theorem is proved.

Theorem 2.1. *There are real analytic, free actions of non-abelian free groups on the following real analytic manifolds:*

(1) *the simply connected, compact Lie group SU_2;*
(2) *the torus $\mathbf{R}^k/\mathbf{Z}^k$ for any $k \geq 2$.*

For the second assertion, it suffices to provide examples in dimension $k = 2$; we refer to Remark 5.1 for such an example. Here, we describe an example in dimension 3 which will work also for abelian threefolds.

Consider the lattice $\mathbf{Z}^3$, together with the standard quadratic form of signature $(1,2)$: $Q(x,y,z) = x^2 - y^2 - z^2$. Its group of isometries $\mathsf{SO}_{1,2}(\mathbf{Z})$ is a lattice in the Lie group $\mathsf{SO}_{1,2}(\mathbf{R})$.

Lemma 2.2. *The group $\mathsf{SO}_{1,2}(\mathbf{Z})$ contains a free subgroup Γ of Schottky type such that the eigenvalues of every element $g \in \Gamma \setminus \{\mathsf{Id}\}$ form a triple of real numbers $\lambda_g > 1 > \lambda_g^{-1}$.*

Proof. Consider the subgroup G_0 of $\mathsf{SO}_{1,2}(\mathbf{R})$ preserving each connected component of $\{Q(x,y,z) = 1\}$. If g is an element of this group, and g has an eigenvalue of modulus > 1, then its eigenvalues are $\lambda_g > 1 > \lambda_g^{-1}$ for some $\lambda_g > 1$; equivalently, g is a loxodromic isometry of the hyperbolic space $\mathbb{H} = \{(x,y,z) |\, Q(x,y,z) = 1 \text{ and } x > 0\}$. Now, take two loxodromic isometries f and

g in $G_0 \cap \mathsf{SO}_{1,2}(\mathbf{Z})$ to which the tennis-table lemma of Fricke and Klein applies (see [4]). Then, the group Γ generated by sufficiently large iterates of f and g is a Schottky group such that each of its element $g \neq \mathsf{Id}$ is loxodromic. $\square$

Choose such a free group Γ, of rank 2, and fix a pair (a,b) of elements generating Γ. In [7], Drumm and Goldman find a non-empty open subset U of $\mathbf{R}^3 \times \mathbf{R}^3$ such that for every $(s,t) \in \mathsf{U}$, the affine transformations

$$A_s(x,y,z) = a(x,y,z) + s, \quad B_t(x,y,z) = b(x,y,z) + t \tag{2.1}$$

generate a free group acting freely and properly on $\mathbf{R}^3$. The group generated by A_s and B_t is an affine deformation $\Gamma_{s,t}$ of Γ; given any reduced word w in a, b, and their inverses, we get an element $w(A_s,B_t)$ in $\Gamma_{s,t}$, which we can write

$$w(A_s,B_t)(x,y,z) = w(a,b)(x,y,z) + L_w(a,b)(s) + R_w(a,b)(t) \tag{2.2}$$

where $L_w(a,b)$ and $R_w(a,b)$ are elements of the algebra generated by a and b in $\mathsf{End}(\mathbf{R}^3)$.

Since a and b are in $\mathsf{SL}_3(\mathbf{Z})$ the group $\Gamma_{s,t}$ acts on the torus $\mathbf{R}^3/\mathbf{Z}^3$. This action is free if, and only if, given any non-trivial reduced word w, and any element (p,q,r) of the lattice $\mathbf{Z}^3$, the equation

$$(\mathsf{Id} - w(a,b))(x,y,z) + (p,q,r) = L_w(a,b)(s) + R_w(a,b)(t) \tag{2.3}$$

has no solution $(x,y,z) \in \mathbf{R}^3$. Fix such a pair $(w,(p,q,r))$. The question becomes: is the vector $L_w(a,b)(s) + R_w(a,b)(t)$ contained in the affine plane $(\mathsf{Id} - w(a,b))(\mathbf{R}^3) + (p,q,r)$? We distinguish two cases. If $(\mathsf{Id} - w(a,b))(\mathbf{R}^3) + (p,q,r)$ is actually a vector subspace, in which case this subspace coincides with $(\mathsf{Id} - w(a,b))(\mathbf{R}^3)$, then we know from Drumm-Goldman result that there is a pair (s,t) for which Equation (2.3) has no solution. If $(\mathsf{Id} - w(a,b))(\mathbf{R}^3) + (p,q,r)$ does not contain the origin, then there is no solution to Equation (2.3) if (s,t) is small enough. Thus, the set $W(w,(p,q,r))$ of parameters $(s,t) \in \mathbf{R}^3 \times \mathbf{R}^3$ such that Equation (2.3) has no solution is non-empty; hence, $W(w,(p,q,r))$ is open and dense (as the complement of a proper affine subspace of $\mathbf{R}^3 \times \mathbf{R}^3$). By Baire theorem, the intersection of all those open dense subsets is non-empty: this precisely means that there are pairs (s,t) such that the free group $\Gamma_{s,t}$ acts freely on $\mathbf{R}^3/\mathbf{Z}^3$.

2.2. **Abelian threefolds.** Consider any lattice $\Lambda_0 \subset \mathbf{C}$, for instance the lattice $\Lambda_0 = \mathbf{Z}[\mathsf{i}]$ with $\mathsf{i}^2 = -1$. Set

$$\Lambda = \Lambda_0 \times \Lambda_0 \times \Lambda_0 \subset \mathbf{C}^3 \tag{2.4}$$

and denote by N the abelian threefold $\mathbf{C}^3/\Lambda$. Now, copy the argument given in the last section (§ 2.1), with the same group $\Gamma_{s,t}$, but viewed as a subgroup of the affine group $\mathsf{SL}_3(\mathbf{Z}) \ltimes \mathbf{C}^3$, acting on $N = \mathbf{C}^3$. We get

Theorem 2.3. *Let $\Lambda_0 \subset \mathbf{C}$ be a cocompact lattice. There is a free action of a non-abelian free group on the abelian threefold $(\mathbf{C}/\Lambda_0)^3$ by holomorphic affine transformations.*

3. PROOF OF THEOREM A

We now prove Theorem A. According to [6, 3], the group $\mathsf{Aut}(M)$ satisfies Tits alternative: *if $\Gamma \subset \mathsf{Aut}(M)$ does not contain a non-abelian free group, then Γ contains a solvable subgroup of finite index and, in particular, is amenable.* Thus, we assume that Γ is a non-abelian free group, acting freely on some compact Kähler manifold M of dimension ≤ 3. We shall use several times the following fact.

Lemma 3.1. *If the group Γ stabilizes a non-empty subset $S \subset M$, the restriction $f \in \Gamma \to f_{|S}$ is an injective homomorphism, and the action of Γ on S is free. If the action of a finite index subgroup $\Gamma_0 \subset \Gamma$ lifts to an action on a finite cover $M' \to M$, then the action of Γ_0 on M' is free.*

3.1. **Kodaira dimension.** If M is a complex manifold, we denote by K_M its canonical bundle.

Lemma 3.2. *Let M be a compact Kähler manifold. If the Kodaira dimension of M is non-negative, either K_M is torsion, or there is a finite index subgroup of $\mathsf{Aut}(M)$ that preserves a proper, non-empty, and irreducible Zariski closed subset $Z \subset M$.*

Proof. We can assume that any proper, Zariski closed, and $\mathsf{Aut}(M)$-invariant subset is empty. Let m be a positive integer. The set $Z(m) \subset M$ of common zeros of all sections Ω of $K_M^{\otimes m}$ is Zariski closed and $\mathsf{Aut}(M)$-invariant. So, $Z(m)$ is empty for all m such that $H^0(M, K_M^{\otimes m}) \neq \{0\}$; for such an m, the pluricanonical map $\Psi_m \colon M \to \mathbb{P}(H^0(M, K_M^{\otimes m})^\vee)$ is a regular morphism; let $B_m \subset \mathbb{P}(H^0(M, K_M^{\otimes m})^\vee)$ denote the image of Ψ_m. Then

(1) Ψ_m is $\mathsf{Aut}(M)$-equivariant: there is a homomorphism $\rho_m \colon f \in \mathsf{Aut}(M) \mapsto \rho_m(f) \in \mathsf{Aut}(B_m)$ such that $\rho_m(f) \circ \Psi_m = \Psi_m \circ \rho_m(f)$ (this homomorphism is given by the linear representation of $\mathsf{Aut}(M)$ on $H^0(M, K_M^{\otimes m})$);
(2) the group $\rho_m(\mathsf{Aut}(X))$ is finite (see [15, Corollary 2.4] and [19]).

Thus, a finite index subgroup of $\mathsf{Aut}(M)$ fixes individually every fiber of Ψ_m. If $\dim(B_m) \geq 1$, this contradicts our standing hypothesis. If $\dim(B_m) = 0$, $H^0(M, K_M^{\otimes m}) = \mathbf{C}\Omega$ for some non-trivial section Ω of $K_M^{\otimes m}$ and Ω does not vanish because $Z(m)$ is empty. So, $K_M^{\otimes m}$ is trivial and K_M is torsion. □

3.2. **Curves and surfaces.**

Lemma 3.3. *If a free group acts freely on a curve, the group is cyclic. If* $\dim(M) \leq 2$ *and* Γ *is a non-abelian free group acting freely on M, then* Γ *does not stabilize any proper, non-empty, Zariski closed subset.*

Proof. Since every automorphism of $\mathbb{P}^1(\mathbf{C})$ has a fixed point, we can assume that the genus of the curve is at least 1, but then its automorphism group is virtually solvable, and any free subgroup is cyclic. The second assertion follows from the first one and from Lemma 3.1. □

Lemma 3.4. *If a free group* Γ *acts freely on a compact, Kähler surface, the group is cyclic.*

Proof. We argue by contradiction, assuming that Γ is not cyclic.

If $h^{2,0}(M) > 0$, Lemmas 3.2 and 3.3 imply that K_M is trivial. If M is a K3 surface, its Euler characteristic is positive, every homeomorphism of M has a periodic point [8], and we get a contradiction. If M is a torus, we get a contradiction from Lemma 1.1. This exhausts all surfaces with K_M trivial.

Now, assume that $h^{2,0}(M) = 0$. Then, from the holomorphic Lefschetz fixed point formula, we must have $h^{1,0}(M) > 0$, since otherwise every automorphism would have a fixed point. Consider the Albanese morphism $\alpha\colon M \to A_M$, where A_M is the Albanese torus of M, and let E be the image of α. Since $h^{1,0}(M) > 0$, we get $\dim(E) \in \{1,2\}$. By Lemma 3.1 and 3.3, every proper Γ-invariant analytic subset of M (resp. of E) is empty. Thus, E is smooth and α is a submersion. First, assume $\dim(E) = 1$. Since $E \subset A_M$ can not be the Riemann sphere, its group of automorphisms is virtually solvable; thus, the kernel of the homomorphism $\Gamma \to \mathsf{Aut}(E)$ is a non-abelian free group, acting freely on the fibers of α, contradicting Lemma 3.3. Thus, $\dim(E) = 2$ and $\alpha\colon M \to E$ is a finite cover. This implies $h^{2,0}(M) > 0$, and we get a contradiction. □

Remark 3.5. Let M be a singular complex projective threefold. Denote by M^0 the singular locus of M, by M^1 the singular locus of M^0, etc. Let Γ be a non-abelian free group acting freely on M by automorphisms. Then, Γ preserves the subvarieties M^i. Since Γ is not cyclic, we deduce from Lemma 3.3 that none of these spaces has dimension 0 or 1. So, the singular locus M^0 is a smooth complex projective surface, and Lemma 3.4 provides a contradiction. So, Theorem A extends to the case of complex projective threefolds.

3.3. **Dimension** 3. Assume $\dim(M) = 3$, M is not a torus, and a non-abelian free group Γ acts freely on M.

a. kod$(M) \geq 0$**. –** First, assume that the Kodaira dimension of M is non-negative. It follows from Lemmas 3.2 to 3.4 that K_M is torsion. Then, after a finite étale cover, K_M is trivial and M is one of the following examples:

(1) a torus of dimension 3;
(2) a (simply connected) Calabi-Yau threefold;
(3) a product of an elliptic curve with a K3 surface.

Lemma 3.6. *If a finite cover of M is a torus, and* $\Gamma \subset \mathsf{Aut}(M)$ *is a non-abelian free group acting freely on M, then M is a torus.*

Thus, the first case is excluded, since we assume that M is not a torus.

Proof. By assumption, there is a torus $A = \mathbf{C}^3/\Lambda$, and a finite group G acting freely on A such that $M = A/G$. By construction, M is a quotient of $\mathbf{C}^3$ by a group of affine transformations $\tilde{G} \subset \mathsf{Aff}(\mathbf{C}^3)$; the group Λ is a finite index subgroup of $\tilde{G}$, and the image of the (linear part) homomorphism $L \colon \tilde{G} \to \mathsf{GL}_3(\mathbf{C})$ is a finite group (isomorphic to G since the action of G on A is free).

The group Γ lifts to a free group of affine transformations of $\mathbf{C}^3$ permuting the orbits of $\tilde{G}$. When f is an element of Γ, we denote by $\hat{f} \colon z \mapsto L(f)z + T(f)$ the corresponding affine transformation. The group $L(\Gamma)$ normalizes $L(\tilde{G})$, and a finite index subgroup $L(\Gamma_0)$ commutes to every element in $L(\tilde{G})$. If G is non-trivial, then $L(\tilde{G})$ contains a non-trivial linear transformation S, S is diagonalizable (because it has finite order), and $L(\Gamma_0)$ preserves its eigenspaces. Since the action of G on A is fixed-point free, the eigenspace E_1 corresponding to the eigenvalue 1 has positive dimension, intersects Λ on a lattice $\Lambda_1 = E_1 \cap \Lambda$, and both E_1 and Λ_1 are invariant under the action of $L(\Gamma_0)$. If S had three distinct eigenvalues, the three eigenlines of S would be $L(\Gamma_0)$ invariant, contradicting the fact that Γ is not virtually solvable. Thus, S has exactly one other eigenvalue α, corresponding to an $L(\Gamma_0)$-invariant eigenspace E_α, with $E_1 \oplus E_\alpha = \mathbf{C}^3$.

Assume $\dim(E_1) = 1$ and $\dim(E_\alpha) = 2$. Since $\mathsf{GL}(E_1)$ is abelian, there is a non-abelian free subgroup Γ_1 of Γ_0 such that $L(\Gamma_2)$ acts trivially on E_1. Then, there is a non-abelian free subgroup Γ_2 of Γ_1 such that the action of $L(\Gamma_2)$ on E_α is made of matrices with eigenvalues $\neq 1$. Take a generating pair $f, g \in \Gamma_2$;

computing the commutator $[\hat{f},\hat{g}]$, we observe that the translation part $T([f,g])$ is contained in E_α ; this implies that $[\hat{f},\hat{g}]$ has a fixed point in $\mathbf{C}^3$, contradicting the assumption on Γ.

The case $\dim(E_1)=2$ is similar: we just have to permute the role of E_1 and E_α. Thus, $L(\tilde{G})$ is trivial and M is actually a torus. □

Assume we are in case (3), with a finite cover M' of M isomorphic to $X\times E$ for some $K3$ surface X; there is a finite group of automorphisms $F\subset \mathsf{Aut}(M')=\mathsf{Aut}(X\times E)$ acting freely on M' such that $M=M'/F$. Every automorphism f of M' preserves the Albanese fibration $\alpha'\colon M'\to E$: this gives a homomorphism $F\to\mathsf{Aut}(E)$, and we denote by F_E its image. The fibration α' determines a fibration $\alpha\colon M\to E/F_E$, and this fibration is Γ-invariant. Two cases may occur. Either E/F_E is a curve of genus 1, its automorphism group is solvable, and we get a contradiction with Lemmas 3.4 and 3.1. Or E/F_E is a Riemann sphere, the fixed points of the elements of F_E correspond to the critical values of α, and a finite index subgroup of Γ preserves the corresponding fibers. Again, Lemma 3.4 provides a contradiction.

We can now assume that we are in case (2), i.e. the universal cover of M is an irreducible Calabi-Yau threefold. Lifting Γ to the universal cover, we can assume that M itself is Calabi-Yau. The action on cohomology gives rise to a homomorphism $\Gamma\to\mathsf{GL}(H^*(M;\mathbf{Z}))$, $f\mapsto f_*=(f^{-1})^*$. Here is the key lemma:

Lemma 3.7. *Let M be a (simply connected) Calabi-Yau threefold. The action of $\mathsf{Aut}(M)$ on the cohomology group $H^3(M;\mathbf{Z})$ factors through a finite group.*

Let f be an automorphism of M. If all orbits of f are infinite then

(1) *the action of f on the cohomology of M is virtually unipotent: there is a positive integer k such that $(f^k)^*$ is unipotent;*
(2) *$h^{2,1}(M)=h^{1,1}(M)$ and the topological Euler characteristic of M is* 0.

Proof. Fix a Kähler class κ on M. Since $h^{p,q}(M)=0$ when $p+q=1$ or 5, we deduce that every class α in $H^{2,1}(M)$ is primitive: $\alpha\wedge\kappa=0$ because $H^{3,2}(M)=0$. Thus, the intersection product $\int_M\alpha\wedge\overline{\alpha}$ determines an $\mathsf{Aut}(X)$-invariant, positive definite quadratic form on the vector space $H^{2,1}(M)$. On $H^{3,0}(M)$, the product $\omega\mapsto\int_M\omega\wedge\overline{\omega}$ is also positive definite. As a consequence, the image of $\mathsf{Aut}(M)$ in $H^3(M;\mathbf{C})$ is contained in a unitary group. Since it preserves the integral structure $H^3(M;\mathbf{Z})$, it is contained in a finite group. This implies that a finite index subgroup of $\mathsf{Aut}(M)$ acts trivially on $H^3(M,\mathbf{Z})$.

Now, apply the holomorphic Lefschetz fixed point theorem: if there is no periodic point, the traces of f^* on $H^{1,1}(M)$ and $H^{2,1}(M)$ satisfy

$$\mathrm{tr}((f^n)^*_{1,1}) = \mathrm{tr}((f^n)^*_{2,1}) \qquad (\forall n \in \mathbf{Z} \setminus \{0\}). \tag{3.1}$$

Changing f in some positive iterate $g = f^m$, we get $\mathrm{tr}((g^n)^*_{2,1}) = h^{2,1}(M)$ for all n, and then we deduce $\mathrm{tr}((g^n)^*_{1,1}) = h^{2,1}(M)$ for all n. This equality implies that $(g^*)_{1,1}$ is unipotent and $h^{2,1}(M) = h^{1,1}(M)$. Thus, $(f^*)_{1,1}$ is virtually unipotent. Since $h^{2,0}(M) = 0$, the action of f^* on $H^*(M,\mathbf{Z})$ is virtually unipotent. □

On a Calabi-Yau threefold, every holomorphic vector field on M is identically 0; this implies that $\mathsf{Aut}(M)^0$ is trivial and that the representation $\mathsf{Aut}(M) \to \mathsf{GL}\,(H^*(M;\mathbf{Z}))$ has a finite kernel (see [14]). So, Γ being a free subgroup of $\mathsf{Aut}(M)$, the representation $\Gamma \to \mathsf{GL}\,(H^*(M;\mathbf{Z}))$ is faithfull. Since Γ acts freely on M, Lemma 3.7 implies that all elements $f^* \in \mathsf{GL}\,(H^*(M;\mathbf{Z}))$, for $f \in \Gamma$, are virtually unipotent: we conclude that Γ is cyclic, because a subgroup of $\mathsf{GL}_m(\mathbf{C})$ all of whose elements are virtually unipotent is a solvable group up to finite index. We obtain the following.

Lemma 3.8. *Let M be a compact Kähler manifold of dimension* 3 *whose Kodaira dimension is non-negative. If there is a non-abelian free subgroup of* $\mathsf{Aut}(M)$ *acting freely on M, then M is a torus.*

b. kod$(M) = -\infty$. – Thus, in what follows, we assume that the Kodaira dimension of M is $-\infty$. In particular, $h^{3,0}(M) = 0$. We prove that non-abelian free groups can not act freely on M.

Assume that $h^{1,0}(M) = 0$, and apply the holomorphic fixed point formula. This gives $\mathrm{tr}((f^n)^*_{2,0}) = -1$ for all elements $f \neq \mathsf{Id}$ of Γ and all integers $n \neq 0$. Again, we get a contradiction. Thus, $h^{1,0}(M) > 0$, and the Albanese map is a non-trivial morphism

$$\alpha \colon M \to A_M, \tag{3.2}$$

where A_M is the Albanese torus of M. This map is equivariant with respect to a homomorphism $\rho \colon \mathsf{Aut}(M) \to \mathsf{Aut}(A_M)$, meaning that $\alpha \circ f = \rho(f) \circ \alpha$ for every $f \in \mathsf{Aut}(M)$. Let $E \subset A_M$ be the image of α. Assuming that the rank of the free group Γ is at least 2, we shall prove successively that

- E is smooth and the map $\alpha \colon M \to E$ is a submersion;
- E has dimension 2,

and then we obtain a contradiction.

If E contains a non-empty Zariski closed proper subset Z which is invariant under the action of $\rho(\Gamma)$, then $\alpha^{-1}(Z)$ is a non-empty, Zariski closed, and Γ-invariant subset of M of dimension is at most 2. If a Zariski closed subset of dimension k is invariant and singular, its singular locus is also invariant and has dimension $\leq k-1$. So Γ preserves a smooth Zariski closed subset of M of dimension at most 2; we get a contradiction because Γ can not act freely on such a subset (Lemmas 3.1, and 3.4). Thus, E is smooth and the critical locus of α is empty, i.e. α is a submersion.

If E is a curve, its genus is ≥ 1 (because E is contained in the torus A_M), its automorphism group is solvable, and there is a non-abelian free group $\Gamma_1 \subset \Gamma$ acting trivially on E, and freely on every fiber of α: again, we get a contradiction from Lemma 3.4.

If $\dim(E) = 3$, then M is a finite cover of E (because α is a submersion). This implies that there is a non-trivial holomorphic 3-form on M, contradicting $h^{3,0}(M) = 0$.

Thus, we assume $\dim(E) = 2$. Let $K \subset A_M$ be the (connected) subtorus of maximal dimension such that $E + K = E$: it is uniquely determined by E, and the projection $p(E)$ of E in the quotient torus A_M/K is a manifold of general type and of dimension $\dim(E) - \dim(K)$ (more precisely, the canonical bundle of $p(E)$ is ample, see [5], §VII). If $K = \{0\}$ is reduced to a point, then, E is a surface of general type, hence $\mathsf{Aut}(E)$ is a finite group and we get a contradiction with Lemma 3.3. If $\dim(K) = 1$ we get $\dim(p(E)) = 1$ and the morphism $p \circ \alpha \colon M \to p(E)$ is invariant under a finite index subgroup of Γ. Since $p(E)$ is a curve of general type, the image of Γ in $\mathsf{Aut}(p(E))$ is finite and, again, we get a contradiction with Lemma 3.4.

Now, assume $\dim(K) = 2$, which means that $E = A_M$ is a 2-dimensional torus. Denote by (x, y) the affine coordinates on $E = \mathbf{C}^2/\Lambda$, where Λ is a lattice in $\mathbf{C}^2$. Let Ω be a holomorphic 2-form on M. Since $h^{3,0}(M) = 0$, we get

$$\Omega \wedge \alpha^*(dx) = \Omega \wedge \alpha^*(dy) = 0. \tag{3.3}$$

This implies that $\Omega = a\alpha^*(dx \wedge dy)$ for some holomorphic function $a \colon M \to \mathbf{C}$; such a function must be a constant, and we conclude that $H^{2,0}(M) = \mathbf{C}\alpha^*(dx \wedge dy)$. In particular, a non-abelian free subgroup Γ_1 of Γ acts trivially on $H^{2,0}(M)$.

The holomorphic Lefschetz fixed point formula gives $\mathrm{tr}((f_{1,0}^*)^n) = 2$ for every $f \in \Gamma_1$ and every n; this shows that $f_{1,0}^*$ is unipotent. Hence, $\rho(\Gamma_1)$ is a subgroup of $\mathsf{Aut}(E)$, all of whose elements are affine automorphisms of $\mathbf{C}^2/\Lambda$

with a unipotent linear part: such a group is solvable. Thus, a non-abelian free subgroup of Γ acts freely on the fibers of α and this contradicts Lemma 3.3.

This concludes the proof of Theorem A.

4. TWO QUESTIONS

4.1. Here is a well known question in algebraic dynamics (it is related to specific forms of the abundance conjecture, on Calabi-Yau manifolds for instance):

Question.– Does there exist an automorphism f of a complex projective manifold X such that all orbits of f are Zariski dense, but X is not an abelian variety?

4.2. Michael Herman proved that there is a real analytic diffeomorphism h of a (real analytic) compact manifold M such that (1) the topological entropy of $f\colon M \to M$ is positive, and (2) f is a minimal transformation of M, meaning that every orbit of f is dense in M for the euclidian topology (see [9]). In [17], Mary Rees constructs similar examples on tori of any dimension ≥ 2, but her examples are only C^0-smooth (in dimension 2, positive entropy forces the existence of periodic orbits for $\mathcal{C}^2$-diffeomorphisms [11]). We don't know whether such an example exists in the context of holomorphic diffeomorphisms of compact Kähler manifolds, even for Calabi-Yau manifolds. The case $\dim(X) \leq 3$ can be dealt with technics of the minimal model program, as in [12].

Question.– Does there exist an automorphism f of a compact Kähler manifold X with positive topological entropy, such that all orbits of f are dense in X for the euclidean topology ?

If the question raised in Section 4.1 has a positive answer, then this last question has a negative answer for any complex projective variety X (an automorphism of an abelian variety with positive entropy always has orbits which are not Zariski dense).

5. COMPACT COMPLEX MANIFOLDS

This section concerns compact complex manifolds which are not Kähler.

5.1. Examples on Hopf surfaces. Consider the surface $Y = (\mathbf{C}^2 \setminus \{0\})/\langle\Phi\rangle$ where Φ is a homothetic contraction $\Phi(x,y) = (\alpha x, \alpha y)$ with $\alpha \in \mathbf{C}^*$ of modulus $|\alpha| < 1$. The group $\mathsf{GL}_2(\mathbf{C})$ acts on $\mathbf{C}^2 \setminus \{0\}$, commuting to Φ, so it acts on Y holomorphically. An element $\gamma \in \mathsf{GL}_2(\mathbf{C})$ induces a fixed point free automorphism of Y if and only if the set of eigenvalues of γ does not intersect $\alpha^{\mathbf{Z}}$. If Γ is a free subgroup of $\mathsf{SU}_2(\mathbf{C})$, this last property is satisfied by every element $\gamma \in \Gamma \setminus \{\mathsf{Id}\}$, so Γ acts freely on Y. Then, by the theorem of Baire, we get: *there is a dense G_δ subset $\mathcal{F}$ of $\mathsf{GL}_2(\mathbf{C})^2$ such that every pair of matrices in $\mathcal{F}$ generates a free subgroup of $\mathsf{Aut}(Y)$ acting freely on Y.*

If we quotient Y by a finite order homothety $\Psi(x,y) = (\xi x, \xi y)$, with ξ some root of unity, we get a secondary Hopf surface with the same property.

Remark 5.1. Let T be the quotient of $\mathbf{R}^2 \setminus \{0\}$ by the homothety $\Phi(x,y) = (2x, 2y)$; this is a real analytic surface diffeomorphic to $\mathbf{R}^2/\mathbf{Z}^2$. The group $\mathsf{SL}(2,\mathbf{Z})$ contains a free subgroup Γ such that every element $\gamma \in \Gamma \setminus \{\mathsf{Id}\}$ is diagonalisable, with two real eigenvalues $\lambda(\gamma) > \lambda(\gamma)^{-1}$. None of these eigenvalues is in $2^{\mathbf{Z}}$, because $\lambda(\gamma)$ is not an integer (it is a quadratic integer). So, Γ acts freely on T by real analytic diffeomorphisms.

5.2. Surfaces. Let S be a compact complex surface. Assume that S is not Kähler, and that S admits an automorphism $f\colon S \to S$ with no periodic orbit. According to Kodaira's classification, such a surface has a unique minimal model $\eta\colon S \to S_0$. If η is not an isomorphism, it contracts a finite number of rational curves $E_i \subset S$, some positive iterate f^m preserves each E_i and has a fixed point on it, thus our assumption implies that S is minimal.

Let $a(S)$ be the algebraic dimension of S, i.e. the transcendance degree of the field of meromorphic functions $\mathcal{M}(S)$. A smooth compact complex surface with $a(X) = 2$ is projective, so $a(S) \in \{0,1\}$.

If $a(S) = 1$ there is a genus 1 fibration $\pi\colon S \to B$, onto some curve B, such that $\mathcal{M}(S) = \pi^*\mathcal{M}(B)$ (see [1], Proposition VI.4.1 and its proof). This fibration is $\mathsf{Aut}(S)$-equivariant: there is a homomorphism $\rho\colon \mathsf{Aut}(S) \to \mathsf{Aut}(B)$ such that $\pi \circ g = \rho(g) \circ \pi$ for all $g \in \mathsf{Aut}(S)$.

If $a(S) = 0$, then S is a surface of class VII_0, i.e. a minimal surface with Kodaira dimension $\mathrm{kod}(S) = -\infty$ and first Betti number $b_1(S) = 1$. Since f has no periodic orbit, the topological Euler characteristic of S vanishes (see [8]); equivalently, $b_2(S) = 0$. According to [18], S must be a Hopf or a Inoue surface (see also [2, 13]).

Theorem B.– *Let S be a smooth connected compact complex surface. There is a non-abelian free group $\Gamma \subset \mathsf{Aut}(S)$ acting freely on S if, and only if S is a Hopf surface obtained as a quotient of $\mathbf{C}^2 \setminus \{0\}$ by a group of homotheties*

$$(x,y) \mapsto (\alpha^m \xi^n x, \alpha^m \xi^n y)$$

with $\alpha \in \mathbf{C}^*$ of modulus < 1 and ξ a root of unity.

Proof. Section 5.1 gives one implication. We now assume that Γ is a non-abelian free group acting freely on S. By Lemma 3.4, S is not Kähler. As seen above, S is minimal, and there are three cases to consider: $\mathsf{Aut}(S)$ preserves a genus 1 fibration $\pi\colon S\to B$; S is a Hopf surface; or S is an Inoue surface.

Assume that $\mathsf{Aut}(S)$ preserves a genus 1 fibration. If $\rho(\Gamma)\subset\mathsf{Aut}(B)$ is virtually solvable, we argue as in Lemma 3.4 and obtain a contradiction. So, B is isomorphic to $\mathbb{P}^1(\mathbf{C})$ and all orbits of $\rho(\Gamma)\subset\mathsf{Aut}(\mathbb{P}^1(\mathbf{C}))$ are infinite; as a consequence, ρ is a submersion and is isotrivial. From [1], Section V.5 and Theorem V.5.4, we know that S is a Hopf surface (S can not be a product since it is not Kähler). So, we only need to prove the following lemma. $\square$

Lemma 5.2. *Let S be a Hopf or Inoue surface. Either* $\mathsf{Aut}(S)$ *contains a finite index solvable subgroup or S is one of the surfaces described in Theorem B.*

Proof. First, assume that S is a Hopf surface. A finite cover S' of S is a primary Hopf surface, i.e. a quotient of $\mathbf{C}^2\setminus\{0\}$ by a holomorphic contraction $\Phi\colon\mathbf{C}^2\to\mathbf{C}^2$ fixing the origin. Namba gave normal forms for Φ and described the automorphism group of primary Hopf surfaces according to this normal form; as a corollary, $\mathsf{Aut}(S')$ and $\mathsf{Aut}(S)$ are solvable groups, except if Φ is (conjugate to) a homothety, in which case $\mathsf{Aut}(S')$ is $\mathsf{GL}_2(\mathbf{C})/\langle\Phi\rangle$ (see [16], as well as [20, Section I]). This last property is based on the following remark.

Remark 5.3. Suppose $\Phi(x,y)=(\alpha x,\alpha y)$ with α of infinite order, and consider an automorphism g of $\mathbf{C}^2\setminus\{0\}$ such that $g\Phi g^{-1}=\Phi^{\pm1}$. By Hartogs theorem, g extends to an automorphism of $\mathbf{C}^2$ fixing the origin. Then the equation $g(\alpha x,\alpha y)=\alpha^{\pm1}g(x,y)$ implies that $g\in\mathsf{GL}_2(\mathbf{C})$ and $g\Phi g^{-1}=\Phi$.

If S is a secondary Hopf surface, $S=S'/G$ for some finite subgroup G of $\mathsf{Aut}(S')$ acting freely on S'. The fundamental group of S is a central extension $0\to\mathbf{Z}\to\Lambda\to G\to1$, where $\mathbf{Z}$ corresponds to $\pi_1(S')$; we can also identify Λ to the group of deck transformations on the universal cover $\mathbf{C}^2\setminus\{0\}$ of S, with $\mathbf{Z}$ generated by the homothety Φ, and from the last remark we see that Λ is a subgroup of $\mathsf{GL}_2(\mathbf{C})$. Note that the action of $\Lambda\subset\mathsf{GL}_2(\mathbf{C})$ on $\mathbb{P}(\mathbf{C}^2)=\mathbb{P}^1(\mathbf{C})$ factorizes as an action of G; we denote by $\mathsf{P}(G)$ its image.

Lifting the group $\mathsf{Aut}(S)$ to the universal cover, we obtain an extension

$$1\to\Lambda\to\tilde{\mathsf{Aut}}(S)\to\mathsf{Aut}(S)\to1. \tag{5.1}$$

The group $\mathsf{Aut}(S)$ normalizes the group Λ, hence also the subgroup generated by Φ^k for some sufficiently divisible integer $k\geq1$. The last remark, applied with Φ^k, shows that $\tilde{\mathsf{Aut}}(S)$ is contained in $\mathsf{GL}_2(\mathbf{C})$.

The subgroup $\mathsf{P}(G)$ is normalized by the image of $\tilde{\mathsf{Aut}}(S)$ in $\mathsf{PGL}_2(\mathbf{C})$, and a finite index subgroup of $\mathsf{P}(\tilde{\mathsf{Aut}}(S))$ commutes to all elements of $\mathsf{P}(G)$. If $\mathsf{P}(G)$ were not trivial, $\tilde{\mathsf{Aut}}(S)$ would be virtually solvable, because the centralizer of a finite order element of $\mathsf{PGL}_2(\mathbf{C})$ if cyclic or dihedral. So, $\mathsf{P}(G)$ is trivial and Λ is contained in the group of homotheties $\mathbf{C}^*$. Since Λ is discrete, it is generated by Φ and a homothety $\Psi(x,y) = (\xi x, \xi y)$ of finite order. We conclude that S is one of the Hopf surfaces of Theorem B.

There are three types of Inoue surfaces, but in all cases the universal cover $\tilde{S}$ is isomorphic to $\mathbb{H} \times \mathbf{C}$, where $\mathbb{H} \subset \mathbf{C}$ is the upper half plane. By Liouville's theorem, every holomorphic function from $\mathbf{C}$ to $\mathbb{H}$ is constant; this implies that every holomorphic diffeomorphism F of $\mathbb{H} \times \mathbf{C}$ preserves the fibration onto $\mathbb{H}$: $F(z,w) = (u_F(z), a_F(z)w + b_F(z))$ where u_F is in $\mathsf{PSL}_2(\mathbf{R})$, acting by homographies on $\mathbb{H}$, a_F and b_F are holomorphic functions of $z \in \mathbb{H}$, and a_F does not vanish. The group of deck transformations $\Lambda \simeq \pi_1(S)$ of the universal cover $\mathbb{H} \times \mathbf{C} \to S$ fixes a unique point on the boundary of $\mathbb{H}$; putting this point at infinity, the action of Λ on $\mathbb{H}$ is generated by real translations $z \mapsto z + a_i$ and a scalar multiplication $z \mapsto \alpha z$ (see § 2 and 3 of [10]). Let $\tilde{\mathsf{Aut}}(S)$ be the group of all possible lifts of automorphisms of S to its universal cover (an extension of $\mathsf{Aut}(S)$ by Λ). This group normalizes the action of Λ, so every element $\tilde{f}$ of $\tilde{\mathsf{Aut}}(S)$ can be written $\tilde{f}(z,w) = (uz + b, a(z)w + b(z))$, with $u \in \mathbf{R}^*$ and $v \in \mathbf{R}$. Hence, $\tilde{\mathsf{Aut}}(S)$ is a solvable group. □

5.3. An example in dimension 3. Let Λ be a lattice in the Lie group $\mathsf{SL}_2(\mathbf{C})$. The quotient space $M = \mathsf{SL}_2(\mathbf{C})/\Lambda$ is a compact complex manifold of dimension 3; it is not Kähler (for instance it contains homologous curves of arbitrary large area). Let γ be an element of $\mathsf{SL}_2(\mathbf{C})$. Its action by left multiplication $g \mapsto \gamma \cdot g$ on $\mathsf{SL}_2(\mathbf{C})$ induces an automorphism, i.e. a holomorphic diffeomorphism, $L_\gamma \colon M \to M$. The point $g\Lambda \in M$ is fixed by L_γ if and only if $g^{-1}\gamma g \in \Lambda$.

Proposition 5.4. *There is a non-abelian free group $\Gamma \subset \mathsf{SL}_2(\mathbf{C})$ acting freely on M by left translation. There is a dense G_δ subset $\mathcal{F}$ in $\mathsf{SL}_2(\mathbf{C}) \times \mathsf{SL}_2(\mathbf{C})$ such that any pair $(\gamma_1, \gamma_2) \in \mathcal{F}$ generates a free subgroup of rank 2 in $\mathsf{SL}_2(\mathbf{C})$ acting freely on M.*

Proof. Let γ be an element of $\mathsf{SU}_2(\mathbf{C})$. If $g^{-1}\gamma g \in \Lambda$ its iterates $g^{-1}\gamma^n g$ form a bounded sequence in the discrete group Λ so that γ has finite order. Thus, any free subgroup of $\mathsf{SU}_2(\mathbf{C})$ acts freely on M by left translation. The second assertion follows from the theorem of Baire. □

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