GENUS AND GONALITY OF SMALL CURVES, DYNAMICAL UNIFORM BOUNDEDNESS, AND BIFURCATION

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Abstract. We prove the Gonality Conjecture in arithmetic dynamics: for any non-isotrivial one-parameter algebraic family of rational maps on $\mathbb{P}^1$, the gonality of distinct dynatomic curves tends to infinity. More generally, outside the flexible Lattès family, every small sequence of horizontal curves has gonality tending to infinity, and its genus grows superlinearly with its degree over the parameter curve. We also obtain higher-dimensional analogues under natural bifurcation and multiplier-genericity hypotheses.

As applications, we prove uniform boundedness results for iterated preimages over number fields and geometric uniform boundedness results for preperiodic points over function fields.

The proof combines arithmetic equidistribution, woven currents, and bifurcation theory; the bifurcation mechanism is what forces the growth of genus and gonality.

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1. Introduction

1.1. **Genus and gonality of small curves.** Let

$$f : \Lambda \times \mathbb{P}^N \to \Lambda \times \mathbb{P}^N$$
$$(t, z) \mapsto (t, f_t(z))$$

be an algebraic family of endomorphisms over $\mathbb{C}$ parametrized by an irreducible quasi-projective Λ, where $f_t : \mathbb{P}^N \to \mathbb{P}^N$ are endomorphisms of fixed algebraic degree $d \geq 2$ over $\mathbb{C}$. The integer d is called the *algebraic degree* of f. Such a family is called a *one-parameter* algebraic family of endomorphisms of $\mathbb{P}^N$ over $\mathbb{C}$. A one-parameter algebraic family is called *isotrivial* if for every $t \in \Lambda(\mathbb{C})$, f_t is conjugate to the same fixed endomorphism.

Let $\pi_1 : \Lambda \times \mathbb{P}^N \to \Lambda$ be the canonical projection. Let Γ be a Zariski closed one-dimensional subvariety in $\Lambda \times \mathbb{P}^N$. We call Γ a *horizontal curve* if Γ is irreducible and $\pi_1|_\Gamma : \Gamma \to \Lambda$ is dominant. The following are two types of examples of dynamically meaningful horizontal curves.

Dynatomic curves: If $n > m \geq 0$ are two integers, the equation $f^n(t, z) = f^m(t, z)$ defines an algebraic curve in $\Lambda \times \mathbb{P}^N$. A *dynatomic curve* is by definition any irreducible component of such curves. In other words, a dynatomic curve is a family of preperiodic points.

Preimage curves: Let Γ_0 be a fixed horizontal curve. If n is a non-negative integer, $f^{-n}(\Gamma_0)$ is an algebraic curve in $\Lambda \times \mathbb{P}^N$. A *preimage curve* is by definition any irreducible component of such curves. In other words, a preimage curve is a family of iterated preimage points.

When f is isotrivial, the geometry of dynatomic curves is very simple. Up to a finite base change $\Lambda' \to \Lambda$, we may assume that f is trivial; then every dynatomic curve is isomorphic to Λ by the canonical projection π_1. When f is non-isotrivial, we expect that the complexity of the geometry of dynatomic curves reflects the complexity of the family of dynamical systems.

The guiding question of this paper is to make this expectation quantitative. We ask whether dynamically defined horizontal curves, such as dynatomic curves and preimage curves, must acquire large genus and gonality as soon as the family genuinely varies. More generally, we study horizontal curves which are small for the canonical dynamical height. In this form, dynatomic curves and preimage curves are treated by the same principle: arithmetic smallness in a non-isotrivial dynamical family should force geometric complexity.

In the case $N = 1$, there have been substantial works on the geometry of dynatomic curves and preimage curves, most of them focusing on the

unicritical family $f_t(z) = z^d + t$ and the flexible Lattès family, while little is known for general one-parameter families. We refer the reader to the survey paper [BIJ+19, Section 9 and 10] for some references. Here a *Lattès map* is by definition an endomorphism of $\mathbb{P}^1$ that is semi-conjugate to an endomorphism of an elliptic curve. A *flexible Lattès family* is by definition a non-isotrivial one-parameter algebraic family of Lattès maps.

Let Γ be an irreducible quasi-projective curve over $\mathbb{C}$. We define $\mathrm{genus}(\Gamma)$ to be the geometric *genus* of any projective compactification of Γ. We define the *gonality* $\mathrm{gonal}(\Gamma)$ to be the minimal degree of a dominant rational map from Γ to $\mathbb{P}^1$. We have $\mathrm{gonal}(\Gamma) \leq \lfloor(\mathrm{genus}(\Gamma)+3)/2\rfloor$, where $\lfloor\cdot\rfloor$ is the floor function. In particular, if a sequence of irreducible quasi-projective curves Γ_n satisfies $\mathrm{gonal}(\Gamma_n) \to +\infty$, then $\mathrm{genus}(\Gamma_n) \to +\infty$.

Fact 1. *Let $\phi : C_1 \to C_2$ be a non-constant morphism between irreducible quasi-projective curves over $\mathbb{C}$ of degree e. Then*

$$\mathrm{gonal}(C_2) \leq \mathrm{gonal}(C_1) \leq e\,\mathrm{gonal}(C_2).$$

This is a standard property of gonality; see for instance [ACGH85, Chapter III]. Indeed, the left inequality follows by taking the norm of a gonal function on C_1 along the extension $\mathbb{C}(C_1)/\mathbb{C}(C_2)$, and the right inequality follows by composing ϕ with a gonal function on C_2.

1.2. **Gonality Conjecture.** An aim of our paper is to solve the following Gonality Conjecture.

Conjecture 1.1. For any non-isotrivial one-parameter algebraic family of endomorphisms of $\mathbb{P}^1$ over $\mathbb{C}$, if Γ_n is a sequence of distinct dynatomic curves, then $\mathrm{gonal}(\Gamma_n) \to +\infty$.

The above Gonality Conjecture is the one-dimensional case of [BIJ+19, Conjecture 10.13 (c)]. It implies that $\mathrm{genus}(\Gamma_n) \to +\infty$, which implies the one-dimensional case of [BIJ+19, Conjecture 10.13 (d)].

Previously, this conjecture was only known for the unicritical family $f_t(z) = z^d + t$ by Doyle-Poonen [DP20], and for the flexible Lattès family by Nguyen-Saito [NS96].

We actually solve a more general version of this conjecture, see Theorem 1.2 below. To formulate our result, we need the notion of *Moriwaki height*. This notion was first introduced by Moriwaki in [Mor00a]. In our paper, we use the version developed by Yuan–Zhang [YZ26] in the framework of adelic line bundles.

Let f be a non-isotrivial one-parameter algebraic family of endomorphisms of $\mathbb{P}^N$ of degree $d \geq 2$ over $\mathbb{C}$. Let K be a subfield of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$ such that f is over K. Following Yuan–Zhang [YZ26] (see Section 6.1 for details), for every polarization $\overline{H} \in \widehat{\mathrm{Pic}}(K/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$, we can define the canonical Moriwaki height of every horizontal curve Γ in $\Lambda \times \mathbb{P}^N$ over $\overline{K}$, denoted by $\hat{h}_f^{\overline{H}}(\Gamma) \in \mathbb{R}_{\geq 0}$, which satisfies the following two properties:

(1) $\hat{h}_f^{\overline{H}}(f(\Gamma)) = d\hat{h}_f^{\overline{H}}(\Gamma)$;

(2)(*Northcott property:*) If $\overline{H}$ is big and nef, then for each $D \geq 0$ and $B \geq 0$, the number of horizontal curves Γ over $\overline{K}$ with $[K(\Gamma) : K] \leq D$ and $\hat{h}_f^{\overline{H}}(\Gamma) \leq B$ is finite.

As a consequence of the above two properties, when $\overline{H}$ is big, $\hat{h}_f^{\overline{H}}(\Gamma) = 0$ if and only if Γ is a dynatomic curve. If Γ is a preimage curve, $f^n(\Gamma) = \Gamma_0$, then $\hat{h}_f^{\overline{H}}(\Gamma) = d^{-n}\hat{h}_f^{\overline{H}}(\Gamma_0)$. In particular, if $(\Gamma_n)_{n\geq 1}$ is a sequence of distinct dynatomic curves or preimage curves, then $\hat{h}_f^{\overline{H}}(\Gamma_n) \to 0$. Following Yuan–Zhang [YZ26, Section 5.4.1], a sequence $\Gamma_n, n \geq 0$ is called *small for* f if for any $\overline{H} \in \widehat{\mathrm{Pic}}(K/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$, $\hat{h}_f^{\overline{H}}(\Gamma_n)$ converges to 0. Any sequence of dynatomic or preimage curves is small for f.

The following Theorem 1.2 and Theorem 1.4 are our first main results. These results in particular hold when $(\Gamma_n)_{n\geq 1}$ is a sequence of dynatomic curves or preimage curves.

Theorem 1.2 (Gonality Conjecture for horizontal curves with small heights)**.** *Let f be a non-isotrivial one-parameter algebraic family of endomorphisms of $\mathbb{P}^1$ of degree $d \geq 2$ over $\mathbb{C}$, which is not the flexible Lattès family. Let K be any subfield of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$ such that f is over K. Let $(\Gamma_n)_{n\geq 1}$ be a sequence of distinct horizontal curves over $\overline{K}$ that is small for f. Then*

$$\mathrm{gonal}(\Gamma_n) \to +\infty;$$

and

$$\frac{\mathrm{genus}(\Gamma_n)}{\deg(\Gamma_n)} \to +\infty.$$

Here $\deg(\Gamma_n)$ is the degree of the canonical projection map $\pi_1|_{\Gamma_n} : \Gamma_n \to \Lambda$.

If Γ_n is a sequence of dynatomic curves and f is the flexible Lattès family, Nguyen-Saito[NS96] proved that $\mathrm{gonal}(\Gamma_n) \to +\infty$. Thus we can prove Conjecture 1.1 by combining Nguyen-Saito's result and Theorem 1.2.

Thus the one-dimensional picture is essentially complete for dynatomic curves. Outside the flexible Lattès family, Theorem 1.2 proves a stronger statement for all small horizontal curves; in the flexible Lattès dynatomic case, the required gonality growth is supplied by Nguyen–Saito.

Let us also emphasize the second conclusion of Theorem 1.2. The divergence of $\mathrm{genus}(\Gamma_n)/\deg(\Gamma_n)$ reflects a genuinely dynamical source of ramification. This behavior is different from the towers naturally arising from abelian schemes: over the good reduction locus such covers are typically étale, and the possible ramification is confined to a fixed bad set; Riemann–Hurwitz then gives at most linear genus growth in the degree. In our setting, bifurcation creates new geometric complexity along the family, and this is what forces the normalized genus to diverge.

The proof of Theorem 1.2 relies on one-dimensional bifurcation theory developed by Mañé–Sad–Sullivan [MSS83], Lyubich [Lyu83, Lyu84] and DeMarco [DeM03]. Our next goal is to extend Theorem 1.2 to higher dimension. A bifurcation theory for families of endomorphisms of $\mathbb{P}^N$ over $\mathbb{C}$ was developed by Berteloot–Bianchi–Dupont for $N \geq 2$ [BBD18]. The *stable set* of a family of endomorphisms of $\mathbb{P}^N$ is by definition the locus where the Lyapunov exponent function is pluriharmonic, and the *bifurcation set* is the complement of the stable set. See Section 4 for more details.

The higher-dimensional bifurcation theory is not yet as complete as its one-dimensional counterpart. In dimension one, stability theory, together with McMullen's rigidity theorem, gives a strong description of stable non-isotrivial families: apart from the flexible Lattès phenomenon, stability forces isotriviality. For endomorphisms of $\mathbb{P}^N$, $N \geq 2$, no comparable classification of stable families is currently known. In dimensions $N \geq 3$, current higher-dimensional bifurcation theory typically requires a genericity condition on periodic multipliers, excluding persistent resonances and non-diagonalizability. This is the origin of the periodically generic condition in Assumption A. We also discuss in Sections 6 and 9 an adelic-line-bundle viewpoint on this rigidity problem, where stability is related to the degeneracy of the bifurcation class on moduli.

We need some additional technical assumptions. Let g be an endomorphism of $\mathbb{P}^N$ of degree $d \geq 2$ over $\mathbb{C}$. Let x be a n-periodic point of g, and let $w_1, \ldots, w_N$ be the eigenvalues of the differential $dg^n(x)$. We say that x is *resonant* if there is a relation $w_1^{m_1} \cdots w_N^{m_N} = w_j$, where

the m_j are non-negative integers with $m_1 + \cdots + m_N \geq 2$. We say that x is *non-diagonalizable* if $dg^n(x)$ is not diagonalizable as a matrix.

Throughout the paper, a Hermitian manifold is always assumed to be **connected**. We say that a holomorphic family f of endomorphisms of $\mathbb{P}^N$ of degree $d \geq 2$ parametrized by a Hermitian manifold Λ is *periodically generic* if the lower density of n-periodic points of f that are not persistently resonant or persistently non-diagonalizable along Λ is positive as $n \to +\infty$. More precisely, let $\mathcal{P}_n$ be the set of irreducible n-periodic subvarieties V in $\Lambda \times \mathbb{P}^N(\mathbb{C})$ such that there exists $x \in V$ which is non-resonant and diagonalizable. Thus f is periodically generic if

$$\limsup_{n \to +\infty} d^{-Nn} \sum_{V \in \mathcal{P}_n} \deg V > 0.$$

By definition, if there exists $t \in \Lambda$ such that the upper density of n-periodic points of f_t that are not resonant and diagonalizable is positive as $n \to +\infty$, then f is periodically generic.

Definition 1.3 (Assumption A)**.** Let f be a holomorphic family of endomorphisms of $\mathbb{P}^N$ of degree $d \geq 2$ parametrized by a Hermitian manifold. We say that f satisfies *Assumption A* if the following hold:

(1) The bifurcation set of f is non-empty;

(2) If $N \geq 3$, we further assume that f is periodically generic.

We note that the assumption that f has non-empty bifurcation set implies that f is non-isotrivial.

Let f be a one-parameter algebraic family of endomorphisms of $\mathbb{P}^N$ over $\mathbb{C}$, parametrized by Λ. A sequence of horizontal curves $(\Gamma_n)_{n \geq 1}$ is called *generic* if for any proper subvariety $H \subset \Lambda \times \mathbb{P}^N$, there are only finitely many Γ_n contained in H. In particular, when $N = 1$, a distinct sequence of horizontal curves is generic.

The following result extends Theorem 1.2 to higher dimension.

Theorem 1.4 (Genus and gonality of horizontal curves with small heights for families of endomorphisms of $\mathbb{P}^N$)**.** *Let f be a one-parameter algebraic family of endomorphisms of $\mathbb{P}^N$ over $\mathbb{C}$ satisfying Assumption A. Let K be any subfield of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$ such that f is over K. Let $(\Gamma_n)_{n \geq 1}$ be a generic sequence of horizontal curves over $\overline{K}$ that is small for f. Then*

$$\operatorname{gonal}(\Gamma_n) \to +\infty;$$

and

$$\frac{\operatorname{genus}(\Gamma_n)}{\deg(\Gamma_n)} \to +\infty.$$

Theorems 1.2 and 1.4 are proved in Section 7.

Remark 1.5. When $N \geq 2$, the assumption "$(\Gamma_n)_{n\geq 1}$ is a generic sequence" cannot be replaced by "$(\Gamma_n)_{n\geq 1}$ is a distinct sequence" due to the possible existence of periodic isotrivial subvariety of positive dimension.

A *Lattès map* on $\mathbb{P}^N$ is by definition an endomorphism of $\mathbb{P}^N$ over $\mathbb{C}$ that is semi-conjugate (by a finite morphism $\pi : A \to \mathbb{P}^N$) to an endomorphism of an abelian variety A. The Lattès locus containing all Lattès maps is a proper subvariety in $\mathrm{End}_d(\mathbb{P}^N)$.

A Lattès map is called *split* if it is semi-conjugate (by a finite morphism $\pi : (\mathbb{P}^1)^N \to \mathbb{P}^N$) to an endomorphism $g : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N$, such that $g = (h, \ldots, h)$ is a split map, where $h : \mathbb{P}^1 \to \mathbb{P}^1$ is a one-dimensional Lattès map.

We have the following examples of one-parameter algebraic families of endomorphisms of $\mathbb{P}^N$ satisfying Assumption A for every $N \geq 1$.

Theorem 1.6. *Let f be a one-parameter algebraic family of endomorphisms of $\mathbb{P}^N$ of degree $d \geq 2$ over $\mathbb{C}$ which is not a Lattès family. If there is a parameter $t \in \Lambda(\mathbb{C})$ such that f_t is a split Lattès map, then f satisfies Assumption A.*

Proposition 1.7. *Let f be a non-isotrivial one-parameter algebraic family of endomorphisms of $\mathbb{P}^N$ of degree $d \geq 2$ over $\mathbb{C}$. Assume that the image of the moduli map of f passes through a very general point of $\mathcal{M}_{d,N}$. Then f satisfies Assumption A.*

Theorem 1.6 and Proposition 1.7 are proved in Section 9.

Theorem 1.2 and 1.4 have consequences toward the Uniform Boundedness Conjecture for iterated preimages and preperiodic points for one parameter families. The link is provided by a theorem of Frey: a curve with infinitely many algebraic points of bounded degree must have bounded gonality. Thus gonality growth turns into uniform boundedness once one organizes preperiodic points or iterated preimages into suitable horizontal curves.

1.3. **Uniform Boundedness Conjecture for iterated preimages over number fields.** Let g be an endomorphism of $\mathbb{P}^N$ over a number field. Let $a \in \mathbb{P}^N(\overline{\mathbb{Q}})$. Let

$$\mathrm{Preim}\,(g, a) := \left\{x \in \mathbb{P}^N(\overline{\mathbb{Q}}) : g^n(x) = a \text{ for some } n \geq 0\right\}.$$

Let $D \geq 1$. We are interested in the set of rational iterated preimages of a with degree bounded by D:

$$\mathrm{Preim}\,(g, a, D) := \{x \in \mathrm{Preim}\,(g, a) : [\mathbb{Q}(x) : \mathbb{Q}] \leq D\}.$$

By standard properties of the canonical height associated to g, for fixed g and a, the cardinality of $\mathrm{Preim}\,(g, a, D)$ is finite. It is a natural problem to see how this cardinality depends on g and a. This problem is reminiscent of the classical problem of bounding torsion points on abelian varieties.

Let $\mathrm{End}_{d,N}$ be the space of all degree d endomorphisms of $\mathbb{P}^N$ over $\mathbb{C}$. The group $\mathrm{PGL}_{N+1}(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^N(\mathbb{C}))$ acts on $\mathrm{End}_{d,N}(\mathbb{C})$ by conjugacy. The geometric quotient $\mathcal{M}_{d,N} := \mathrm{End}_{d,N}/\mathrm{PGL}_{N+1}$ is the (coarse) *moduli space*, which is a complex affine variety. Let $\pi : \mathrm{End}_{d,N} \to \mathcal{M}_{d,N}$ be the canonical projection. If f is an algebraic family of degree d endomorphisms of $\mathbb{P}^N$ over $\mathbb{C}$, parametrized by a quasi-projective variety Λ, then the canonical projection π induces a morphism $\pi_\Lambda : \Lambda \to \mathcal{M}_{d,N}$. We say the family f is *maximal varying* if $\pi_\Lambda : \Lambda \to \mathcal{M}_{d,N}$ is quasi-finite. If $\dim \Lambda = 1$, then maximal varying is equivalent to non-isotrivial. By a *marked point*, we mean a morphism $a : \Lambda \to \mathbb{P}^N$, $t \mapsto a_t$.

For $N = 1$, a slightly weaker version of the following conjecture was proposed by A. Levin [Lev12]. Instead of for iterated preimages, a similar Uniform Boundedness Conjecture for preperiodic points was proposed by Morton-Silverman [MS94]. Fakhruddin [Fak03] showed that Morton-Silverman's conjecture [MS94] implies the Uniform Boundedness Conjecture for torsion points on abelian varieties.

Conjecture 1.8 (Uniform Boundedness for iterated preimages)**.** Let f be a maximal varying algebraic family of degree d endomorphisms of $\mathbb{P}^N$ parametrized by a quasi-projective variety Λ and let a be a marked point, both over a number field. Then for each $D \geq 1$, there exists $C = C(f, a, D) > 0$ and a proper subvariety $V_{f,a}$ in $\Lambda \times \mathbb{P}^N$ over $\overline{\mathbb{Q}}$ depending only on (f, a), such that

$$\#\left(\mathrm{Preim}\,(f_t, a_t, D) \setminus V_{f,a}(\overline{\mathbb{Q}})\right) \leq C$$

for every $t \in \Lambda(\overline{\mathbb{Q}})$ such that $[\mathbb{Q}(t) : \mathbb{Q}] \leq D$. When $N = 1$, we require that $V_{f,a} = \emptyset$.

As a consequence of Theorem 1.2, we completely solve Conjecture 1.8 for one-parameter families of endomorphisms of $\mathbb{P}^1$, i.e. when $N = 1$ and $\dim \Lambda = 1$.

Theorem 1.9. *Let f be a non-isotrivial one-parameter algebraic family of degree d endomorphisms of $\mathbb{P}^1$ parametrized by a quasi-projective curve Λ and let a be a marked point, both over a number field. Then for each $D \geq 1$, there exists $C = C(f, a, D) > 0$ such that*

$$\#\mathrm{Preim}\,(f_t, a_t, D) \leq C$$

for every $t \in \Lambda(\overline{\mathbb{Q}})$ such that $[\mathbb{Q}(t) : \mathbb{Q}] \leq D$.

For $N \geq 1$ and $\dim \Lambda = 1$, as a consequence of Theorem 1.4, we solve Conjecture 1.8 under the assumption that f satisfies Assumption A and the marked point has Zariski dense forward orbit.

Theorem 1.10. *Let $N \geq 1$. Let f be a one-parameter algebraic family of degree d endomorphisms of $\mathbb{P}^N$ parametrized by a quasi-projective curve Λ and let a be a marked point, both over a number field. Assume f satisfies Assumption A. Let $\Gamma_a \subset \Lambda \times \mathbb{P}^N$ be the graph of a. Assume that the forward orbit $\{f^n(\Gamma_a)\}_{n\geq 0}$ is Zariski dense in $\Lambda \times \mathbb{P}^N$. Then for each $D \geq 1$, there exists $C = C(f, a, D) > 0$ such that*

$$\#\mathrm{Preim}\,(f_t, a_t, D) \leq C$$

for every $t \in \Lambda(\overline{\mathbb{Q}})$ such that $[\mathbb{Q}(t) : \mathbb{Q}] \leq D$.

Theorems 1.9 and 1.10 are proved in Section 8.

Remark 1.11. (1) The assumption that f is non-isotrivial in Theorem 1.9 cannot be removed. Let g be an endomorphism of $\mathbb{P}^1$ over a number field. Let f be the trivial one-parameter algebraic family parametrized by $\mathbb{P}^1$, i.e. $f_t = g$ for every $t \in \mathbb{P}^1(\overline{\mathbb{Q}})$. Let $a : \mathbb{P}^1 \to \mathbb{P}^1$ be the identity morphism. Let $t_0 \in \mathbb{P}^1(\overline{\mathbb{Q}})$ with infinite g-orbit. Let $D := [\mathbb{Q}(t) : \mathbb{Q}]$. For $n \geq 0$, let $t_n := g^n(t_0)$. Since $\{t_0, \ldots, t_n\} \subset \mathrm{Preim}\,(f_{t_n}, a_{t_n}, D)$, we have $\#\mathrm{Preim}\,(f_{t_n}, a_{t_n}, D) \geq n+1$. This implies that the constant C in Theorem 1.9 does not exist.

(2) The Zariski dense orbit assumption in Theorem 1.10 is used to remove the exceptional subvariety from the conclusion. Without this assumption, when $N \geq 2$, one should in general allow an exceptional subvariety due to the possible existence of periodic isotrivial subvarieties of positive dimension.

1.4. **The Geometric Uniform Boundedness Conjecture.** Let B be an irreducible quasi-projective curve over $\mathbb{C}$, and let $k = \mathbb{C}(B)$ be the corresponding complex function field. Throughout the paper, by a complex function field we mean a function field over a complex quasi-projective curve. We define $\mathrm{gonal}(k) := \mathrm{gonal}(B)$. An endomorphism f on $\mathbb{P}^N$ over k can be naturally seen as a one-parameter algebraic family of endomorphisms of $\mathbb{P}^N$, parametrized by a Zariski open subset of B. We say that f is *isotrivial* if the associated one-parameter algebraic family is isotrivial. For each $D \geq 1$, we define

$$\mathrm{Preper}\,(f, D) := \left\{x \in \mathbb{P}^N(\overline{k}) : x \text{ is } f\text{-preperiodic}, \mathrm{gonal}(k(x)) \leq D\right\}.$$

Theorem 1.2 and Theorem 1.4 have consequences on the Geometric Uniform Boundedness Conjecture (GUBC). The original Uniform

Boundedness Conjecture proposed by Morton-Silverman [MS94] is for number fields, the following is a function field analogue of Morton-Silverman's conjecture.

Conjecture 1.12 (Geometric Uniform Boundedness Conjecture)**.** Given integers $D \geq 1$, $N \geq 1$ and $d \geq 2$, there exists $C = C(d, N, D) > 0$ such that if f is a degree d non-isotrivial endomorphism of $\mathbb{P}^N$ over a complex function field k, then there exists a proper subvariety $V_{f,D} \subset \mathbb{P}^N$ over $\overline{k}$ such that

$$\#\left(\mathrm{Preper}\,(f, D) \setminus V_{f,D}(\overline{k})\right) \leq C.$$

When $N = 1$, we require that $V_{f,D} = \emptyset$.

Remark. In view of Fact 1, if $\mathrm{Preper}\,(f, D)$ is non-empty, then one automatically has $\mathrm{gonal}(k) \leq D$. Thus, in the above conjecture, one only needs to consider complex function fields k of gonality at most D.

When $N \geq 2$, in general $V_{f,D}$ can be non-empty due to the possible existence of periodic isotrivial subvariety of positive dimension.

Conjecture 1.12 was previously only known for two special one-parameter families, namely the unicritical family $f_t(z) = z^d + t$ and the flexible Lattès family.

Let us be more precise about **the meaning of saying that GUBC holds along a one-parameter algebraic family.** Let $\mathcal{M}_{d,N}$ be the moduli space of degree d endomorphisms of $\mathbb{P}^N$ over $\mathbb{C}$. Let $S \subset \mathcal{M}_{d,N}$ be an irreducible algebraic curve over $\mathbb{C}$. Let f be a degree d non-isotrivial endomorphism of $\mathbb{P}^N$ over a complex function field $k = \mathbb{C}(B)$. After shrinking B, the canonical projection π induces a canonical morphism $\pi_B : B \to \mathcal{M}_{d,N}$. We say that f *lies over* S if $\pi_B(B)$ is a Zariski open subset of S.

Definition 1.13. Let $S \subset \mathcal{M}_{d,N}$ be an irreducible algebraic curve over $\mathbb{C}$. We say that the GUBC holds along S if, for every integer $D \geq 1$, there exists $C = C(D, S) > 0$ such that if f is a degree d non-isotrivial endomorphism of $\mathbb{P}^N$ over a complex function field k which lies over S, then there exists a proper subvariety $V_{f,D} \subset \mathbb{P}^N$ over $\overline{k}$ such that

$$\#\left(\mathrm{Preper}\,(f, D) \setminus V_{f,D}(\overline{k})\right) \leq C.$$

Moreover when $N = 1$, we require that $V_{f,D} = \emptyset$.

As a consequence of Theorem 1.2, we establish GUBC for every one-parameter algebraic family of endomorphisms of $\mathbb{P}^1$.

Theorem 1.14 (One-parameter Geometric Uniform Boundedness for endomorphisms of $\mathbb{P}^1$)**.** *The GUBC holds along every irreducible algebraic curve $S \subset \mathcal{M}_{d,1}$ over $\mathbb{C}$.*

Since the two conditions in Assumption A is invariant under conjugacy, one can also define Assumption A for irreducible algebraic curve $S \subset \mathcal{M}_{d,N}$. As a consequence of Theorem 1.4, we have

Theorem 1.15 (One-parameter Geometric Uniform Boundedness for endomorphisms of $\mathbb{P}^N$)**.** *Let $S \subset \mathcal{M}_{d,N}$ be an irreducible algebraic curve over $\mathbb{C}$ such that S satisfies Assumption A. Then the GUBC holds along S.*

Theorems 1.14 and 1.15 are proved in Section 8.

1.5. **Previous results.**

Results on dynatomic curves and the Gonality Conjecture. For the unicritical family $f_t(z) = z^d + t$, the genus and topology of periodic and preperiodic dynatomic curves were studied by Bousch [Bou92], Morton [Mor96], and Gao [Gao16]. In particular, Morton obtained genus estimates for periodic dynatomic curves, and Gao computed the genera of preperiodic dynatomic curves. The gonality growth for the unicritical family was proved by Doyle–Poonen [DP20], using these geometric inputs together with finite-field arguments and the Castelnuovo–Severi inequality. The flexible Lattès family was treated by Nguyen–Saito [NS96]. By contrast, the mechanism in this paper is bifurcational: in a genuinely varying dynamical family, bifurcation forces geometric complexity. Arithmetic equidistribution and woven currents are used to connect this bifurcation mechanism with the geometry of small curves.

Results for the Uniform Boundedness Conjecture for iterated preimages(=Conjecture 1.8). All previous results for Conjecture 1.8 concern one-parameter families of endomorphisms of $\mathbb{P}^1$.

(i) When f is the quadratic polynomial family $f_t(z) = z^2 + t$ and a is a constant marked point, Faber et al [FHI+09] proved Conjecture 1.8 for all but finitely many $a \in \overline{\mathbb{Q}}$.
(ii) Ingram [Ing11] proved Conjecture 1.8 for the one-parameter flexible Lattès family on $\mathbb{P}^1$.
(iii) In the setting of abelian varieties over number fields, Cadoret–Tamagawa [CT12, Theorem 1.1] proved the uniform boundedness of ℓ-primary torsion points for abelian varieties over a fixed base curve. Ji–Song–Xie [JSX26, Theorem 1.6] gave a geometric proof of Cadoret–Tamagawa's theorem and extended it to the setting of iterated preimages of the $\times\ell$ map.

(iv) A. Levin [Lev12] and Ingram [Ing13] obtained partial results for Conjecture 1.8 for certain one-parameter families of endomorphisms of $\mathbb{P}^1$ when the number of places of bad reduction is controlled.

(v) Explicit bounds on the number of rational iterated preimages for the quadratic polynomial family $f_t(z) = z^2 + t$ were studied by Faber [Fab10], Faber–Hutz–Stoll [FHS11], Hutz–Hyde–Krause [HHK11] and Sano [San25]. Sano [San25] also obtained explicit bounds for the family $f_t(z) = z^d + t$, $d \geq 3$.

Results for the Geometric Uniform Boundedness Conjecture for preperiodic points(=Conjecture 1.12). All of the following results concern endomorphisms of $\mathbb{P}^1$, except the results for abelian varieties:

(i) Nguyen-Saito [NS96] proved that GUBC holds along the flexible Lattès family.

(ii) Doyle-Poonen [DP20] proved that GUBC holds along the unicritical family $f_t(z) = z^d + t$. They also obtained parallel results over function fields of positive characteristic.

(iii) Also in the positive characteristic case, Doyle–Faber [DF24] recently extended Doyle–Poonen's result to some other one-parameter families.

(iv) Assuming the *abcd* Conjecture (a strong form of the *abc* conjecture), Looper [Loo21a] [Loo21b] proved Conjecture 1.12 in the set of polynomials over complex function fields.

(v) Benedetto [Ben07], Canci [Can15] and Canci-Paladino[CP16] obtained partial results of Conjecture 1.12 concerning the case when the number of the places of bad reduction of f is uniformly bounded from above.

(vi) In the setting of abelian varieties over complex function fields, Ji–Song–Xie [JSX26, Theorem 1.4] proved that the genus of a generic sequence of horizontal curves with small Moriwaki heights over a fixed base curve goes to infinity. This extends previous results of Cadoret–Tamagawa [CT12, Theorem 1.2], [CT11, Theorem 1.3]. Bakker–Tsimerman proved the Geometric Torsion Conjecture for abelian varieties with real multiplication [BT18]. Remarkably, Looper–Yap recently completely solved the Geometric Torsion Conjecture for abelian varieties [LY26].

Results for the Uniform Boundedness Conjecture for preperiodic points over number fields. This is the number-field analogue of Conjecture 1.12.

Conjecture 1.16 (Morton-Silverman [MS94])**.** Let $d \geq 2$, $N \geq 1$. Let k be a number field. Then there exists a constant $C = C(d, N, [k : \mathbb{Q}]) > 0$ such that if f is an endomorphism of $\mathbb{P}^N$ of degree d over k, then

$$\operatorname{Preper}(f, k) \leq C,$$

where

$$\operatorname{Preper}(f, k) := \left\{x \in \mathbb{P}^N(k) : x \text{ is } f\text{-preperiodic}\right\}.$$

All known results for the above conjecture are for endomorphisms of $\mathbb{P}^1$.

(i) Using deep results of Mazur [Maz78], Kamienny [Kam92], and Merel [Mer96], one can show that the above conjecture holds for the set of Lattès maps on $\mathbb{P}^1$.
(ii) Assuming the *abcd* Conjecture (a strong form of the *abc* conjecture), Looper [Loo21a] [Loo21b] proved that the above conjecture holds for the set of polynomials on $\mathbb{P}^1$.
(iii) Besides the above two results, many other works provide evidence of Morton-Silverman's conjecture. We recommend [DF24] and [BIJ+19] for a survey of those results.

1.6. **Strategy of the proof.** We first explain the proof of Theorem 1.2, which contains the main ideas of the paper. The argument is by contradiction. Suppose that there is a small sequence of distinct horizontal curves Γ_n for which the desired genus or gonality growth fails. Arithmetic equidistribution connects the smallness of the curves with the relative dynamical Green current. This provides the arithmetic input through which bifurcation enters the argument.

The main analytic idea is to bring the language of woven currents into arithmetic dynamics. The limiting objects produced by the curves are positive closed currents, but the assumptions of low genus or low gonality force these limits to remember how they were built from curves. A uniformly woven current is, roughly speaking, an integral of currents of integration over analytic curves with uniformly bounded geometry. We also use weaker forms of this structure, obtained by approximation or by taking images of uniformly woven currents. The point of this language is that it turns a numerical bound on curves into an internal geometric structure of the limiting currents.

The genus and gonality parts produce this structure in different ways. For the genus statement, the argument is carried out directly

on $\Lambda \times \mathbb{P}^1$: if $\mathrm{genus}(\Gamma_n)/\deg(\Gamma_n)$ does not tend to infinity, then genus-volume estimates give a weak woven-type structure for the limiting currents. Our work builds on woven current theory, originally developed by Bedford–Lyubich–Smillie [BLS93], and its further developments due to Cantat [Can01], Dujardin [Duj03, Duj04, Duj12, Duj13], De Thélin [dT04, dT08] and Dinh [Din05]. See also [Duj23] for a nice introduction of woven currents theory.

For the gonality statement, one first passes through symmetric products. A bounded-gonality map gives families of effective zero-cycles, and the resulting geometry yields an analogous weak woven-type structure after pushing back to the original family, see Theorem 3.9. Theorem 3.9 is of independent interest in the theory of woven currents: we relax the bounded genus assumption to bounded gonality.

In both cases, this weak woven structure produces normal families of curves in the dynamical space.

The actual source of complexity is bifurcation. In a genuinely varying one-dimensional family which is not flexible Lattès, normality of the type produced above is incompatible with the motion of repelling cycles on the bifurcation locus. Using the stability theory of Mañé–Sad–Sullivan and Lyubich, together with McMullen's rigidity theorem, this incompatibility rules out the low-genus and low-gonality alternatives. This proves Theorem 1.2.

The higher-dimensional theorem follows the same philosophy, but the role of one-dimensional stability theory is replaced by the bifurcation theory for endomorphisms of $\mathbb{P}^N$ and the multiplier-genericity condition in Assumption A. The uniform boundedness results for iterated preimages are then obtained from the gonality theorems by Frey's theorem, one works with exact preimage curves and then applies Northcott's theorem on a fixed fiber. For preperiodic points over function fields, the same gonality input gives the required uniform bound, with an exceptional subvariety in higher dimension.

1.7. **Organization of the paper.** In Sections 2 and 3 we develop the analytic theory of woven currents needed in the proof. Section 2 establishes the basic compactness and decomposition properties. Sections 3 study the limiting behavior of woven currents arising from algebraic curves and intersection theory. In Section 4 we recall the bifurcation theory. In Section 5 we recall the slicing theory. In Section we recall the height theory for polarized dynamical systems over finitely generated fields, following Yuan–Zhang, and discusses the adelic viewpoint on bifurcation classes. In Section 7 we prove the gonality and genus-growth

theorems (Theorem 1.2 and 1.4). Section 8 derives the uniform boundedness results over number fields and complex function fields (Theorem 1.9, 1.10, 1.14 and 1.15). Finally, In Section 9 we prove the examples and genericity statements related to Assumption A (Theorem 1.6 and Proposition 1.7).

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2. Uniformly woven currents

2.1. **Basic properties of uniformly woven currents.** In this subsection, we consider positive closed currents that admit a decomposition as an integral of closed irreducible analytic subvarieties with uniformly bounded volume from above on any compact subset, which are called *uniformly woven currents*.

Let (X,ω) be a Hermitian manifold of dimension N. The volume of an analytic subvariety V (resp. more generally when S is a positive current) of X is computed with respect to ω and is denoted by $\mathrm{vol}(V)$ (resp. $\mathrm{vol}(S)$). For a subset $K\subset X$, we denote $\mathbf{M}_K(S)$ the volume of the positive current S on K.

Let $K_1\subset K_2\subset\cdots$ be an exhaustion by compact sets of X, i.e. $K_i\subset K_{i+1}^{\circ}$ and $\cup_{i=1}^{\infty}K_i=X$, where K° is the interior of K. We fix this exhaustion once for all.

Let $0\leq p\leq N$. For a compact subset $K\subset X$, let $\mathcal{D}^p(K)$ be the Banach space of continuous (p,p)-forms with support contained in K, where the norm is the supremum norm. Let $\mathcal{D}^p(X)$ be the space of continuous (p,p)-forms with compact support, i.e. $\mathcal{D}^p(X)=$

$\cup_{i=1}^{\infty}\mathcal{D}^p(K_i)$. Via the continuous embedding $j_n : \mathcal{D}^p(K_n) \to \mathcal{D}^p(K_{n+1})$, $\mathcal{D}^p(X)$ is a *strict LB space* (i.e. strict inductive limit of an increasing sequence of Banach spaces), see [SW99, Page 58]. Let $\mathcal{D}'_p(X)$ be the dual space of $\mathcal{D}^p(X)$.

Let $\mathcal{C}_p$ be the space of positive closed bidimension (p,p)-currents. Since positive currents has order 0 [Dem, Proposition 1.14], $\mathcal{C}_p$ is a subset of $\mathcal{D}'_p(X)$. In the following we discuss the metrizability of certain subset of the space of $\mathcal{C}_p$.

Let $M_i > 0, i \geq 1$ be a sequence of real numbers. Let $\mathcal{C}_{p,M_i}$ be the subset of $\mathcal{C}_p$ containing elements S such that $\mathbf{M}_{K_i}S \leq M_i$. Then

$$\tilde{\mathcal{C}_p} := \cap_{i=1}^{\infty}\, \mathcal{C}_{p,M_i}$$

is pointwise bounded, i.e. for every $\Phi \in \mathcal{D}^p(X)$, the set

$$\left\{\langle S, \Phi\rangle : S \in \tilde{\mathcal{C}_p}\right\}$$

is bounded in $\mathbb{C}$. For a strict LB space E, a pointwise bounded subset in the dual space E' is equicontinuous [SW99, Page 83, Theorem 4.2]. Thus $\tilde{\mathcal{C}_p}$ is equicontinuous. For a separable topological vector space E, the weak-* topology on an equicontinuous subset of E' is metrizable [SW99, Page 87 Theorem 4.7]. Since $\mathcal{D}^p(X)$ is separable, we get

Proposition 2.1. *The weak-* topology on $\tilde{\mathcal{C}_p}$ is metrizable.*

We now focus our attention on analytic subvarieties with bounded volume on compact subsets, which are subsets of $\tilde{\mathcal{C}_p}$.

Definition 2.2. An *effective analytic cycle V of pure dimension p* is a finite summation

$$V = \sum_W n_W W,$$

where $n_W \in \mathbb{Z}_{\geq 1}$ and each W in the summation is a closed irreducible analytic subvariety of X of dimension p.

We denote $\operatorname{supp} V := \cup_W W$ the support of V, which is a (maybe reducible) closed analytic subvariety of X of pure dimension p. We define $\operatorname{vol}(V) := \sum_W n_W \operatorname{vol}(W)$. We denote $[V]$ the current induced by V.

Definition 2.3. Let $M_i > 0, i \geq 1$ be a sequence of real numbers. A *bounded volume space* of dimension p, denoted by $\mathcal{V}_p$, is a collection of effective analytic cycles V of pure dimension p such that for every $i \geq 1$,

$$\mathbf{M}_{K_i}(V) \leq M_i.$$

Here we specifically emphasize that by our definition, the zero current also belongs to $\mathcal{V}_p$.

The topology on $\mathcal{V}_p$: Since $\mathcal{V}_p$ can be naturally seen as a subset of $\tilde{\mathcal{C}}_p$, we let the topology on $\mathcal{V}_p$ be the subspace topology induced by the weak-* topology on $\tilde{\mathcal{C}}_p$. With this topology, $\mathcal{V}_p$ is a metrizable space since $\tilde{\mathcal{C}}_p$ is a metrizable space.

Recall that the Hausdorff topology on subsets of X is defined by the following basis of neighborhoods: $\mathcal{U}_{K,\varepsilon}(A)$ consists of subsets B such that the set $B \cap K$ is contained in the ε-neighborhood of $A \cap K$, and the other way around, where $K \subset X$ is a compact subset.

Let $K \subset X$ be any fixed compact subset. Let $\mathcal{V}_{p,K}$ be the closed subset of $\mathcal{V}_p$ containing elements V such that $\operatorname{supp} V \cap K \neq \emptyset$.

We have the following Bishop's theorem [Bis64], see [Chi89, Section 16.1, Proposition 1] for details. Recall that on a metrizable space, compactness and sequential compactness are equivalent.

Theorem 2.4. *The space $\mathcal{V}_p$ is a compact metrizable space. For each compact subset $K \subset X$, $\mathcal{V}_{p,K}$ is compact. Moreover if $V_n \to V$ in $\mathcal{V}_p$, then* $\operatorname{supp}(V_n) \to \operatorname{supp} V$ *in the sense of Hausdorff topology.*

It is clear that if $K_1 \subset K_2 \subset \cdots$ is an exhaustion by compact sets of X, then $\mathcal{V}_{p,K_1} \subset \mathcal{V}_{p,K_2} \cdots$ is an exhaustion by compact sets of $\mathcal{V}_p$.

Definition 2.5. Let $\mathcal{V}^*_{p,K}$ be the subset of $\mathcal{V}_{p,K}$ containing all elements $V = \sum n_W W$ such that:

(1) there is only one irreducible component W_0 such that $W_0 \cap K \neq \emptyset$;

(2) moreover $n_{W_0} = 1$.

By Theorem 2.4, $\mathcal{V}^*_{p,K}$ is an open subset of $\mathcal{V}_{p,K}$.

The spaces $\mathcal{V}_p$, $\mathcal{V}_{p,K}$ and $\mathcal{V}^*_{p,K}$ are locally compact Polish spaces (a Polish space is a separable completely metrizable space). In this class of spaces, measure theory exhibits very favorable behavior in the following sense:

(1) On a Polish space, a finite Borel measure is Radon [Kec95, Page 107, Theorem 17.11];

(2) On a locally compact Hausdorff space, the Riesz representation theorem holds.

We will frequently make use of the two properties stated above throughout this article. For simplicity, a positive measure on a topological space will always mean a Borel measure in this article.

Definition 2.6 (Uniformly woven current)**.** A positive closed bidimension (p,p) current S on X is called *uniformly woven* if there is a bounded volume space $\mathcal{V}_p$ such that we can write

$$S = \int_{\mathcal{V}_p} [V_a]\, d\nu(a),$$

where each V_a is an element in $\mathcal{V}_p$ and ν is a finite measure on $\mathcal{V}_p$.

A quick application of the measure theory on $\mathcal{V}^*_{p,K}$ is the following:

Proposition 2.7. *Let S be a bidimension (p,p) uniformly woven current on X,*

$$S = \int_{\mathcal{V}_p} [V_a]\, d\nu(a).$$

Let $K \subset X$ be a compact subset. Then there exist a finite measure μ on $\mathcal{V}^*_{p,K}$ such that for every continuous test form Φ satisfying $\operatorname{supp}\Phi \subset K$*, we have*

$$\langle S, \Phi\rangle = \int_{\mathcal{V}^*_{p,K}} \langle [V_a], \Phi\rangle \; d\mu(a).$$

Proof. Let $C_c(\mathcal{V}^*_{p,K})$ be the space of continuous functions on $\mathcal{V}^*_{p,K}$ with compact support. Let $\chi \in C_c(\mathcal{V}^*_{p,K})$. Then χ naturally induces a function ψ_χ on $\mathcal{V}_p$: for each $V \in \mathcal{V}_p$, there is a unique way to write

$$V = \sum_{i\in I} n_i W_i + \sum_{j\in J} n_j Z_j$$

as a finite summation of irreducible subvarieties, where each W_i satisfies $W_i \cap K \neq \emptyset$, and each Z_j satisfies $Z_j \cap K = \emptyset$. Define

$$\psi_\chi(V) := \sum_{i\in I} n_i \chi(W_i). \tag{2.1}$$

Let $U \Subset X$ be a relatively compact open subset of X such that $K \subset U$. By Lelong's theorem [Chi89, Page 189, Proposition 1], there exists $c_U > 0$ such that every irreducible closed analytic subvariety W in U satisfying $W \cap K \neq \emptyset$ has $\mathbf{M}_U(W) \geq c_U$. On the other hand, by the definition of the bounded volume space $\mathcal{V}_p$, there exists $M_U < +\infty$ such that $\mathbf{M}_U(V) \leq M_U$ for every $V \in \mathcal{V}_p$. Hence in the summation (2.1), we have $\sum_{i\in I} n_i \leq M_U/c_U$, so $\sum_{i\in I} n_i$ is uniformly bounded from above. This implies ψ_χ is a bounded function. We denote $\eta_K(V)$ by $\eta_K(V) := \sum_{i\in I} n_i$, which is a non-negative integer.

We claim that ψ_χ is a Borel measurable function. For each $n \geq 0$, let $\mathcal{V}^{(n)}_{p,K}$ be the subset of $\mathcal{V}_p$ containing all elements V such that $\eta_K(V) = n$.

Then each $\mathcal{V}_{p,K}^{(n)}$ is a Borel measurable subset, and we can write $\mathcal{V}_p$ as a disjoint union of Borel measurable subsets as

$$\mathcal{V}_p = \sqcup_{n=0}^{\infty} \mathcal{V}_{p,K}^{(n)}.$$

In fact by Lelong's theorem, $\mathcal{V}_{p,K}^{(n)} = \emptyset$ when n is large enough. To show ψ_χ is a Borel measurable function, it suffices to show ψ_χ restricted on each $\mathcal{V}_{p,K}^{(n)}$ is continuous. Let $V_m \in \mathcal{V}_{p,K}^{(n)}$, $m \geq 0$ be a sequence such that $V_m \to \tilde{V} \in \mathcal{V}_{p,K}^{(n)}$. Let $\tilde{W}$ be an irreducible component of $\tilde{V}$ such that $\tilde{W} \cap K \neq \emptyset$, let $n_{\tilde{W}}$ be its multiplicity. Let Ω be a sufficiently small neighborhood of $\tilde{W}$ in X. Let F_m be the set containing all irreducible components W of V_m such that $W \subset \Omega$. Let $F'_m \subset F_m$ be the subset containing elements W such that $W \cap K \neq \emptyset$. Let $F''_m := F_m \setminus F'_m$. For m large enough, since multiplicity does not decrease under limits, we have

$$\sum_{W \in F'_m} n_W \leq \sum_{W \in F_m} n_W \leq n_{\tilde{W}},$$

where n_W is the multiplicity of W in V_m.

On the other hand, since all V_m and $\tilde{V}$ are in $\mathcal{V}_{p,K}^{(n)}$, for m large enough we have

$$\sum_{\tilde{W}} \sum_{W \in F'_m} n_W = \sum_{\tilde{W}} n_{\tilde{W}} = n,$$

and

$$V_m = \sum_{\tilde{W}} \sum_{W \in F'_m} n_W W,$$

where the summation is taken over all irreducible component of $\tilde{W}$ of $\tilde{V}$ such that $\tilde{W} \cap K \neq \emptyset$.

As a consequence, we have

$$\sum_{W \in F'_m} n_W = n_{\tilde{W}}.$$

Thus for any sequence W_m such that $W_m \in F'_m$, we have

$$W_m \to \tilde{W}$$

in the space $\mathcal{V}_{p,K}^*$. This implies $\psi_\chi(V_m) \to \psi_\chi(\tilde{V})$. Hence ψ_χ is continuous on $\mathcal{V}_{p,K}^{(n)}$. This implies ψ_χ is Borel measurable on $\mathcal{V}_p$.

Since ψ_χ is a bounded Borel measurable function, ψ_χ is integrable with respect to ν, i.e. $\int |\psi_\chi| \, d\nu < +\infty$. The map $\chi \mapsto \int \psi_\chi \, d\nu$ defines a positive linear functional on $C_c(\mathcal{V}_{p,K}^*)$. Since $\mathcal{V}_{p,K}^*$ is a locally compact

Hausdorff space, by Riesz representation theorem, there exists a unique Radon measure μ on $\mathcal{V}_{p,K}^*$ such that for every $\chi \in C_c(\mathcal{V}_{p,K}^*)$,

$$\int \chi \, d\mu = \int \psi_\chi \, d\nu. \tag{2.2}$$

We first show that μ is finite. Indeed, the integer-valued function

$$\eta_K(V) := \sum_{i \in I} n_i$$

is uniformly bounded on $\mathcal{V}_p$, by the Lelong lower bound for the volume of each irreducible component meeting K and by the uniform volume upper bound in the bounded volume space. If $\chi \in C_c(\mathcal{V}_{p,K}^*)$ satisfies $0 \le \chi \le 1$, then $0 \le \psi_\chi \le \eta_K$. Hence

$$\int_{\mathcal{V}_{p,K}^*} \chi \, d\mu = \int_{\mathcal{V}_p} \psi_\chi \, d\nu \le \left(\sup_{\mathcal{V}_p} \eta_K \right) \nu(\mathcal{V}_p) < +\infty.$$

Since $\mathcal{V}_{p,K}^*$ is locally compact and σ-compact, we can select a monotonically increasing sequence of continuous functions with compact support χ_n such that

$$\mu(\mathcal{V}_{p,K}^*) = \lim_{n \to +\infty} \int_{\mathcal{V}_{p,K}^*} \chi_n \, d\mu,$$

which implies that $\mu(\mathcal{V}_{p,K}^*) < +\infty$.

We now extend (2.2) from compactly supported continuous functions to bounded continuous functions on $\mathcal{V}_{p,K}^*$. Let χ be such a function. Choose continuous functions $\rho_m \in C_c(\mathcal{V}_{p,K}^*)$ such that $0 \le \rho_m \le 1$ and $\rho_m \to 1$ pointwise. Applying (2.2) to $\rho_m \chi$ and letting $m \to +\infty$, the dominated convergence theorem gives

$$\int_{\mathcal{V}_{p,K}^*} \chi \, d\mu = \int_{\mathcal{V}_p} \psi_\chi \, d\nu,$$

where on the right hand side we use the same definition of ψ_χ as in (2.1); the domination follows from $|\psi_{\rho_m \chi}| \le \|\chi\|_\infty \eta_K$.

Let Φ be a bidegree (p,p) continuous test form on X such that $\operatorname{supp} \Phi \subset K$. The function

$$\chi_\Phi(W) := \langle [W], \Phi \rangle, \qquad W \in \mathcal{V}_{p,K}^*,$$

is bounded and continuous. Since Φ is supported in K, for every $V \in \mathcal{V}_p$ we have

$$\psi_{\chi_\Phi}(V) = \langle [V], \Phi \rangle.$$

Therefore

$$\langle S, \Phi\rangle = \int_{\mathcal{V}_p} \langle [V_b], \Phi\rangle \; d\nu(b) = \int_{\mathcal{V}^*_{p,K}} \langle [V_a], \Phi\rangle \; d\mu(a).$$

This completes the proof.

□

Remark 2.8. In Proposition 2.7, if we let $K = \overline{\Omega}$, where $\Omega \Subset X$ is a relatively compact open subset, then the restriction $S|_\Omega$ satisfies

$$S|_\Omega = \int_{\mathcal{V}^*_{p,K}} [V_a] \; d\mu(a).$$

Let S be a positive closed (p,p)-current on X. The *trace measure* of S is by definition the measure $\sigma_S := S \wedge \omega^{N-p}$. Let R be a positive closed $(1,1)$-current on X. On a local chart, we can write $R = dd^c u$, where u is a p.s.h. function. If $u \in L^1_{loc}(\sigma_S)$, then we define $dd^c u \wedge S := dd^c(uS)$. If on each local chart, the local potential of R satisfies the integrability condition, then we can globally define $R \wedge S$.

Definition 2.9. In this case we say that $R \wedge S$ is *admissible*.

If R has bounded local potential, then $R \wedge S$ is always admissible.

Let $\mathcal{M}(\mathcal{V}_p)$ be the space of all positive measures on $\mathcal{V}_p$ with mass smaller or equal to one. Recall that the weak-* topology on $\mathcal{M}(\mathcal{V}_p)$ is defined by the weakest topology such that for every continuous function ϕ on $\mathcal{V}_p$ with compact support, the map $\nu \mapsto \int \phi \, d\nu$ is continuous. We have the following result.

Proposition 2.10. *Let X be a Hermitian manifold of dimension N. Let ν_n be a sequence of measures in $\mathcal{M}(\mathcal{V}_p)$ such that $\nu_n \to \nu$ in the weak-* sense. Let S_n and S be the corresponding uniformly woven currents. Let T be a positive closed $(1,1)$-current on X with continuous local potential. Then $S_n \wedge T \to S \wedge T$ in the sense of currents.*

Proof. We need to show for every real smooth $(p-1,p-1)$-form Ω with compact support, we have $\langle S_n \wedge T, \Omega\rangle \to \langle S \wedge T, \Omega\rangle$. Since

$$\langle S \wedge T, \Omega\rangle = \int_{\mathcal{V}_p} \langle [V_a] \cap T, \Omega\rangle \; d\nu(a),$$

and by the similar formula for $\langle S_n \wedge T, \Omega\rangle$, we only need to show that the function $I_{T,\Omega} : \mathcal{V}_p \to \mathbb{R}$, $V \mapsto \langle [V] \wedge T, \Omega\rangle$ is continuous with compact support. First we show $I_{T,\Omega}$ has compact support. Let $\mathcal{U} \subset \mathcal{V}_p$ be the subset of elements V such that $\operatorname{supp} V \cap \operatorname{supp} \Omega \neq \emptyset$. Since $\operatorname{supp} \Omega$ is compact, by Theorem 2.4, $\mathcal{U}$ is compact. It is clear that for $v \notin \mathcal{U}$,

$I_{T,\Omega}(V) = 0$. This implies that $I_{T,\Omega}$ has compact support. It remains to show $I_{T,\Omega}$ is continuous. Let $V_n \to V$ in $\mathcal{V}_p$. The problem is local, so we assume $T = dd^c u$, where u is a continuous p.s.h. function. Then

$$I_{T,\Omega}(V_n) = \langle [V_n] \wedge dd^c u, \Omega \rangle = \langle dd^c(u[V_n]), \Omega \rangle = \langle [V_n], u\, dd^c \Omega \rangle.$$

Similarly $I_{T,\Omega}(V) = \langle [V], u\, dd^c \Omega \rangle$. Since u is continuous and $V_n \to V$ in $\mathcal{V}_p$, we have $I_{T,\Omega}(V_n) \to I_{T,\Omega}(V)$, hence $I_{T,\Omega}$ is continuous, which finishes the proof. □

The following result holds for the intersections of integral of positive closed currents, in particular it holds for the intersections of uniformly woven currents. The case $N = 2$ was discussed in [DDG11, Lemma 2.7].

Proposition 2.11. *Let (X, ω) be a Hermitian manifold of dimension N, $N \geq 2$. Let S be a positive closed (p,p)-current on X and let R be a positive closed $(1,1)$-current on X such that $S \wedge R$ is admissible. Assume that S and R admit decompositions $S = \int S_a \, d\nu(a)$ and $R = \int R_b \, d\lambda(b)$. Here ν (resp. λ) is a positive measure on the space of positive closed (p,p)-currents (resp. $(1,1)$-currents) on X, and each S_a (resp. R_b) is a positive closed (p,p)-current (resp. $(1,1)$-current) on X. Then for $\nu \otimes \lambda$-a.e. (a,b), $S_a \wedge R_b$ is admissible, and we have*

$$S \wedge R = \int (S_a \wedge R_b) \, d(\nu \otimes \lambda)(a,b). \tag{2.3}$$

Proof. The result is local, so we can assume that S and R are positive closed currents on the unit ball $B \subset \mathbb{C}^N$ with finite mass.

We first show that for ν-a.e. a, $S_a \wedge R$ is admissible. Let u be a potential of R, i.e. $dd^c u = R$ on B. Let $B' \subset\subset B$ be a smaller ball. By our assumption, $\int_{B'} |u| \, d\sigma_S < +\infty$, where σ_S is the trace measure. Since σ_S admits a decomposition $\sigma_S = \int \sigma_{S_a} \, d\nu(a)$, for ν-a.e. a, we have $\int_{B'} |u| \, d\sigma_{S_a} < +\infty$. This implies that for ν-a.e. a, $S_a \wedge R$ is admissible.

Let $B' \subset\subset B$ be a smaller ball compactly contained in B. By a classical method of constructing p.s.h. functions via integral kernels [ME00, Section 2] (see also [DDG11, Lemma 2.8], [DNS08, Theorem 2.7]), there is a constant $C = C(B') > 0$ such that for every positive closed $(1,1)$-current T on B with finite mass, there is a canonical p.s.h. function u on B' such that:

(i) $dd^c u = T$ on B';
(ii) $u \leq 0$ on B';
(iii) $||u||_{L^1(B')} \leq C\mathbf{M}(T)$,

where $\mathbf{M}(T)$ is the mass of σ_T. We call u the canonical potential of T on B'.

Next we show that if $S_a \wedge R$ is admissible, then for λ-a.e. b, $S_a \wedge R_b$ is admissible. Let u_b be the canonical potential of R_b on B'. Let $u := \int u_b \, d\lambda(b)$. Then u is a well-defined non-positive p.s.h. function on B' such that $dd^c u = R$ on B'. Since by our assumption $\int -u \, d\sigma_{S_a} < +\infty$, for λ-a.e. b, we have $\int -u_b \, d\sigma_{S_a} < +\infty$. This implies that for λ-a.e. b, $S_a \wedge R_b$ is admissible.

The above discussion implies that for $\nu \otimes \lambda$-a.e. (a, b), $S_a \wedge R_b$ is admissible.

Finally we show the integral formula (2.3) holds. On B' we have

$$uS = \int (u_b S_a) \, d(\nu \otimes \lambda)(a, b).$$

Apply dd^c to the both sides we have

$$S \wedge R = \int (S_a \wedge R_b) \, d(\nu \otimes \lambda)(a, b)$$

on B'. Since this holds on each ball $B' \Subset B$, we get the desired formula (2.3) holds on B, which finishes the proof. □

2.2. **Uniformly woven currents on a complex analytic space.** In this subsection we discuss uniformly woven currents on a (maybe singular) complex analytic space.

A complex analytic space is as defined in [Dem, Definition 5.2]. Examples include all quasi-projective varieties over $\mathbb{C}$.

Let X be an irreducible complex analytic space of dimension N. Smooth functions and forms on X are understood on X^{reg} and are required, near every singular point and after a local embedding $j : X \underset{\text{loc.}}{\hookrightarrow} \mathbb{C}^l$, to be restrictions of smooth functions and forms on an open subset of $\mathbb{C}^l$. A *Hermitian form* on X is defined similarly: it is a Hermitian $(1,1)$-form on X^{reg} which, under such a local embedding, is the restriction of a Hermitian form on an ambient open subset. A complex analytic space together with a Hermitian form is called a *Hermitian analytic space*. These notions are independent of the choice of local embedding; see [Dem85, Section 1]. The volume of subvarieties in X is computed with respect to this Hermitian form.

Currents and the operators d, d^c and dd^c on X are defined by duality; we refer to [Dem85] for the standard background. A function $u : X \to \mathbb{R} \cup \{-\infty\}$ is p.s.h. if locally, under a local embedding into some $\mathbb{C}^l$, it is the restriction of a p.s.h. function on the ambient space. A

positive closed $(1,1)$-current R on X has *continuous* (resp. *bounded*) local potential if locally $R = dd^c u$ for a continuous (resp. bounded) p.s.h. function u on X. Let S be a positive closed bidimension (p,p)-current on X, $1 \leq p \leq N-1$. If u is locally integrable with respect to the trace measure of S, we define

$$S \wedge R := dd^c(uS),$$

which is independent of the choice of local potential u and is a positive closed bidimension $(p-1,p-1)$ current.

Let $\mathcal{O}_X$ be the ring of holomorphic functions on X. We call X *Stein* if X can be embedded into a Stein manifold as a closed analytic subvariety. Assume X is Stein. An open subset $D \subset X$ is called *Runge* if $\mathcal{O}_X$ is dense in $\mathcal{O}_U$ with respect to the compact-open topology.

Based on Coltoiu's [Col91] construction of the extension of p.s.h. function, we show that for a bounded p.s.h. function on X, locally there always exists an extension that remains a bounded p.s.h. function.

Proposition 2.12. *Let u be a bounded p.s.h. function on X. Then for every $x \in X$, there exists a local embedding $j : X \underset{\text{loc.}}{\hookrightarrow} U$ near x, where U is an open subset of $\mathbb{C}^l$, and a bounded p.s.h. function $\tilde{u}$ on U, such that $\tilde{u}|_{U\cap X} = u$.*

Proof. We construct $\tilde{u}$ by the method of Coltoiu's [Col91]. Without loss of generality we may assume that X is a closed analytic subvariety in $\mathbb{D}^l \subset \mathbb{C}^l$ with $x = 0$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. Then X is a Stein space since $\mathbb{D}^l$ is Stein. By adding a constant on u, we may further assume that $u > 0$ on X. Consider the following open subset $B \subset X \times \mathbb{D}$:

$$B := \{(z,w) \in X \times \mathbb{D} : \log|w| + u(z) < 0\},$$

Since $\log|w| + u(z)$ is a p.s.h. function on $X \times \mathbb{D}$, B is Runge in $X \times \mathbb{D}$.

By [Col91, Theorem 3], there exists a Runge open subset $\Omega \subset \mathbb{D}^l \times \mathbb{D}$ with $\Omega \cap (X \times \mathbb{D}) = B$. For any point $(z,w) \in \Omega$, consider the distance $\delta_\Omega(z,w)$ to the boundary $\partial\Omega$ along the w-direction. Since Ω is Runge in $\mathbb{D}^l \times \mathbb{D}$ (hence Stein), the function $-\log \delta_\Omega$ is p.s.h. on Ω [hor73]. For any $z \in \mathbb{D}^l$, set

$$\tilde{u}(z) := -\log \delta_\Omega(z,0).$$

It is clear that $(z,0) \in \Omega$. Then $\tilde{u}$ is a p.s.h. function on $\mathbb{D}^l$. Since $\Omega \cap (X \times \mathbb{D}) = B$, from the definition of $\tilde{u}$ we get $\tilde{u} = u$ on X. Since $\delta_\Omega(z,0) < 1$ for every $z \in \mathbb{D}^l$, $\tilde{u} > 0$ on $\mathbb{D}^l$. Since a p.s.h. function is bounded above on compact subsets, $\tilde{u}$ is a bounded p.s.h. function on $\frac{1}{2}\mathbb{D}^l$. This completes the proof by setting $U := \frac{1}{2}\mathbb{D}^l$. □

On a Hermitian analytic space X, the bounded volume space $\mathcal{V}_p$, $\mathcal{V}_{p,K}$ and $\mathcal{V}^*_{p,K}$ are defined as in Section 2.1, where we replace the Hermitian manifold X by the Hermitian analytic space X. The definition of uniformly woven currents on X is also defined following the same principle, see Definition 2.6 .

2.3. **Pullback of uniformly woven currents by a holomorphic finite quotient map.** In general, it is a delicate matter to define the pullback operater acting on positive closed currents under a holomorphic map between complex analytic spaces, especially in the presence of singularities. We will provide a well-defined notion of pullback operater for uniformly woven currents in the case of a holomorphic finite quotient map.

On the other hand, the pushforward of current by a proper (i.e. the preimage of a compact subset is compact) holomorphic map is always well-defined. In fact, via duality, it suffices to define the pullback of a smooth form with compact support by a proper holomorphic map, which is always a smooth form with compact support.

Let (Y, ω_Y) be a Hermitian manifold of dimension N. Denote $\mathrm{Aut}(Y)$ the group of holomorphic automorphisms of Y. Let G be a finite subgroup of $\mathrm{Aut}(Y)$. Let

$$Z := Y/G$$

be the quotient space, which is an irreducible normal complex analytic space of dimension N, see [Car57]. Let $\pi : Y \to Z$ be the quotient map. Then π is a finite map (i.e. proper with finite fibers). The topological degree $\deg \pi$ is equal to $|G|$. Let ω_Z be a Hermitian form on Z so that (Z, ω_Z) becomes a Hermitian analytic space.

Let $0 \leq p \leq N$. Let V be a p-dimensional closed irreducible analytic subvariety of Z. Thus $\pi^{-1}(V)$ is a maybe reducible p-dimensional closed analytic subvariety of Y. Assume $\pi^{-1}(V) = \sum_{a \in F} W_a$, where F is a finite set and each W_a is a p-dimensional closed irreducible analytic subvariety of Y. Thus G acts transitively on F. Let G_a be the stabilizer of W_a, which is a finite subgroup of $\mathrm{Aut}(W_a)$.

Lemma 2.13. *Let W be an irreducible complex analytic space, let G be a finite subgroup of the holomorphic automorphism group $\mathrm{Aut}(W)$. For each $x \in W$, let G_x be the stabilizer of x. Then there exists a unique subgroup H of G such that*

$$\{x \in W : G_x = H\}$$

is a Zariski open subset.

Proof. The uniqueness is clear since W is irreducible. We will show that such subgroup exists. Let

$$H := \{g \in G : g(x) = x \text{ for every } x \in W\}.$$

Then H is a subgroup of G. For each $g \notin H$, the set

$$\mathrm{Fix}(g) := \{x \in W : g(x) = x\}$$

is a closed proper analytic subvariety of W. Then we have

$$\{x \in W : G_x = H\} = W \setminus \cup_{g \notin H} \mathrm{Fix}(g)$$

is Zariski open in W. □

Come back to our setting. The finite group G_a acts holomorphically on W_a. Let H_a be the subgroup of G_a given in Lemma 2.13. Since G acts transitively on F, all H_a are conjugate in G, therefore H_a has constant cardinality. Let $\eta_V := |H_a|$. We may thus define:

Definition 2.14 (Pullback of a closed analytic subvariety)**.** The pullback π^*V of a p-dimensional closed irreducible analytic subvariety of Z is an effective analytic cycle of pure dimension p, such that

$$\pi^*V := \sum_{a \in F} \eta_V W_a.$$

If $V = \sum_W n_W$ is an effective cycle of pure dimension p in Z, we define

$$\pi^*V := \sum_W n_W \pi^*(W),$$

which is an effective cycle of pure dimension p in Y.

The following proposition is crucial.

Proposition 2.15. *Let V be an effective analytic cycle of pure dimension p in Z. Then π^*V is the unique effective analytic cycle R of pure dimension p such that:*

(1) $\pi_*([R]) = \deg \pi\ [V]$*;*

(2) $g_*([R]) = [R]$ *for every* $g \in G$*.*

Proof. Let $V = \sum_{i=1}^n n_i V_i$, where each V_i is a p-dimensional closed irreducible analytic subvariety of Z. Let R be an effective analytic cycle R of pure dimension p satisfying the above two properties. Then we can decompose $R = \sum_{i=1}^n R_i$, where each R_i is an effective analytic cycle of pure dimension p such that $\mathrm{supp}\, R_i = \pi^{-1}(V_i)$. Then each R_i still satisfies the above two properties. Thus without loss of generality, we may assume V is a p-dimensional closed irreducible analytic subvariety of Z.

Let R be an effective analytic cycle R of pure dimension p satisfying the above two properties. Let F be the set of all irreducible components of $\pi^{-1}(V)$. Property (1) implies that we can write $R = \sum_{a\in F} \phi_a W_a$, where $\phi_a \in \mathbb{Z}_{\geq 0}$. For each $a \in F$,

$$\pi_*([W_a]) = \deg(\pi|_{W_a})\,[V],$$

where $\deg(\pi|_{W_a})$ is the topological degree of $\pi|_{W_a}$, which is equal to the length of the G_a-orbit of a generic point $x \in W_a$. Since G acts transitively on F, all G_a are conjugate in G, therefore G_a has constant cardinality. Assume $L_V := |G_a| \in \mathbb{Z}_{\geq 1}$. By the orbit–stabilizer theorem,

$$\deg(\pi|_{W_a}) = \frac{|G_a|}{|H_a|} = \frac{L_V}{\eta_V},$$

where H_a, η_V are given in Lemma 2.13.

Again since G acts transitively on F, property (2) implies that ϕ_a is a constant. Assume $\phi_a \equiv \phi \in \mathbb{Z}_{\geq 1}$. Thus property (1) gives

$$\pi_*([R]) = \sum_{a\in F} \phi \deg(\pi|_{W_a})\,[V] = |G|\,[V],$$

which is equivalent to

$$\phi \cdot |F| \cdot \frac{L_V}{\eta_V} = |G|. \tag{2.4}$$

Since G acts transitively on F, By the orbit–stabilizer theorem we have $|G| = L_V|F|$, thus (2.4) is equivalent to $\phi = \eta_V$. Thus we must have $R = \sum_{a\in F} \eta_V W_a = \pi^* V$, which finishes the proof. □

If $\dim V = 0$, one can pullback a point. We can thus define pushforward of a function via duality:

Definition 2.16 (Pushforward of a function)**.** The pushforward of a function $\phi : Y \to \mathbb{R}$ is a function $\pi_*\phi : Z \to \mathbb{R}$ given by

$$\pi_*\phi(x) := \sum_{y\in\pi^{-1}(x)} \eta_y \phi(y),$$

where $\pi^* x = \sum_{y\in\pi^{-1}(x)} \eta_y\, y$ is the effective zero-cycle given by Definition 2.14.

We have the following basic property of pushforward of a function.

Proposition 2.17. *If $\phi : Y \to \mathbb{R}$ is a continuous function, then $\pi_*\phi$ is a continuous function on Z.*

Proof. Let $z_n \to z$ in Z. We need to show $\pi_*\phi(z_n) \to \pi_*\phi(z)$. By definition, $\pi_*\phi(z_n) = \langle [\pi^* z_n], \phi\rangle$ and $\pi_*\phi(z) = \langle [\pi^* z], \phi\rangle$, where $[\pi^* z_n]$ (resp. $[\pi^* z]$) are associated finite positive measures supported on $\pi^{-1}(z_n)$ (resp. $\pi^{-1}(z)$). We can multiply ϕ by a cut off function supported on a compact neighborhood of $\pi^{-1}(z)$ to make ϕ compactly supported, without affecting the value of $\langle [\pi^* z_n], \phi\rangle$ and $\langle [\pi^* z], \phi\rangle$. Thus it suffices to show $[\pi^* z_n] \to [\pi^* z]$ in the weak-* sense. Assume by contradiction that $[\pi^* z_n] \not\to [\pi^* z]$. Since the masses of $\pi^* z_n$ are constant ($= |G|$), by passing to a subsequence we may assume $\pi^* z_n \to \mu \neq [\pi^* z]$ in the weak-* sense.

For each $n \geq 1$, we have $\pi_*([\pi^* z_n]) = \deg \pi\ [z_n]$, and $g_*([\pi^* z_n]) = [\pi^* z_n]$ for every $g \in G$. Since $\pi^* z_n \to \mu \neq [\pi^* z]$ in the weak-* sense, μ is a finite sum of Dirac masses satisfying $\pi_*(\mu) = \deg \pi\ \delta_z$, and $g_*\mu = \mu$ for every $g \in G$. By Proposition 2.15, we have $\mu = [\pi^* z]$, which is a contradiction. Thus $\pi_*\phi$ is a continuous function on Z. □

Apply Chern-Levine-Nirenberg inequality and Proposition 2.12, we get:

Lemma 2.18. *Let T be a positive closed $(1,1)$-current on Z with bounded local potential. Let $U \Subset Z$ be an open subset. Let $K \subset U$ be a compact subset. Then there exists a constant $C = C(U, K, T) > 0$ such that for every positive closed bidimension (p,p) current S on Z, we have*

$$\mathbf{M}_K(S \wedge T^p) \leq C\ \mathbf{M}_U(S \wedge \omega_Z^p).$$

Proof. The problem is local, so we may assume that Z is embedded in an open subset $\Omega \subset \mathbb{C}^l$ as a closed analytic subvariety and $T = dd^c u$, where u is a p.s.h. function on Ω. We may further assume that there exists a Hermitian form ω on Ω with $\omega|_Z = \omega_Z$. By Proposition 2.12 we may further assume that u is bounded in Ω. For a compact subset $K \subset \Omega$, by Chern-Levine-Nirenberg inequality [Dem, Page 146], there is a constant $C = C(\Omega, K) > 0$ such that

$$\mathbf{M}_K(S \wedge (dd^c u)^p) \leq C||u||^p_{\infty,U}\mathbf{M}_U(S \wedge \omega^p).$$

This completes the proof. □

Let $L_1 \subset L_2 \subset \cdots$ be an exhaustion by compact sets of Z, i.e. $L_i \subset L_{i+1}^\circ$ andf $\cup_{i=1}^\infty L_i = Z$. Define $K_i := \pi^{-1}(L_i)$. Thus $K_1 \subset K_2 \subset \cdots$ is an exhaustion by compact sets of Y. We fix these two exhaustions by compact sets. Bounded volume space are defined with respect to those exhaustions by compact sets. The following Proposition is important.

Proposition 2.19. *Assume in addition that (Y, ω_Y) is Kähler. Let $\mathcal{V}_{p,Z}$ be a p-dimensional bounded volume space of Z. Then there exists a p-dimensional bounded volume space $\mathcal{V}_{p,Y}$ of Y such that $\pi^*V \in \mathcal{V}_{p,Y}$ for every $V \in \mathcal{V}_{p,Z}$. Moreover the pullback operator*

$$\Pi : \mathcal{V}_{p,Z} \to \mathcal{V}_{p,Y},\ V \mapsto \pi^*V$$

is continuous.

Proof. Let $T := \pi_*(\omega_Y)$. Then T is a positive closed $(1,1)$-current on Z. Let u be a local potential of ω_Y. By Proposition 2.17, π_*u is continuous. By [Dem85, Corollaire 1.12 and Proposition 1.13], π_*u is a p.s.h. function on an open subset of Z and π_*u is a local potential of T, i.e. $dd^c(\pi_*u) = T$ locally. Thus T has continuous local potential.

Let $M_i' > 0$ in the definition of $\mathcal{V}_{p,Z}$ such that for every element $V \in \mathcal{V}_{p,Z}$, $\mathbf{M}_{L_i}V \leq M_i'$. Since T has continuous local potential, its local potentials are bounded on every relatively compact open subset. Hence Lemma 2.18 applies on L_{i+1}, and for each $i \geq 1$ there exists a constant $C_i > 0$ such that for every $V \in \mathcal{V}_{p,Z}$,

(2.5)
$$\mathbf{M}_{K_i}([\pi^*V] \wedge \omega_Y^p) = \mathbf{M}_{L_i}([V] \wedge T^p) \leq C_i \mathbf{M}_{L_{i+1}}([V] \wedge \omega_Z^p) \leq C_i M_{i+1}'.$$

Let $M_i := C_i M_{i+1}'$. We define $\mathcal{V}_{p,Y}$ as the set of all effective analytic cycles V of pure dimension p such that $\mathbf{M}_{K_i}V \leq M_i$. Then by (2.5), $\pi^*V \in \mathcal{V}_{p,Y}$ for every $V \in \mathcal{V}_{p,Z}$.

It remains to show that the pullback operator $\Pi : \mathcal{V}_{p,Z} \to \mathcal{V}_{p,Y}$ is continuous. Since $\mathcal{V}_{p,Y}$ and $\mathcal{V}_{p,Z}$ are both metric spaces, we need to show that for any sequence V_n in $\mathcal{V}_{p,Z}$ such that $V_n \to V \in \mathcal{V}_{p,Z}$, $\pi^*(V_n) \to \pi^*V$ in $\mathcal{V}_{p,Y}$. By Theorem 2.4, $\mathcal{V}_{p,Y}$ is compact. Let R be a subsequential limit of $\pi^*(V_n)$ in $\mathcal{V}_{p,Y}$. It suffices to show that $R = \pi^*V$. By Proposition 2.15, for each $n \geq 1$,

(1) $\pi_*([\pi^*(V_n)]) = \deg \pi\ [V_n]$;

(2) $g_*([\pi^*(V_n)]) = [\pi^*(V_n)]$ for every $g \in G$.

Since the pushforward operater π_* and g_* are continuous, we get:

(i) $\pi_*([R]) = \deg \pi\ [V]$;

(ii) $g_*([R]) = [R]$ for every $g \in G$.

Then by the uniqueness part of Proposition 2.15, we have $R = \pi^*V$. This completes the proof. □

For a finite measure μ on $\mathcal{V}_{p,Z}$, the pushforward $\Pi_*\mu$ is a finite measure on $\mathcal{V}_{p,Y}$ since Π is continuous (Proposition 2.19). Thus any woven representation of a uniformly woven current on Z admits a natural lift to a uniformly woven current on Y.

Definition 2.20. Assume in addition that (Y, ω_Y) is Kähler. Let

$$S = \int_{\mathcal{V}_{p,Z}} [V_a]\, d\mu(a)$$

be a uniformly woven current on Z. For this woven representation of S, we define its lifted current on Y by

$$\pi_\mu^* S := \int_{\mathcal{V}_{p,Y}} [V_b]\, d(\Pi_* \mu)(b),$$

where $\Pi : \mathcal{V}_{p,Z} \to \mathcal{V}_{p,Y}$ is the pullback operator and $\mathcal{V}_{p,Y}$ is given in Proposition 2.19.

Remark 2.21. The lifted current $\pi_\mu^* S$ is defined here relative to the chosen woven representation of S. We do not use representation-independence in this paper; the above construction is sufficient for our applications. We will address the question of the independence of this woven representation in future work.

3. Bidimension (1,1) woven currents and intersection theory

3.1. **Currents that are approximated by algebraic curves satisfying genus-volume relation.** In this subsection, we consider positive closed bidimension $(1,1)$ currents in a quasi-projective variety X over $\mathbb{C}$ that can be approximated by effective algebraic cycles of pure dimension one satisfying a genus-volume relation. Dujardin [Duj03], [Duj04] developed an intersection theory for woven currents of bidimension $(1,1)$ in two dimensions, and in [Duj12] he generalized this two-dimensional result to higher dimensions. The results we present in this section can be regarded as an extension of Dujardin's work: while [Duj12] considers currents that arise as limits of rational curves, in this section we treat currents that are limits of curves satisfying a genus–volume relation. Our method of proof follows that of Dujardin.

Definition 3.1 (Genus-volume relation)**.** Let $(V_n)_{n\geq 1}$ be a sequence of effective algebraic cycles of pure dimension one in $\mathbb{P}^N$ over $\mathbb{C}$. We say that $(V_n)_{n\geq 1}$ satisfies the *genus-volume relation* if there is a constant $C > 0$ such that for every $n \geq 1$, we have

$$\operatorname{genus}(V_n) \leq C \operatorname{vol}(V_n).$$

Here for $V = \sum_W n_W W$ an effective algebraic cycle of pure dimension one, where each W is an irreducible algebraic curve in $\mathbb{P}^N$ and $n_W \in \mathbb{Z}_{\geq 1}$, we define $\operatorname{genus}(V) := \sum_W n_W \operatorname{genus}(W)$. By genus of W

we mean the geometric genus of any projective compactification of W. The volume is computed using Fubini-Study form on $\mathbb{P}^N(\mathbb{C})$.

Let I be a linear subspace of $\mathbb{P}^N(\mathbb{C})$ of dimension $N-2$, and let V be a projective line in $\mathbb{P}^N(\mathbb{C})$ such that $V \cap I = \emptyset$. Let $\pi : \mathbb{P}^N(\mathbb{C}) \setminus I \to V \cong \mathbb{P}^1(\mathbb{C})$ be the projection defined as follows: for $x \in \mathbb{P}^N(\mathbb{C}) \setminus I$, let H_x be the unique hyperplane passing through x and I; then $\pi(x)$ is the unique point in $H_x \cap V$.

Let Γ be an irreducible algebraic curve in $\mathbb{P}^N(\mathbb{C})$ of genus g such that $\Gamma \cap I = \emptyset$. Let U be a simply connected open subset of $\mathbb{P}^1(\mathbb{C})$, and let W be an irreducible component of $\Gamma \cap \pi^{-1}(U)$. We say that W is a *good component* if the map $\pi : W \to U$ is biholomorphic. Equivalently, this means that W is a graph over U. Otherwise W is called a *bad component*. For a bad component W, the multiplicity $m(W,U)$ is by definition the topological degree of $\pi : W \to U$. We denote by $b(\Gamma, U)$ the number of bad components in Γ over U counted with multiplicity, i.e. $b(\Gamma, U) = \sum_W m(W,U)$, where the summation takes over all irreducible components W of $\Gamma \cap \pi^{-1}(U)$.

Lemma 3.2. *Let Γ be an irreducible algebraic curve in $\mathbb{P}^N(\mathbb{C})$ of genus g. Let I be a linear subspace of $\mathbb{P}^N(\mathbb{C})$ of dimension $N-2$ such that $\Gamma \cap I = \emptyset$. let $(U_i)_{1 \le i \le l}$ be a disjoint collection of simply connected open subset of $\mathbb{P}^1(\mathbb{C})$. Then*

$$\sum_{i=1}^{l} b(\Gamma, U_i) \le 4\mathrm{vol}(\Gamma) + 4g - 4,$$

where $\mathrm{vol}(\Gamma)$ *is computed with respect to the Fubini-Study form $\omega_{\mathbb{P}^N}$.*

Proof. Let $p : \hat{\Gamma} \to \Gamma$ be the normalization of Γ, hence $\hat{\Gamma}$ is a complex smooth algebraic curve of genus g. Considering the branched covering $\pi \circ p : \hat{\Gamma} \to \mathbb{P}^1(\mathbb{C})$, by Riemann-Hurwitz formula we have

$$2g - 2 = -2\deg(\pi \circ p) + \sum_{x \in R} (e_x - 1),$$

where R is the critical set of $\pi \circ p : \hat{\Gamma} \to \mathbb{P}^1(\mathbb{C})$, and e_x is the critical multiplicity at x.

Since $\deg(\pi \circ p) = \deg(\pi|_\Gamma) = \mathrm{vol}(\Gamma)$ (both are equal to the number of the intersection points of Γ with a generic hyperplane), and since $e_x \ge 2$ when $x \in R$, we have

$$\sum_{x \in R} e_x \le 4\mathrm{vol}(\Gamma) + 4g - 4.$$

If $\pi : W \to U_i$ is a bad component, then the multiplicity $m(W, U_i)$ is the same as $\deg((\pi \circ p)|_{\hat{W}})$, where $\hat{W} := p^{-1}(W)$. Since U_i is simply connected, we have

$$\deg((\pi \circ p)|_{\hat{W}}) \leq \sum_{x \in R \cap \hat{W}} e_x.$$

For the proof, see the last paragraph in page 750 of the proof of [Duj03, Lemma 3.2].

Since U_i, $1 \leq i \leq l$ are disjoint, we have

$$\sum_{i=1}^{l} b(\Gamma, U_i) = \sum_{i=1}^{l} \sum_{W} m(W, U_i) \leq \sum_{x \in R} e_x \leq 4\mathrm{vol}(\Gamma) + 4g - 4,$$

where, in the second term, the sum is taken over all irreducible components W of $\Gamma \cap \pi^{-1}(U_i)$. This completes the proof. □

Definition 3.3 (Affine coordinate)**.** An affine coordinate is the following data:

(i) H is a hyperplane of $\mathbb{P}^N(\mathbb{C})$;
(ii) $I_1, \ldots, I_N$ are N linear subspaces in H of dimension $N-2$, which are in general positions.
(iii) For each i, $\pi_i : \mathbb{P}^N(\mathbb{C}) \setminus I_i \to \mathbb{P}^1(\mathbb{C})$ is a projection as in Subsection 3.1 such that $\pi_i^{-1}(\infty) \in H$.

In the above definition, the map

$$(\pi_1, \ldots, \pi_N) : \mathbb{P}^N(\mathbb{C}) \setminus H \to \mathbb{C}^N$$

is an isomorphism, which gives an affine coordinate via those π_i. In particular, for each $z \in \mathbb{C}$, the fiber $\pi_i^{-1}(z)$ is isomorphic to $\mathbb{C}^{N-1}$.

Let $r > 0$. For each $1 \leq i \leq N$, let $\mathcal{Q}_i$ be a subdivision of $\mathbb{C}$ into *affine cubes* of size r (with respect to the Euclidean metric $\omega_{\mathbb{C}}$). Here by an affine cube we mean an open affine cube, and by subdivision we mean that the union of affine cubes form an open dense subset of $\mathbb{C}$, and two different affine cubes have empty intersection. An affine cube in $\mathbb{C}^N$ is by definition the set $\cap_{1 \leq i \leq N} \pi_i^{-1}(Q_i)$, where Q_i is an affine cube in $\mathbb{C}$ subordinate to the subdivision $\mathcal{Q}_i$. Let $\mathcal{Q}$ be the subdivion of $\mathbb{C}^N$ given by the above affine cubes in $\mathbb{C}^N$. Let $\Omega_{\mathcal{Q}}$ be the union of all affine cubes in $\mathbb{C}^N$ subordinate to the subdivision $\mathcal{Q}$, whcih is an open and dense subset of $\mathbb{C}^N$. Let $\partial\mathcal{Q} := \mathbb{C}^N \setminus \Omega_{\mathcal{Q}}$ be the boundary of $\mathcal{Q}$.

Let B be an open ball of $\mathbb{C}^N$. We fix the ball B, an affine coordinate, subdivisions $\mathcal{Q}_i$ of $\mathbb{C}$ of size r and subdivision $\mathcal{Q}$ of $\mathbb{C}^n$ of size r.

The following lemma is crucial.

Lemma 3.4. *Let $(V_n)_{n\geq 1}$ be a sequence of effective algebraic cycles of pure dimension one in $\mathbb{P}^N$ satisfying the genus-volume relation. Assume that $S_n := [V_n]/\mathrm{vol}(V_n) \to S$ in the sense of currents. Assume that for each i and n, the projection center $I_i \cap V_n = \emptyset$ and S puts no mass on $\partial\mathcal{Q}$, i.e.*

$$(S \wedge \omega_{\mathbb{P}^N})(\partial\mathcal{Q}) = 0.$$

Then there is a uniformly woven current $S_{\mathcal{Q}}$ on $\Omega_{\mathcal{Q}}$ such that $S_{\mathcal{Q}} \leq S$ and the mass of the measure $(S - S_{\mathcal{Q}}) \wedge \omega_{\mathbb{P}^N}$ on B satisfies

$$\mathbf{M}_B((S - S_{\mathcal{Q}}) \wedge \omega_{\mathbb{P}^N}) \leq Cr^2, \tag{3.1}$$

where C is a constant independent of $\mathcal{Q}$.

Let T be a positive closed current on an open subset $\Omega \subset \mathbb{P}^N(\mathbb{C})$. The zero extension of T is a positive current on $\mathbb{P}^N(\mathbb{C})$. By abuse of notation, we also denote this extension by T. In general T is not closed. So in Lemma 3.4, $S_{\mathcal{Q}}$ can be seen as a positive but not necessarily closed current on $\mathbb{P}^N(\mathbb{C})$.

For any open subset $\Omega \subset \mathbb{P}^N(\mathbb{C})$, we denote by $\mathcal{V}_\Omega$ the space of all effective analytic cycles of pure dimension one with volume bounded by 1 from above.

Proof of Lemma 3.4. Consider one projection $\pi_i : \mathbb{P}^N(\mathbb{C}) \setminus I_i \to \mathbb{P}^1(\mathbb{C})$. Let $Q \in \mathcal{Q}_i$ be an affine cube. For each $n \geq 1$, let $V_n = \sum_{j=1}^{l} V_{n,j}$ be the decomposition of V_n into irreducible components. Here we allow $V_{n,j}$ to be repeated. Let $b(V_{n,j}, Q)$ be the number of bad components of $\pi_i : V_{n,j} \to \mathbb{P}^1(\mathbb{C})$ over Q, counted with multiplicity. Let $b(V_n, Q) := \sum_{j=1}^{l} b(V_{n,j}, Q)$, and let $b(V_n, \mathcal{Q}_i) := \sum_{Q\in\mathcal{Q}_i} b(V_n, Q)$. Since $(V_n)_{n\geq 1}$ satisfy the genus-volume relation, by Lemma 3.2, there is a constant $C_1 > 0$ independent of n and $\mathcal{Q}$ such that

$$b(V_n, \mathcal{Q}_i) \leq C_1 \mathrm{vol}(V_n). \tag{3.2}$$

Next we shall construct the uniformly woven current $S_{\mathcal{Q}}$ using good components of $\pi_i : V_n \to \mathbb{P}^1(\mathbb{C})$, $1 \leq i \leq N$. Let $V_{n,\mathcal{Q}_i}$ be the sum of all good components Γ with respect to π_i with volume bounded by 1 from above; moreover $\Gamma \cap \overline{B} \neq \emptyset$. There exists a compact subset $K \subset \mathbb{C}$ independent of n such that $\pi_i(\Gamma) \subset K$ for every such Γ.

Let $\Omega_i := \cup_{Q\in\mathcal{Q}_i} \pi_i^{-1}(Q)$. Let $S_{n,\mathcal{Q}_i}$ be the closed current on Ω_i,

$$S_{n,\mathcal{Q}_i} := \frac{1}{\mathrm{vol}(V_n)}[V_{n,\mathcal{Q}_i}].$$

We can write $S_{n,\mathcal{Q}_i}$ as a uniformly woven current

$$S_{n,\mathcal{Q}_i} = \int_{\mathcal{V}_{\Omega_i}} [\Gamma_a]\, d\nu_{n,i}(a),$$

and each Γ_a is a graph with respect to π_i over an affine cube $Q \subset \mathbb{C}$, $\nu_{n,i}$ is a finite average of Dirac masses on $\mathcal{V}_{\Omega_i}$. The condition $\Gamma \cap \overline{B} \neq \emptyset$ implies that the volume of Γ_a is bounded uniformly from below by a constant independent of n, since such Γ is a graph over a cube $Q \subset K$. Thus the mass of $\nu_{n,i}$ is uniformly bounded from above by a constant independent of n. It is clear that $S_{n,\mathcal{Q}_i} \leq S_n$ on Ω_i.

Let $V_{n,\mathcal{Q}}$ be the sum of all irreducible component Γ of $V_n|_{\Omega_\mathcal{Q}}$, such that Γ has volume bounded by 1, and there exists at least $1 \leq i \leq N$ such that Γ is contained in a component of $V_{n,\mathcal{Q}_i}$.

Let $S_{n,\mathcal{Q}}$ be the closed current on $\Omega_\mathcal{Q}$,

$$S_{n,\mathcal{Q}} := \frac{1}{\mathrm{vol}(V_n)}[V_{n,\mathcal{Q}}].$$

Define $B' := B \setminus \partial\mathcal{Q}$, which is an open dense subset of B . It is clear that for each $1 \leq i \leq N$,

$$S_{n,\mathcal{Q}} \leq S_n \text{ on } B, \text{ and } S_{n,\mathcal{Q}} \geq S_{n,\mathcal{Q}_i} \text{ on } B'.$$

The total number of good components Γ of $\pi_i : V_n \to \mathbb{P}^1(\mathbb{C})$ over some $Q \in \mathcal{Q}_i$ such that $\mathrm{vol}(\Gamma) \geq 1$ is at most $\mathrm{vol}(V_n)$. Together with (3.2), this implies that there is a constant $C_2 > 0$ independent of n and $\mathcal{Q}$ such that

$$\mathbf{M}_B((S_n - S_{n,\mathcal{Q}_i}) \wedge \pi_i^*(\omega_\mathbb{C})) \leq C_2 r^2.$$

Since $S_{n,\mathcal{Q}} \geq S_{n,\mathcal{Q}_i}$ on B' and $S_{n,\mathcal{Q}} \leq S_n$ on B, we have

$$\mathbf{M}_{B'}((S_n - S_{n,\mathcal{Q}}) \wedge \pi_i^*(\omega_\mathbb{C})) \leq C_2 r^2.$$

Since $\sum_{i=1}^N \pi_i^*(\omega_\mathbb{C})$ restricted on B is equivalent to $\omega_{\mathbb{P}^N}$ restricted on B, there is a constant $C_3 > 0$ independent of n and $\mathcal{Q}$ such that

$$\mathbf{M}_{B'}\left((S_n - S_{n,\mathcal{Q}}) \wedge \omega_{\mathbb{P}^N}\right) \leq C_3 r^2. \tag{3.3}$$

Write

$$S_{n,\mathcal{Q}} = \int_{\mathcal{V}_{\Omega_\mathcal{Q}}} [\Gamma_a]\, d\nu_n(a),$$

where ν_n is a finite measure on $\mathcal{V}_{\Omega_\mathcal{Q}}$ with mass $\mathbf{M}(\nu_n) \leq \sum_{i=1}^N \mathbf{M}(\nu_{n,i})$, which is uniformly bounded from above by a constant independent of n.

The space $\mathcal{V}_{\Omega_\mathcal{Q}}$ is a compact Hausdorff space by Theorem 2.4. Since the masses of all ν_n are bounded by N, by passing to a subsequence we

may assume $\nu_n \to \nu$ in the weak-* sense, where ν is a finite measure on $\mathcal{V}_{\Omega_\mathcal{Q}}$. Let

$$S_\mathcal{Q} := \int_{\mathcal{V}_{\Omega_\mathcal{Q}}} [\Gamma_a]\, d\nu(a).$$

By Proposition 2.10, we have $S_{n,\mathcal{Q}} \to S_\mathcal{Q}$ in the sense of currents. Since $S_{n,\mathcal{Q}} \leq S_n$ and $S_n \to S$ in the sense of currents, we have $S_\mathcal{Q} \leq S$. By (3.3) we have

$$\mathbf{M}_{B'}\left((S - S_\mathcal{Q}) \wedge \omega_{\mathbb{P}^N}\right) \leq C_3 r^2.$$

Since $0 \leq S - S_\mathcal{Q} \leq S$ and $S \wedge \omega_{\mathbb{P}^N}$ gives no mass to $B \setminus B'$, we have

$$\mathbf{M}_{B}\left((S - S_\mathcal{Q}) \wedge \omega_{\mathbb{P}^N}\right) \leq C_3 r^2.$$

This implies (3.1), which finishes the proof. □

Let $\mathcal{Q}_0$ be a fixed subdivision of $\mathbb{C}^N$ into affine cubes of size r. Let $\tilde{Q} \in \mathcal{Q}_0$ be one of the affine cubes. Let $x \in \tilde{Q}$, let $\mathcal{Q}_x := \mathcal{Q}_0 + x$ be the translation of $\mathcal{Q}_0$, which is also a subdivision of $\mathbb{C}^N$ of size r. Let $0 < \lambda < 1$. If Q is an affine cube of size r, we let λQ be the cube with the same center as Q, homothetic to it by the factor λ. Let m be the Lebesgue measure on $\mathbb{C}^N$.

Lemma 3.5. *Let μ be a probability measure on $\mathbb{C}^N$ and $0 < \lambda < 1$, $0 < \delta < 1$. Then there exists a Borel measurable subset $A \subset \tilde{Q}$ such that $m(A) \geq (1-\delta)m(\tilde{Q})$, and for $x \in A$,*

$$\mu\left(\mathbb{C}^N \setminus \cup_{Q \in \mathcal{Q}_x} \lambda Q\right) \leq 1 - \delta\lambda^{2N},$$

moreover μ put no mass on $\partial\mathcal{Q}_x$.

Proof. Consider the product space $\mathbb{C}^N \times \mathbb{C}^N$, and the product meassure $m \otimes \mu$ on it. Let $W \subset \mathbb{C}^N \times \mathbb{C}^N$ be the subset containing points (x, y) such that $x \in \tilde{Q}$ and $y \in \cup_{Q \in \mathcal{Q}_x} \lambda Q$. By Fubini's theorem

$$\int_{\tilde{Q}} \mu\left(\cup_{Q \in \mathcal{Q}_x} \lambda Q\right)\, dm(x) = \int_{\mathbb{C}^N \times \mathbb{C}^N} \mathbb{1}_W\, d(m \otimes \mu)(x, y).$$

The slicing of W with respect to $y \in \mathbb{C}^N$ gives the set $y + \lambda\tilde{Q}$. Again by Fubini's theorem

$$\int_{\mathbb{C}^N \times \mathbb{C}^N} \mathbb{1}_W\, d(m \otimes \mu)(x, y) = \int_{\mathbb{C}^N} m(y + \lambda\tilde{Q})\, d\mu(y) = \lambda^{2N} m(\tilde{Q}).$$

Let $A' \subset \tilde{Q}$ be the subset containing points x such that $\mu\left(\cup_{Q \in \mathcal{Q}_x} \lambda Q\right) \geq \delta\lambda^{2N}$. Since $\int_{\tilde{Q}} \mu\left(\cup_{Q \in \mathcal{Q}_x} \lambda Q\right)\, dm(x) = \lambda^{2N} m(\tilde{Q})$, we have $m(A') \geq (1-\delta)m(\tilde{Q})$.

Let $A'' \subset \tilde{Q}$ be the subset containing points x such that $\mu\left(\cup_{Q\in\mathcal{Q}_x} Q\right) = 1$. Since $\int_{\tilde{Q}} \mu\left(\cup_{Q\in\mathcal{Q}_x} Q\right)\ dm(x) = m(\tilde{Q})$ and $\mu\left(\cup_{Q\in\mathcal{Q}_x} Q\right) \leq 1$, we have $m(A'') = m(\tilde{Q})$. For $x \in A''$, μ put no mass on $\partial\mathcal{Q}_x$.

The set $A := A' \cap A''$ is what we want. □

Let X be a complex quasi-projective variety, embedded in $\mathbb{P}^N(\mathbb{C})$. Let ω_X be the restriction of the Fubini-Study form $\omega_{\mathbb{P}^N}$ on $\mathbb{P}^N(\mathbb{C})$ to X. The volumes and diameters on X are computed with respect to ω_X. Let B be a ball inside an affine coordinate $\mathbb{C}^N$ such that $B_X := B \cap X$ is compactly contained in X.

The following is the main result of this section.

Theorem 3.6. *Let $(V_n)_{n\geq 1}$ be a sequence of effective algebraic cycles of pure dimension one in X satisfying the genus-volume relation. Assume that $[V_n]/\mathrm{vol}(V_n) \to S$ in the sense of currents. Let R be a positive closed $(1,1)$-current on X with continuous local potential. Then there exists an affine coordinate (Definition 3.3) such that for each $r > 0$, there exists a subdivision $\mathcal{Q}$ of $\mathbb{C}^N$ of size r, a uniformly woven current S_r on $\Omega_{\mathcal{Q}}$ such that $S_r \leq S$ and*

$$\mathbf{M}_{B_X}((S - S_r) \wedge R) \to 0 \tag{3.4}$$

when $r \to 0$.

Proof. We choose the affine coordinate in Definition 3.3 after B has been fixed. More precisely, choose a hyperplane $H \subset \mathbb{P}^N(\mathbb{C})$ such that $H \cap \overline{B} = \emptyset$ and no irreducible component of any V_n is contained in H. Then choose N linear subspaces $I_1, \ldots, I_N \subset H$ in general position such that $I_i \cap V_n = \emptyset$ for every i and every $n \geq 1$; this is possible by avoiding a countable union of proper algebraic subsets in the corresponding parameter spaces. We choose N projections $(\pi_i)_{1\leq i\leq N}$ associated to these I_i. In the following affine cubes in $\mathbb{C}^N$ are with respect to $(\pi_i)_{1\leq i\leq N}$. To show the theorem, it suffices to show for every $\varepsilon > 0$, there exists $r_0 = r_0(\varepsilon) > 0$ such that for $0 < r < r_0$, there exists a subdivision $\mathcal{Q}$ of $\mathbb{C}^N$ of size r and currents $T_r, (T_i)_{1\leq i\leq N}$ satisfying conditions in Theorem 3.6 such that $\mathbf{M}_{B_X}((S - S_r) \wedge R) < 2\varepsilon$.

Let μ_0 be the restriction of the measure $S \wedge R$ on B_X, which is a finite measure. Let μ be the zero extension of μ_0 on $\mathbb{C}^N$. By Lemma 3.5, for every $\varepsilon > 0$, there exists $0 < \lambda = \lambda(\varepsilon) < 1$ sufficiently close to 1 such that for every $r > 0$, there is a subdivision $\mathcal{Q}$ of $\mathbb{C}^N$ of size r such that:

(i) μ put no mass on $\partial\mathcal{Q}$.;
(ii) $\mu\left(\mathbb{C}^N \setminus \cup_{Q\in\mathcal{Q}} \lambda Q\right) < \varepsilon$.

A crucial point here is that (ii) holds uniformly in r. Note that the subdivision $\mathcal{Q}$ satisfies the conditions in Lemma 3.4. Let S_r be the current $S_{\mathcal{Q}}$ in Lemma 3.4. It is clear that $S_r \leq S$. It remains to show $\mathbf{M}_{B_X}((S-S_r)\wedge R) < 2\varepsilon$ when $0 < r < r_0$, where $r_0 = r_0(\varepsilon) > 0$ is a constant to be defined later.

For each $r > 0$, let $\mathcal{Q}$ be the subdivision of $\mathbb{C}^N$ of size r constructed above. The problem is local, so we can write $R = dd^c u$ on B_X, where u is a continuous p.s.h. function on a neighborhood of $\overline{B_X}$. Let ψ be a smooth cut-off function on $\cup_{Q\in\mathcal{Q}} Q$, such that $0 \leq \psi_{\mathcal{Q}} \leq 1$, $\psi = 1$ on λQ for each $Q \in \mathcal{Q}$, and we have $||\psi||_{C^2} \lesssim r^{-2}$. Here $\lesssim$ means that $||\psi||_{C^2} \leq Cr^{-2}$ for a constant independent of r. Hence we can write $dd^c\psi \wedge [X] = S^+ - S^-$, where $S^\pm$ are positive closed currents such that $S^\pm \wedge [X] \lesssim r^{-2}\omega_{\mathbb{P}^N} \wedge [X]$ on B. On each affine cube $Q \in \mathcal{Q}$, let $c_Q \in Q$ be the center of Q. By Lemma 3.4 (3.1), we have

$$\begin{aligned}
\int_{B_X} \psi(S-S_r)\wedge R &= \int_{B_X} \psi(S-S_r)\wedge dd^c u \\
&= \int_{B_X} [u-u(c_Q)](S-S_r)\wedge dd^c\psi \wedge [X] \\
&\lesssim r^{-2}\omega(u,2r)\mathbf{M}_B((S-S_r)\wedge\omega_{\mathbb{P}^N}) \\
&\lesssim \omega(u,2r),
\end{aligned}$$

where

$$\omega(u,2r) := \sup_{z_1,z_2\in\overline{B_X}, |z_1-z_2|\leq 2r} |u(z_1)-u(z_2)|$$

is the modulus of continuity of u of size $2r$. Since u is continuous on a neighborhood of $\overline{B_X}$, there is a constant $r_0 = r_0(\varepsilon) > 0$ such that $\int_{B_X} \psi(S-S_r)\wedge R < \varepsilon$ when $0 < r < r_0$.

Since $\mu\left(\mathbb{C}^N \setminus \cup_{Q\in\mathcal{Q}}\lambda Q\right) < \varepsilon$, we have

$$\begin{aligned}
\mathbf{M}_{B_X}((S-S_r)\wedge R) &\leq \mathbf{M}_{B_X\setminus\cup_{Q\in\mathcal{Q}}\lambda Q}\, S\wedge R + \int_{B_X} \psi(S-S_r)\wedge R \\
&= \mu\left(\mathbb{C}^N \setminus \cup_{Q\in\mathcal{Q}}\lambda Q\right) + \int_{B_X} \psi(S-S_r)\wedge R \\
&< 2\varepsilon,
\end{aligned}$$

when $0 < r < r_0$. This completes the proof. □

3.2. **Currents that are approximated by algebraic curves with constant gonality.** In this subsection, we consider positive closed bidimension $(1,1)$ currents on a quasi-projective variety X over $\mathbb{C}$ that can be approximated by algebraic curves with constant gonality. Our is of independent interest in the theory of woven currents: we relax the bounded genus assumption to bounded gonality. Our result is new even in dimension two.

Let X be a complex quasi-projective variety.

Definition 3.7. Let $V = \sum_W n_W W$ be an effective algebraic cycle of pure dimension one in X. We define

$$\operatorname{gonal}(V) := \max_W \operatorname{gonal}(W),$$

where the maximum is taken over the irreducible components appearing in V.

In our main applications V is a Galois orbit, and then all components have the same gonality.

Let $m \geq 1$. Let X^m be the m-th product space. The permutation group S_m acts on X^m by permuting different coordinates. Let X^m/S_m be the quotient variety. Let $\pi : V^m \to V^m/S_m$ be the canonical projection and let $p : X^m \to X$ be the projection onto the first coordinate.

The following Lemma is crucial.

Lemma 3.8. *Let V be an irreducible complex quasi-projective curve with $\operatorname{gonal}(V) = m$. Let $p : V^m \to V$ be the projection onto the first coordinate. Then*

(1) There exists an algebraic curve Γ in V^m/S_m which is birational to $\mathbb{P}^1$;

(2) Let $W := \pi^{-1}(\Gamma)$. Then p is dominant when restricted to any irreducible component of W, and, as cycles on V,

$$p_*[W] = (m-1)![V].$$

Proof. Since $\operatorname{gonal}(V) = m$, there exists a degree m rational map $\phi : V \dashrightarrow \mathbb{P}^1$. Let $U \subset \mathbb{P}^1$ be the Zariski open set containing all $t \in \mathbb{P}^1$ such that $\phi^{-1}(t)$ has m points counted with multiplicity. Let $\psi : \mathbb{P}^1 \dashrightarrow V^m/S_m$ be the rational map given by the following: for $t \in U$, $\phi^{-1}(t)$ has m points counted with multiplicity, define $\psi(t) := \{x_1, \ldots, x_m\} \in V^m/S_m$ be the unordered list of elements in $\phi^{-1}(t)$. It is clear that ψ is injective on U, hence the Zariski closure $\Gamma := \overline{\psi(U)}^{\text{zar}} \subset V^m/S_m$ is birational to $\mathbb{P}^1$.

For the second assertion, observe that W is invariant under the action of S_m. Let $x \in W$ be a generic point such that all coordinates of x

take different values. Then $y \in W$ such that $p(x) = p(y)$ if and only if $\pi(y) = \pi(x)$ and $p(x) = p(y)$, if and only if $y = \sigma(x)$ for $\sigma \in G$, where G is the subgroup of S_m that fix the first coordinate. Thus a generic point of V has exactly $|G| = (m-1)!$ preimages in W, counted in the cycle $[W]$. Hence $p_*[W] = (m-1)![V]$. □

Let X be a complex quasi-projective variety. We embedd X, X^m and X^m/S_m into some $\mathbb{P}^N$ with N large. Volume of subvarieties are computed with respect to the Fubini-Study form on $\mathbb{P}^N(\mathbb{C})$.

Theorem 3.9. *Let V_n be a sequence of effective algebraic cycles of pure dimension one in X such that every irreducible component of V_n has gonality m for every $n \geq 1$. Assume that $[V_n]/\mathrm{vol}(V_n) \to S$ in the sense of currents. Then there exists a positive closed bidimension $(1,1)$ current S' on X^m such that the following hold:*

(i) *$p_*(S') = S$;*

(ii) *Let R be a positive closed $(1,1)$-current on X^m with continuous local potential and B be any ball compactly contained in X^m. Then for each $r > 0$, there is a uniformly woven current S'_r on an open subset $\Omega_r \subset X^m$, such that $S'_r \leq S'$ and*

$$\mathbf{M}_B((S' - S'_r) \wedge R) \to 0$$

when $r \to 0$.

Proof. We may assume $S \neq 0$, since otherwise the result is trivial. Write $V_n = \sum_{j=1}^{l} V_{n,j}$ as a sum of irreducible components. For each j, let $\Gamma_{n,j} \subset X^m/S_m$ be the curve associated to $V_{n,j}$ by Lemma 3.8, and set

$$\Gamma_n := \sum_{j=1}^{l} \Gamma_{n,j}, \qquad V'_n := \pi^*\Gamma_n.$$

By Lemma 3.8, $p_*[V'_n] = (m-1)![V_n]$. Passing to a subsequence, assume

$$\frac{[V'_n]}{\mathrm{vol}(V'_n)} \to S''.$$

Since $[V_n]/\mathrm{vol}(V_n) \to S \neq 0$, after passing to a further subsequence there is $\alpha > 0$ such that

$$\frac{\mathrm{vol}(V_n)}{\mathrm{vol}(V'_n)} \to \alpha.$$

Then $p_*S'' = (m-1)!\alpha S$. Put

$$c := \frac{1}{(m-1)!\alpha}, \qquad S' := cS''.$$

Then $p_*S' = S$.

It remains to prove (ii). Let R be as in the statement and let $B \Subset X^m$ be a ball. Set $\tilde{R} := \pi_* R$, and choose a relatively compact subset $\tilde{B} \subset X^m/S_m$ with $B \subset \pi^{-1}(\tilde{B})$. The current $\tilde{R}$ has continuous local potential by Proposition 2.17. By Lemma 2.18, the masses of $[\Gamma_n]/\mathrm{vol}(V_n')$ are uniformly bounded on compact subsets. Since every irreducible component of Γ_n is birational to $\mathbb{P}^1$, the proof of Theorem 3.6 applies, with the normalization $\mathrm{vol}(V_n')$, to the sequence Γ_n, the test current $\tilde{R}$, and the compact set $\tilde{B}$. Thus, for every $r > 0$, after passing to a subsequence, we get an open set $\tilde{\Omega}_r \subset X^m/S_m$ and subcycles

$$\Gamma_{n,r} \le \Gamma_n|_{\tilde{\Omega}_r}$$

whose components have uniformly bounded volume on compact subsets, such that the cluster limit of $[\Gamma_{n,r}]/\mathrm{vol}(V_n')$ is uniformly woven on $\tilde{\Omega}_r$, and

$$\limsup_{n\to\infty} \mathbf{M}_{\tilde{B}}\left(\frac{[\Gamma_n] - [\Gamma_{n,r}]}{\mathrm{vol}(V_n')} \wedge \tilde{R}\right) \le \varepsilon(r), \tag{3.5}$$

where $\varepsilon(r) \to 0$ as $r \to 0$.

Let $V_{n,r}' := \pi^*\Gamma_{n,r}$. Since $0 \le V_{n,r}' \le V_n'$, after passing to a subsequence we may assume

$$\frac{[V_{n,r}']}{\mathrm{vol}(V_n')} \to S_r''$$

with $S_r'' \le S''$. Moreover, by the bounded-volume property of the curves $\Gamma_{n,r}$ and Proposition 2.19, the current

$$S_r' := cS_r''$$

is uniformly woven on $\Omega_r := \pi^{-1}(\tilde{\Omega}_r)$, and $S_r' \le S'$.

Finally, using the projection formula for analytic cycles at finite level and then letting $n \to \infty$, we get

$$\begin{aligned}\mathbf{M}_B((S' - S_r') \wedge R) &\le c \liminf_{n\to\infty} \mathbf{M}_{\pi^{-1}(\tilde{B})}\left(\frac{[V_n'] - [V_{n,r}']}{\mathrm{vol}(V_n')} \wedge R\right) \\ &= c \liminf_{n\to\infty} \mathbf{M}_{\tilde{B}}\left(\frac{[\Gamma_n] - [\Gamma_{n,r}]}{\mathrm{vol}(V_n')} \wedge \tilde{R}\right) \le c\,\varepsilon(r)\end{aligned}$$

by (3.5). Since $\varepsilon(r) \to 0$ as $r \to 0$, this proves (ii), and the theorem follows. □

4. Bifurcation theory for families of endomorphisms of $\mathbb{P}^N$

4.1. **The bifurcation current and bifurcation set.** Let f be a holomorphic family of endomorphisms on $\mathbb{P}^N(\mathbb{C})$ of degree $d \geq 2$, parametrized by a Hermitian manifold $(\Lambda, \omega_\Lambda)$, i.e. f is a holomorphic map

$$f : \Lambda \times \mathbb{P}^N(\mathbb{C}) \to \Lambda \times \mathbb{P}^N(\mathbb{C})$$
$$(t, z) \mapsto (t, f_t(z)),$$

such that for every $t \in \Lambda$, f_t is a degree d endomorphism of $\mathbb{P}^N$ over $\mathbb{C}$. Let $\pi_1 : \Lambda \times \mathbb{P}^N(\mathbb{C}) \to \Lambda$ and $\pi_2 : \Lambda \times \mathbb{P}^N(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C})$ be the canonical projections. Let $\omega_{\mathbb{P}^N}$ be the Fubini-Study form on $\mathbb{P}^N(\mathbb{C})$. Let $\omega_1 := \pi_1^*(\omega_\Lambda)$ and $\omega_2 := \pi_2^*(\omega_{\mathbb{P}^N})$. The *relative Green current* of f is defined by

$$T_f := \lim_{n \to +\infty} d^{-n}(f^n)^*(\omega_2) = \omega_2 + dd^c u,$$

where u is a Hölder continuous quasi-p.s.h. function [DS10, Lemma 1.19].

For every $t \in \Lambda$, we have $T_f^N \wedge [\pi_1^{-1}(t)] = \mu_t$, where T_f^N is the N-times self intersection of T_f and μ_t is the unique maximal entropy measure of f_t.

Let $[C_f]$ be the current of integration on the critical set C_f of f taking into account the multiplicities. Define the *bifurcation current* $T_{\text{bif}} := (\pi_1)_*(T_f^N \wedge [C_f])$, which has finite mass on Λ [GV25, Proposition 3.3] if Λ is quasi-projective and f is an algebraic family.

The current T_{bif} has a global potential function $L(t)$, which is the sum of the Lyapunov exponents of the measure μ_t [BB07, Corollary 4.6]. We have $dd^c L = T_{\text{bif}}$. The function L is Hölder continuous [DS10, Theorem 2.47].

Definition 4.1. The *bifurcation set* $\text{Bif}(f)$ is the support of the bifurcation current T_{bif}. The *stable set* is $\text{Stab}(f) := \Lambda \setminus \text{Bif}(f)$.

Let g be an endomorphism of $\mathbb{P}^N$ over $\mathbb{C}$ of degree $d \geq 2$. Let μ_g be the unique maximal entropy measure, and let J_g be the support of μ_g. A periodic point x of g is called *J-repelling* if x is repelling and $x \in J_g$.

When $N = 1$, a repelling periodic point is automatically J-repelling, counterexamples appear when $N \geq 2$ [HP94, Proposition 6.4].

Let $s \in \Lambda$ and let x be a J-repelling periodic point of f_s. Let $U \subset \Lambda$ be an open set containing s. We say that x *moves holomorphically* on U if there exists a holomorphic map $p : U \to \mathbb{P}^N(\mathbb{C})$ such that:

(1) $p(s) = x$;
(2) For every $t \in U$, $p(t)$ is a J-repelling periodic point of f_t.

4.2. **The equilibrium web and equilibrium graph.** In [BBD18], Berteloot–Bianchi–Dupont introduced a key notion called *equilibrium web*, which will be very useful for our purpose. We now give the definition.

Let $U \subset \Lambda$ be an open set. Let

$$\mathcal{G} := \left\{\gamma : U \to \mathbb{P}^N : \gamma \text{ is holomorphic}\right\}$$

be the set of graphs over U. We endow $\mathcal{G}$ with the topology of local uniform convergence. The family f induces a natural continuous map $\mathcal{F} : \mathcal{G} \to \mathcal{G}$ given by $(\mathcal{F}\gamma)(t) := f_t(\gamma(t))$. For each $t \in U$, there is a canonical continuous projection π_t from $\mathcal{G}$ to $\mathbb{P}^N(\mathbb{C})$ given by $\pi_t(\gamma) := \gamma(t)$.

In [BBD18], the authors considered the following closed subset of $\mathcal{G}$: let

$$\mathcal{J} := \left\{\gamma : U \to \mathbb{P}^N : \gamma \text{ is holomorphic and } \gamma(t) \in J_t \text{ for every } t \in U\right\}.$$

Definition 4.2. An equilibrium web (over U) is a probability measure σ on $\mathcal{J}$ such that:
(1) σ is $\mathcal{F}$-invariant and has compact support;
(2) For each $t \in U$, the probability measure $(\pi_t)_*\sigma$ is equal to μ_t.

Note that the assumption that σ is $\mathcal{F}$-invariant and has compact support implies that for every $\gamma \in \operatorname{supp}\sigma$, the orbit $\{\mathcal{F}^n(\gamma)\}_{n\geq 0}$ is contained in a compact subset of $\mathcal{G}$. Hence $\{f^n|_\Gamma\}_{n\geq 1}$ is a normal family, where Γ is the graph of γ; equivalently, $[\Gamma] \wedge T_f = 0$ by [GV25, Remark 2.13].

Next we introduce the notion of *equilibrium graph*. We will show that the existence of equilibrium graph implies the existence of equilibrium web.

Let g be an endomorphism of $\mathbb{P}^N$ over $\mathbb{C}$ of degree $d \geq 2$. A point $x \in \mathbb{P}^N(\mathbb{C})$ is called *typical* if the Birkhoff average

$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{g^i(x)} \to \mu_g$$

in weak-* topology, where δ_y is the Dirac mass at y and μ_g is the maximal entropy measure of g.

Since μ_g is ergodic [DS10, Theorem 1.82], by Birkhoff ergodic theorem, μ_g-a.e. points are typical for g.

Definition 4.3. An equilibrium graph (over U) is a holomorphic map $\gamma : U \to \mathbb{P}^N(\mathbb{C})$ such that:

(1) There is a dense subset $E \subset U$ such that for every $t \in E$, $\gamma(t)$ is a typical point of f_t;

(2) $[\Gamma] \wedge T_f = 0$, where $\Gamma \subset U \times \mathbb{P}^N(\mathbb{C})$ is the graph of γ.

Lemma 4.4. *Let f be a holomorphic family of endomorphisms of $\mathbb{P}^N(\mathbb{C})$ of degree $d \geq 2$, parametrized by a Hermitian manifold Λ. Let $U \subset \Lambda$ be an open subset. If there exists an equilibrium graph γ over U, then there exists an equilibrium web σ over U.*

Proof. For each $n \geq 0$, let $\gamma_n : U \to \mathbb{P}^N(\mathbb{C})$ be the holomorphic map defined by $\gamma_n(t) := f_t^n(\gamma(t))$. We define a probability measure σ_n on $\mathcal{G}$:

$$\sigma_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\gamma_i}.$$

It is well known that $[\Gamma] \wedge T_f = 0$ on U implies that $\{\gamma_n\}_{n \geq 1}$ is a normal family over U, see [GV25, Remark 2.13]. This implies that there is a compact subset $\mathcal{K} \subset \mathcal{G}$ such that $\operatorname{supp} \sigma_n \subset \mathcal{K}$ for every $n \geq 1$. After passing to a subsequence, we may assume that σ_n converges to a probability measure σ supported on $\mathcal{K}$. Moreover σ is $\mathcal{F}$-invariant. We will show that σ is an equilibrium web over U. It remains to show that for every $t \in U$ we have

(1) $(\pi_t)_*(\sigma) = \mu_t$;

(2) For every $\delta \in \operatorname{supp} \sigma$, $\pi_t(\delta) \in \mathcal{J}_t$.

For every $t \in U$ and $n \geq 1$, we have $(\pi_t)_*(\sigma_n) = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{\gamma_i(t)}$. Since π_t is continuous, $\sigma_n \to \sigma$ implies that $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{\gamma_i(t)} \to (\pi_t)_*(\sigma)$. When $t \in E$, we have $\frac{1}{n} \sum_{i=0}^{n-1} \delta_{\gamma_i(t)} \to \mu_t$. This implies $(\pi_t)_*(\sigma) = \mu_t$ when $t \in E$. Since μ_t is the intersection of T_f^N with the fiber of t, the fact that T_f has continuous local potentials implies that the map $t \mapsto \mu_t$ from U to $\mathcal{P}(\mathbb{P}^N(\mathbb{C}))$ is continuous, where $\mathcal{P}(\mathbb{P}^N(\mathbb{C}))$ denotes the space of probability measures on $\mathbb{P}^N(\mathbb{C})$ equipped with the weak-* topology. On the other hand, since σ has compact support, the map $t \mapsto (\pi_t)_*(\sigma)$ from U to $\mathcal{P}(\mathbb{P}^N(\mathbb{C}))$ is also continuous. Since E is dense in U, and these two maps coincide on E, the two maps are equal. This proves (1).

Since for every $t \in U$, $\pi_t : \operatorname{supp} \sigma \to \mathbb{P}^N(\mathbb{C})$ is continuous and $(\pi_t)_*(\sigma) = \mu_t$, we have $\pi_t(\operatorname{supp} \sigma) = \operatorname{supp}(\mu_t) = \mathcal{J}_t$. This proves (2), and the proof is complete. □

Remark 4.5. The converse, namely that the existence of an equilibrium web implies the existence of an equilibrium graph, is also true. Since we do not need this direction, we leave the proof to the reader.

Let f be a holomorphic family of endomorphisms of $\mathbb{P}^N(\mathbb{C})$ of degree $d \geq 2$, parametrized by a connected Hermitian manifold Λ. We have the following result in [Ber18, Main Theorem]:

Lemma 4.6. *Let $s \in \Lambda$. Assume that there exists an open set U containing s such that, for every $n \geq 1$, there is a subset P_n of n-periodic J-repelling points of f_s satisfying $\limsup_{n\to+\infty} \frac{\#P_n}{d^{Nn}} > 0$, and every point $x \in P_n$ moves holomorphically over U. Then $s \in \mathrm{Stab}(f)$.*

Let $U \subset \Lambda$ be a connected open subset and assume there exists an equilibrium web σ over U. Let $s \in U$ and let x be a J-repelling n-periodic point of f_s.

Lemma 4.7. *There exists a unique holomorphic map $\gamma : U \to \mathbb{P}^N(\mathbb{C})$ such that $\gamma \in \operatorname{supp}\sigma$ and $\gamma(s) = x$. Moreover, $\gamma(t)$ is a n-periodic point of f_t for each $t \in U$.*

Proof. By the definition of equilibrium web, there exists a holomorphic map $\gamma : U \to \mathbb{P}^N(\mathbb{C})$ such that $\gamma \in \operatorname{supp}\sigma$ and $\gamma(s) = x$. In particular $\{f^n|_\Gamma\}_{n\geq 1}$ is a normal family over U, where Γ is the graph of γ. On the other hand, since x is a repelling n-periodic point of f_s, the implicit function theorem applied to $f_t^n(z) = z$ gives a small neighborhood $U' \subset U$ of s and a unique holomorphic map $\tilde{\gamma} : U' \to \mathbb{P}^N(\mathbb{C})$ such that $\tilde{\gamma}(s) = x$ and $\tilde{\gamma}(t)$ is a repelling n-periodic point of f_t for every $t \in U'$. Thus γ coincides with $\tilde{\gamma}$ on U'. Since U is connected, this proves the uniqueness of γ. Since $\gamma(t)$ is an n-periodic point of f_t for each $t \in U'$ and U is connected, this property holds for $t \in U$ as well. This completes the proof. □

We have the following result [BBD18, Proposition 5.6]:

Lemma 4.8. *Let γ be the holomorphic map given in Lemma 4.7, and let Γ be the graph of γ. Assume that the resonant points and the non-diagonalizable points in Γ form proper analytic subsets of Γ. Then x moves holomorphically on U, i.e. all periodic points in Γ are J-repelling.*

As a corollary of Lemma 4.8 and Lemma 4.6, we have:

Proposition 4.9. *Let f be a holomorphic family of endomorphisms of $\mathbb{P}^N(\mathbb{C})$ of degree $d \geq 2$, parametrized by a Hermitian manifold Λ.*

When $N \geq 3$, assume in addition that f is periodically generic. Let $s \in \Lambda$. Then the following are equivalent:

(1) $s \in \mathrm{Stab}(f)$;

(2) There exists an open set U containing s and an equilibrium web over U.

Proof. If $N \leq 2$, this is [BBD18, Theorem 1.1]. If $N \geq 3$, (1) implies (2) was proved in [BBD18, Theorem 1.6]. It remains to show (2) implies (1) when $N \geq 3$. Let R_n be the set of J-repelling n-periodic points of f_s. For each $x \in R_n$, we let $\gamma_x : U \to \mathbb{P}^N(\mathbb{C})$ be the unique holomorphic map given in Lemma 4.7. Let Γ_x be the graph of γ_x. We define a subset $P_n \subset R_n$ as follows: a point $x \in R_n$ belongs to P_n if the points in Γ_x that are resonant or non-diagonalizable form proper analytic subsets of Γ_x. Since f is periodically generic and $\lim_{n\to+\infty} \frac{\#R_n}{d^{nN}} = 1$, we have:

(1)

$$\limsup_{n\to+\infty} \frac{\#P_n}{d^{Nn}} > 0.$$

By Lemma 4.8, we have:

(2) For every $x \in P_n$, x moves holomorphically on U.

Since (1) and (2) hold simultaneously, by Lemma 4.6, we have $s \in \mathrm{Stab}(f)$, which finishes the proof. □

4.3. **Uniformly woven currents and bifurcation theory.** Let f be a holomorphic family of endomorphisms of $\mathbb{P}^N(\mathbb{C})$ of degree $d \geq 2$, parametrized by the unit disk $\mathbb{D}$. For every $t \in \mathbb{D}$, let R_t be the current of integration on the fiber $\pi_1^{-1}(t)$. Let $\Omega \subset \mathbb{D} \times \mathbb{P}^N(\mathbb{C})$ be an open subset.

Theorem 4.10. *Let f be a holomorphic family of endomorphisms of $\mathbb{P}^N(\mathbb{C})$ of degree $d \geq 2$, parametrized by the unit disk $\mathbb{D}$. If $N \geq 3$, assume in addition that f is periodically generic. Let S be a bidimension $(1,1)$ uniformly woven current on Ω such that*

(1) $S \wedge T_f = 0$;

(2) For every $t \in \mathbb{D}$ such that $S \wedge R_t$ is admissible, $S \wedge R_t \leq \mu_t$, where μ_t is the maximal entropy measure of f_t;

(3) There exists $s \in \mathbb{D}$ such that $S \wedge R_s$ is admissible and $S \wedge R_s > 0$.

Then $s \in \mathrm{Stab}(f)$.

Proof. We write S as

$$S = \int_{\mathcal{V}} [\Gamma_a]\, d\nu(a),$$

where $\mathcal{V}$ is a bounded volume space, and ν is a finite measure on $\mathcal{V}$, see Definition 2.6.

Let $\Omega' \subset \Omega$ be a relatively compact open subset such that the restriction $S' := S|_{\Omega'}$ still satisfies the three conditions in Theorem 4.10. Let $K := \overline{\Omega'}$, which is compact.

Since we only consider curves in this paragraph, write $\mathcal{V}_K^* := \mathcal{V}_{1,K}^*$. By Proposition 2.7, there exists a finite measure ν_1 on $\mathcal{V}_K^*$ such that

$$S' = \int_{\mathcal{V}_K^*} [\Gamma_a] \, d\nu_1(a).$$

Since ν_1 is a finite measure on a locally compact separable metrizable space, ν_1 is a Radon measure. Replacing ν_1 by its restriction to a compact subset of $\mathcal{V}_K^*$ of positive ν_1-measure, chosen so that $S' \wedge R_s$ remains non-zero, we may assume ν_1 has compact support in $\mathcal{V}_K^*$ and S' still satisfies the three conditions in Theorem 4.10. Now for each $a \in \operatorname{supp} \nu_1$, Γ_a is a closed irreducible analytic curve in Ω'.

Since $S' \wedge R_s > 0$, there is a positive ν_1-measure subset $\mathcal{W} \subset \mathcal{V}_K^*$ such that for every $a \in \mathcal{W}$, Γ_a intersects $\pi_1^{-1}(s)$ properly. Pick $a \in \mathcal{W}$. Let $D \subset \mathbb{D}$ be a small disk containing s such that for every $t \in D$, Γ_a intersects $\pi_1^{-1}(t)$ properly. A parameter $t \in D$ is called *bad* if either there exists a singular point $x \in \Gamma_a$ such that $\pi_1(x) = t$, or there exists a ramification point x of the map $\pi_1|_{\Gamma_a} : \Gamma_a \to D$ such that $\pi_1(x) = t$. Bad parameters form a finite set in D.

Assume by contradiction that $s \in \operatorname{Bif}(f)$. Since $\operatorname{Bif}(f)$ is the support of the bifurcation measure, which has continuous local potentials, $\operatorname{Bif}(f)$ has no isolated point. Thus there exists $s' \in \operatorname{Bif}(f) \cap D$ such that s' is a good parameter, i.e. Γ_a intersects $\pi_1^{-1}(s')$ transversely. By the stability of transverse intersections, replacing D by a smaller disk centered at s', we can choose $\mathcal{B}$ to be a small neighborhood of a such that for every $b \in \mathcal{B}$ and every $t \in D$, Γ_b intersects $\pi_1^{-1}(t)$ transversely. By shrinking Ω', we may further assume the intersection $\Gamma_b \cap \pi_1^{-1}(t)$ is always unique for $b \in \mathcal{B}$ and $t \in D$. Replacing ν_1 by its restriction to $\mathcal{B}$ and replacing s by s', the new uniformly woven current S' and the point s' still satisfy the three conditions in Theorem 4.10. Moreover $S' \wedge R_t > 0$ is admissible for every $t \in D$. For each $a \in \operatorname{supp} \nu_1$, Γ_a is the graph of a holomorphic map $\gamma_a : D \to \mathbb{P}^N(\mathbb{C})$.

Let E be a countable dense subset of D. For every $t \in E$, the set of typical points for f_t has full μ_t measure. By condition (2), $0 < S' \wedge R_t \leq \mu_t$. Hence there exists a full ν_1-measure subset $Y_t \subset \mathcal{V}_K^*$ such that for every $a \in Y_t$, $\gamma_a(t)$ is a typical point for f_t. Since E is countable, there exists a full ν_1-measure subset $Y \subset \mathcal{V}_K^*$ such that for every $a \in Y$ and $t \in E$, $\gamma_a(t)$ is a typical point for f_t. By Proposition 2.11 and condition (1), there exists a full ν_1-measure subset $Z \subset \mathcal{V}_K^*$ such that $[\Gamma_a] \wedge T_f = 0$ for every $a \in Z$. We pick a $\Gamma \in Y \cap Z$. Let

$\gamma : D \to \mathbb{P}^N(\mathbb{C})$ be the holomorphic map such that Γ is the graph of γ. Then γ is an equilibrium graph. By Lemma 4.4 and Proposition 4.9, $s' \in \mathrm{Stab}(f)$, which contradicts $s' \in \mathrm{Bif} \cap D$. This completes the proof. □

5. Review of slicing theory

Let X be a Hermitian manifold of dimension m and (B, ω_B) be a Hermitian manifold of dimension l. Let $\pi : X \to B$ be a surjective holomorphic map. Let $\pi' : X \times \mathbb{P}^N(\mathbb{C}) \to B$ be the surjective holomorphic map given by $\pi'(t, x) := \pi(t)$. We recall Federer's slicing theory [Fed69, Section 4.3] applied in our setting, see also Dinh-Sibony [DS10, Page 280]. Let y denote the coordinates in a chart of B. Let S be a positive closed bidimension (m, m) current on $X \times \mathbb{P}^N(\mathbb{C})$. Let ψ be a positive smooth function on this chart with compact support such that $\int \psi\, \omega_B^l = 1$. Define $\psi_\varepsilon(y) := \varepsilon^{-2l}\psi(\varepsilon^{-1}y)$ and $\psi_{\theta,\varepsilon}(y) := \psi_\varepsilon(y-\theta)$. The measure $\psi_{\theta,\varepsilon}\omega_B^l$ converges in the weak-* sense to the Dirac mass at θ. We say that the *slicing* S_θ at θ with respect to π' is well defined as a positive closed bidimension $(m-l, m-l)$ current on $X \times \mathbb{P}^N(\mathbb{C})$, if for every function ψ satisfying above properties and every smooth test form Φ of bidegree $(m-l, m-l)$ on $X \times \mathbb{P}^N(\mathbb{C})$, we have

$$\langle S_\theta, \Phi\rangle = \lim_{\varepsilon \to 0} \left\langle S \wedge (\pi')^*(\psi_{\theta,\varepsilon}\omega_B^l), \Phi \right\rangle.$$

By slicing theory, S_θ is well defined for Lebesgue-a.e. point $\theta \in B$. Slicing satisfies the following two properties.

(1) Assume the slicing S_θ is well defined at θ. If Ω is a smooth form on $X \times \mathbb{P}^N(\mathbb{C})$, then the slicing $(S \wedge \Omega)_\theta$ is also well defined and equals $S_\theta \wedge \Omega_\theta$ as a current on $(\pi')^{-1}(\theta) = \pi^{-1}(\theta) \times \mathbb{P}^N(\mathbb{C})$, where Ω_θ is the restriction of Ω to $\pi^{-1}(\theta) \times \mathbb{P}^N(\mathbb{C})$.

(2) Let ρ be a smooth probability measure on B. Then

$$S \wedge (\pi')^*\rho = \int S_\theta \, d\rho(\theta).$$

6. Background on height theory

Throughout this subsection, we follow the language and conventions of [YZ26]. Let F be a number field. For any quasi-projective variety V over F and $v \in M_F$, we denote by V_v^{an} the analytification of V over F_v. When v is archimedean, V_v^{an} is the complex analytic variety $V(\mathbb{C})$ with the euclidean topology induced by v. When v is non-archimedean, we use Berkovich's analytification. In our paper, we will only concern the archimedean case.

Let $\pi : U \to X$ be a flat and projective morphism of relative dimension n of quasi-projective varieties over F. Set $m := \dim X + 1$ and $K = F(X)$. Let $Y \to \operatorname{Spec} K$ be the generic fiber of π.

6.1. **Moriwaki height and geometric height.** Let $\overline{H}$ be any element in $\widehat{\operatorname{Pic}}(K/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$. For any point $x \in Y(\overline{K})$ and $\overline{L} \in \widehat{\operatorname{Pic}}(U/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$, define the *Moriwaki height* of x for $\overline{L}$ and $\overline{H}$ by

$$h_{\overline{L}}^{\overline{H}}(x) := h_{\overline{L}}^{\overline{H}}(x') := \frac{\overline{L}|_{x'} \cdot \overline{H}^{m-1}}{\deg(x')} \in \mathbb{R}.$$

Here x' is the scheme theoretic image of x. This gives a height function

$$h_{\overline{L}}^{\overline{H}} : Y(\overline{K}) \longrightarrow \mathbb{R}.$$

More generally, for any closed $\overline{K}$-subvariety Z of Y, the *Moriwaki height* of Z for $\widetilde{L}$ and $\widetilde{H}$ is

$$h_{\overline{L}}^{\overline{H}}(Z) := h_{\overline{L}}^{\overline{H}}(Z') := \frac{\left(\overline{L}|_{Z'}\right)^{\dim Z+1} \cdot \overline{H}^{m-1}}{(\dim Z + 1)\deg_{\widetilde{L}}(Z'/K)}.$$

Here Z' is the scheme theoretic image of Z.

Next we recall the geometric height. Let $\widetilde{H}$ be the image of $\overline{H}$ in $\widehat{\operatorname{Pic}}(X/F)_{\mathrm{nef}}$ and $\widetilde{L}$ be the image of $\overline{L}$ in $\widehat{\operatorname{Pic}}(U/F)_{\mathrm{nef}}$. We define the *geometric height* of x for $\widetilde{L}$ and $\widetilde{H}$ by

$$h_{\widetilde{L}}^{\widetilde{H}}(x) := h_{\widetilde{L}}^{\widetilde{H}}(x') := \frac{\widetilde{L}|_{x'} \cdot \widetilde{H}^{m-2}}{\deg(x')} \in \mathbb{R}.$$

Here x' is the scheme theoretic image of x. This gives a geometric height function

$$h_{\widetilde{L}}^{\widetilde{H}} : Y(\overline{K}) \longrightarrow \mathbb{R}.$$

More generally, for any closed $\overline{K}$-subvariety Z of Y, the *geometric height* of Z for $\overline{L}$ and $\overline{H}$ is

$$h_{\widetilde{L}}^{\widetilde{H}}(Z) := h_{\widetilde{L}}^{\widetilde{H}}(Z') := \frac{\left(\widetilde{L}|_{Z'}\right)^{\dim Z+1} \cdot \widetilde{H}^{m-2}}{(\dim Z + 1)\deg_{\widetilde{L}}(Z'/K)}.$$

Here Z' is the scheme theoretic image of Z.

We have the following comparison between the Moriwaki and the geometric heights.

Proposition 6.1. *Assume that $\overline{H}$ is big and nef. There exists a constant $C > 0$ such that for every $\overline{K}$-subvariety Z, we have*

$$h_{\widetilde{L}}^{\widetilde{H}}(Z) \leq C h_{\overline{L}}^{\overline{H}}(Z).$$

This result is well-known for experts. A proof can be found in [JSX26, Proposition 2.7]. In [JSX26, Proposition 2.7], $\pi : U \to X$ is an abelian scheme, however its proof works in general.

The following result gives an analytic interpretation of the geometric height. The version we need was proved by Yuan-Zhang in [YZ26, Lemma 5.4.4], see also [GV25, Guo25]. In our paper, we only need the case where v is archimedean.

Theorem 6.2. *For any place v of F, We have*

$$h_{\widetilde{L}}^{\widetilde{H}}(Z) = \frac{\int_{(Z')_v^{\mathrm{an}}} c_1(\overline{L})_v^{\dim Z+1} \wedge c_1(\overline{H})_v^{m-2}}{(\dim Z + 1)\deg_{\widetilde{L}}(Z'/K)}.$$

6.2. **The relative equidistribution theorem.** A sequence of points $x_i, i \geq 0$ in $Y(\overline{K})$ is called *generic* if $x_i, i \geq 0$ tends to the generic point of X in the Zariski topology. We recall Yuan-Zhang's relative equidistribution theorem.

Theorem 6.3. [Mor00b, Theorem 6.1][YZ26, Theorem 5.4.6][CM24] *Let $\overline{H}$ be an element of $\widehat{\mathrm{Pic}}(X/\mathbb{Z})_{\mathrm{nef}}$ that $\overline{H}$ is nef and $\widetilde{H}^{m-1} > 0$. Here $\widetilde{H}$ is the image of $\overline{H}$ in $\widehat{\mathrm{Pic}}(X/F)_{\mathrm{nef}}$. Let $\overline{L}$ be an element of $\widehat{\mathrm{Pic}}(U/\mathbb{Z})_{\mathrm{nef}}$ such that $\deg_{\widetilde{L}}(Y/K) > 0$. Here $\widetilde{L}$ is the image of $\overline{L}$ under the canonical composition*

$$\widehat{\mathrm{Pic}}(U/\mathbb{Z})_{\mathrm{nef}} \longrightarrow \widehat{\mathrm{Pic}}(Y/\mathbb{Z})_{\mathrm{nef}} \longrightarrow \widehat{\mathrm{Pic}}(Y/F)_{\mathrm{nef}}.$$

Let $x_i, i \geq 0$ be a generic sequence in $Y(\overline{K})$ such that $h_{\overline{L}}^{\overline{H}}(x_i)$ converges to $h_{\overline{L}}^{\overline{H}}(Y)$. Then for any place v of F, there is a weak convergence

$$\frac{1}{\deg(x_i)}\delta_{\Delta(x_i),v} c_1(\pi^*\overline{H})_v^{m-1} \longrightarrow \frac{1}{\deg_{\widetilde{L}}(Y/F)} c_1(\overline{L})_v^n c_1(\pi^*\overline{H})_v^{m-1}$$

of measures on U_v^{an}. Here $\Delta(x_i) \subset U$ denotes the Zariski closure of the image of x_i in U, and $\delta_{\Delta(x_i),v}$ denotes the Dirac current of $\Delta(x_i)_{K_v}^{\mathrm{an}}$ in U_v^{an}.

Remark 6.4. The equidistribution theorem in this relative setting was first proved in [Mor00b, Theorem 6.1] under an additional hypothesis called *Moriwaki condition* i.e. $\overline{H}^m = 0$. A more general result for quasi-projective varieties was proved by Yuan–Zhang in [YZ26, Theorem 5.4.6]. Chen–Moriwaki removed the Moriwaki condition in [CM24, Theorem F], proving a more general form of the theorem in the setting of adelic curves. In this article, we only require the above special case, corresponding to a polarized adelic structure in the sense of Chen–Moriwaki.

The following definition is given by Yuan-Zhang in [YZ26, Section 5.4.1].

Definition 6.5. The sequence $x_i, i \geq 0$ is called *small* if for any $\overline{H} \in \widehat{\mathrm{Pic}}(K/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$, $h_{\overline{L}}^{\overline{H}}(x_i)$ converges to $h_{\overline{L}}^{\overline{H}}(Y)$.

6.3. **Polarized dynamical systems.** Let X be a quasi-projective variety over F. Let $K := F(X)$. Let $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ be an endomorphism of algebraic degree $d \geq 2$. After shrinking X, we may extend f to a finite endomorphism $X \times \mathbb{P}^N \to X \times \mathbb{P}^N$, we still denote it by f. Let $\pi_1 : X \times \mathbb{P}^N \to X$ and $\pi_2 : X \times \mathbb{P}^N \to \mathbb{P}^N$ be the first and the second projections. Let $L_2 := \pi_2^*(O_{\mathbb{P}^N}(1))$. After shrinking X, we may assume that $f^*L_2 = dL_2$.

The triple $(\pi_1 : X \times \mathbb{P}^N \to X, f, L_2)$ is a *d-polarized dynamical system over X* in the sense of Yuan-Zhang [YZ26]. In [YZ26, Theorem 6.1.1], by Tate's limiting argument, Yuan-Zhang constructed an adelic line bundle $\overline{L_{2f}} \in \widehat{\mathrm{Pic}}(X \times \mathbb{P}^N/F)_{\mathrm{nef}}$ extending L_2 such that $f^*\overline{L_{2f}} = d\overline{L_{2f}}$.

Let V be an irreducible closed subvariety of $X \times \mathbb{P}^N$ over F (resp. $\overline{F}$) of dimension $m = \dim X$. We call V *horizontal* if $\pi_1 : V \to X$ (resp. $\pi_1 : V \to X_{\overline{F}}$) is generically finite and surjective. The *degree* of V is by definition the topological degree of $\pi_1|_V$. Let $x_V \in \mathbb{P}^N(\overline{K})$ be any geometric generic point of V.

Definition 6.6 (Canonical height)**.** The *canonical height* of a horizontal subvariety V is defined as

$$\hat{h}_f^{\overline{H}}(V) := h_{\overline{L_{2f}}}^{\overline{H}}(x_V).$$

Definition 6.7 (Small horizontal subvarieties)**.** A sequence of horizontal subvarieties $V_n, n \geq 0$ (over F or $\overline{F}$) is called *small for f* if for any $\overline{H} \in \widehat{\mathrm{Pic}}(K/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$, we have $\hat{h}_f^{\overline{H}}(V_n) \to 0$ as $n \to \infty$.

Now we fix an embedding $F \hookrightarrow \mathbb{C}$. It gives an archimedean place $v \in M_F$. Let T_f be the relative Green current of $f : (X \times \mathbb{P}^N)(\mathbb{C}) \to (X \times \mathbb{P}^N)(\mathbb{C})$, see Section 4. We have $T_f = c_1(\overline{L_{2f}})_v$.

Let V be a horizontal subvariety over F or $\overline{F}$. Let x_V be any geometric generic point of V. Let $\overline{H} \in \widehat{\mathrm{Pic}}(K/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$.

Definition 6.8 (Canonical geometric height)**.** The *canonical geometric height* of V is defined as

$$\hat{h}_{\mathrm{geom}}^{\widetilde{H}}(V) := h_{\widetilde{L_{2f}}}^{\widetilde{H}}(x_V).$$

After shrinking X, assume that $\overline{H} \in \widehat{\mathrm{Pic}}(X/F)_{\mathrm{nef}}$. Let $\omega_X := c_1(\overline{H})_v$. It is a positive closed $(1,1)$-current on $\overline{X}(\mathbb{C})$ with continuous potential. Set $\omega_1 := \pi_1^*\omega_X$. By Theorem 6.2, we have

Corollary 6.9. *The following holds:*

$$\hat{h}^{\widetilde{H}}_{\mathrm{geom}}(V) = \frac{1}{\deg V}\int_{X\times\mathbb{P}^N(\mathbb{C})} [V(\mathbb{C})]\wedge T_f \wedge \omega_1^{m-1}.$$

6.4. **Adelic bifurcation classes and stable families.** We recall here an algebro-geometric interpretation of stability in terms of the geometric part of the bifurcation adelic line bundle. We keep the notation and terminology of Yuan-Zhang [YZ26]; in particular, we use adelic line bundles, their geometric parts, and their intersection theory without recalling the general construction.

Let $\mathcal{M}_{d,N}$ be the moduli space of degree d endomorphisms of $\mathbb{P}^N$. On the parameter space $\mathrm{End}_{d,N}$ there is a tautological polarized dynamical system; the bifurcation adelic line bundle obtained from it by Deligne pairing is PGL_{N+1}-invariant and descends to a nef adelic line bundle

$$\overline{L}_{\mathrm{bif}} \in \widehat{\mathrm{Pic}}(\mathcal{M}_{d,N}/\mathbb{C})_{\mathrm{nef},\mathbb{Q}}$$

whose archimedean curvature is the bifurcation current on $\mathcal{M}_{d,N}(\mathbb{C})$. Let $\widetilde{L}_{\mathrm{bif}}$ be the geometric part of $\overline{L}_{\mathrm{bif}}$.

We will use the following facts: $\widetilde{L}_{\mathrm{bif}}$ is nef in the sense of Yuan-Zhang, and the support of the archimedean curvature current of $\overline{L}_{\mathrm{bif}}$ is the bifurcation locus. Moreover, by Yuan-Zhang [YZ26, Theorem 6.3.6 and Lemma 5.4.4], the top self-intersection of $\widetilde{L}_{\mathrm{bif}}$ is the total mass of the bifurcation measure; see also Guo [Guo25, Theorem 1.2] for a local integration formula in the quasi-projective setting. This mass is positive by [GTV26, Proposition 2.8]. Hence $\widetilde{L}_{\mathrm{bif}}$ is big. For an irreducible algebraic curve $C \subset \mathcal{M}_{d,N}$, the adelic intersection number

$$\deg_{\widetilde{L}_{\mathrm{bif}}}(C) := \widetilde{L}_{\mathrm{bif}} \cdot C$$

is well-defined. If a one-parameter algebraic family F lies over C, in the sense that the image of its moduli map is a Zariski open subset of C, then F is stable if and only if $\deg_{\widetilde{L}_{\mathrm{bif}}}(C) = 0$. Indeed, stability is equivalent to the vanishing of the pullback bifurcation current on C, and by Yuan–Zhang [YZ26, Theorem 6.3.6 and Lemma 5.4.4] (or Guo [Guo25, Theorem 1.2]) this degree is the total mass of that positive current.

Proposition 6.10. *There exists a proper Zariski closed subset $Z \subset \mathcal{M}_{d,N}$ such that if $C \subset \mathcal{M}_{d,N}$ is an irreducible algebraic curve not*

contained in Z, then every non-isotrivial one-parameter algebraic family lying over C has non-empty bifurcation locus. In particular, every non-isotrivial one-parameter algebraic family whose moduli curve passes through a general point of $\mathcal{M}_{d,N}$ is not stable.

Proof. Since $\widetilde{L}_{\mathrm{bif}}$ is big, by the definition of bigness for adelic line bundles on quasi-projective varieties, after passing to a projective model $\rho : \mathcal{M}' \to \mathcal{M}_{d,N}$ we may choose an ample $\mathbb{Q}$-line bundle A on $\mathcal{M}'$ and an effective geometric adelic class E such that $\widetilde{L}_{\mathrm{bif}} \geq A + E$ in the sense of Yuan–Zhang. Choose an effective representative D of E on a projective model dominating $\mathcal{M}'$, and let $Z \subset \mathcal{M}_{d,N}$ be the image of Supp (D) together with the corresponding exceptional locus. Then Z is contained in a proper Zariski closed subset of $\mathcal{M}_{d,N}$; replacing Z by this closed subset, we may assume that Z is proper and Zariski closed.

Let $C \subset \mathcal{M}_{d,N}$ be an irreducible algebraic curve not contained in Z, and let C' be its strict transform on the model carrying D. Then C' is not contained in Supp (D), hence $D \cdot C' \geq 0$. Since the pullback of A to this model is still big and nef and has positive degree on C', we have $A \cdot C' > 0$. Therefore

$$\deg_{\widetilde{L}_{\mathrm{bif}}}(C) = \widetilde{L}_{\mathrm{bif}} \cdot C \geq A \cdot C' > 0.$$

Thus the bifurcation degree of C is positive. If a one-parameter algebraic family lies over C, the bifurcation current restricted to the base has positive mass, so the family has non-empty bifurcation locus. This proves the proposition. □

In dimension $N = 1$, McMullen's rigidity theorem [McM87] implies that the only non-isotrivial stable families of rational maps are the flexible Lattès families. It is natural to ask:

Question 6.11. Classify the irreducible algebraic curves $C \subset \mathcal{M}_{d,N}$ such that $\deg_{\widetilde{L}_{\mathrm{bif}}}(C) = 0$, or equivalently, such that one-parameter algebraic families lying over C are stable. More generally, which positive-dimensional subvarieties of $\mathcal{M}_{d,N}$ support stable algebraic families?

7. Genus and gonality of small curves: proof of Theorem 1.2 and 1.4

Let f be a one-parameter algebraic family of endomorphisms of $\mathbb{P}^N$ of algebraic degree $d \geq 2$ over $\mathbb{C}$, parametrized by a smooth quasi-projective curve Λ. Let $\overline{\Lambda}$ be a smooth projective compactification of Λ. Let K be a subfield of $\mathbb{C}$ which is finitely generated over $\mathbb{Q}$ such that f is over K.

Let $m-1$ be the transcendental degree of the field extension $K/\mathbb{Q}$, $m \geq 1$. There exist a number field F, a smooth irreducible projective variety $\overline{B}$ defined over F of dimension $m-1$, a surjective flat morphism $\pi : \overline{X} \to \overline{B}$ between smooth projective varieties over F with relative dimension 1, and $\theta_0 \in \overline{B}(\mathbb{C})$ such that the specialization of $\overline{X}_{\mathbb{C}}$ at θ_0 is $\overline{\Lambda}$. Moreover θ_0 is *fully transcendental* with respect to $\overline{B}/F$ in the sense that the image of the generic point of $\operatorname{Spec}\mathbb{C}$ under

$$\operatorname{Spec}\mathbb{C} \xrightarrow{\theta_0} \overline{B}_{\mathbb{C}} \to \overline{B}$$

is the generic point of $\overline{B}$. Let $\pi' : \overline{X} \times \mathbb{P}^N \to \overline{B}$ be the flat morphism given by $\pi'(t,x) := \pi(t)$.

The set of non fully-transcendental points in $\overline{B}(\mathbb{C})$ is contained in a countable union of proper subvarieties, in particular it has Lebesgue measure zero.

The generic fiber of π is connected and smooth. Let B be a Zariski open subset of $\overline{B}$ such that $\theta_0 \in B(\mathbb{C})$, and for every $\theta \in B(\mathbb{C})$, $\pi^{-1}(\theta)$ is connected and smooth.

By abuse of notation, we let f be the family of endomorphisms of $\mathbb{P}^N$ over F, parametrized by a Zariski open subset $X \subset \overline{X}$. Let $\pi_1 : \overline{X} \times \mathbb{P}^N \to \overline{X}$ and $\pi_2 : \overline{X} \times \mathbb{P}^N \to \mathbb{P}^N$ be the two projections. We have $\pi' = \pi \circ \pi_1$.

7.1. **Relative arithmetic equidistribution.** The composition of embedding $F \hookrightarrow K \hookrightarrow \mathbb{C}$ gives a place $v \in M_F$.

Let L_X be a very ample line bundle on $\overline{X}$. Let $\overline{L_X}$ be a big and nef adelic line bundle in $\widehat{\operatorname{Pic}}(X/F)_{\mathrm{nef}}$ whose underlying line bundle is L_X such that $\omega_X := c_1(\overline{L_X})_v$ is a Kähler form on $\overline{X}$. Let $L_2 := \pi_2^*(O_{\mathbb{P}^N}(1))$, $L_1 := \pi_1^* L_X$, $\omega_1 := \pi_1^*\omega_X$ and $\omega_2 := \pi_2^*\omega_{\mathbb{P}^N}$. Then $\omega := \omega_1 + \omega_2$ is a Kähler form on $\overline{X} \times \mathbb{P}^N$ in $c_1(L_1 + L_2)$. For every horizontal subvariety V, the volume of V with respect to ω is

$$\operatorname{vol}(V) := \int_{V(\mathbb{C})} \omega^m = (V \cdot (L_1 + L_2)^m).$$

Proposition 7.1. *There exists a constant $C > 0$ such that for every irreducible horizontal subvariety V (over F or $\overline{F}$), we have*

$$\operatorname{vol}(V) \geq \deg(V) \geq \frac{C}{(1 + \hat{h}^{L_X}_{\mathrm{geom}}(V))^m} \operatorname{vol}(V).$$

Proof. Only need to treat the case over F. Pick a smooth projective compactification Y of $X \times \mathbb{P}^N$ such that endomorphisms id and f on $X \times \mathbb{P}^N$ extend to morphisms $p_1, p_2 : Y \to \overline{X} \times \mathbb{P}^N$. Denote by $\phi := \pi_1 \circ p_1 : Y \to \overline{X}$. We have $p_2^* L_1 = p_1^* L_1$ and $p_2^* L_2 = d p_1^* L_2 + E$, where

E is a divisor on Y such that $\phi(\operatorname{supp} E) \neq \overline{X}$. As L_X is ample on $\overline{X}$, there is an effective $\mathbb{Q}$-divisor D on $\overline{X}$ representing L_X such that $\phi(\operatorname{supp} E) \subseteq \operatorname{supp} D$. There is $C_1 > 0$ such that $C_1\phi^* D + E$ is effective.

Let V be a horizontal subvariety. Denote by $\overline{V}$ the Zariski closure of V in $\overline{X} \times \mathbb{P}^N$ and by V' the Zariski closure of V in Y. Write

$$h_2(V) := (\overline{V} \cdot L_2 \cdot L_1^{m-1})/\deg V.$$

Tate's limiting argument shows that

$$\hat{h}^{L_X}_{\mathrm{geom}}(V) = \lim_{n\to\infty} h_2(f^n(V))/d^n. \tag{7.1}$$

Observe that

$$(\overline{V} \cdot L_1^m)/\deg V = \deg(L_X^m) \geq 1.$$

As L_2 is nef, we have

$$\operatorname{vol}(V) = (\overline{V} \cdot (L_1 + L_2)^m) \geq (\overline{V} \cdot L_1^m) \geq \deg V.$$

This proves the first inequality in Proposition 7.1.

Denote by f_*V the cycle

$$f_*V := \deg(f|_V : V \to f(V))f(V).$$

We have

$$\deg(f_*V) := \deg(f|_V : V \to f(V)) \deg(f(V)) = \deg(V).$$

We have

$$h_2(f(V)) = \deg(f_*V)^{-1}(\overline{f_*(V)} \cdot L_2 \cdot L_1^{m-1}) = \deg(V)^{-1}(V' \cdot p_2^* L_2 \cdot p_1^* L_1^{m-1}).$$

Since $C_1\phi^* D + E$ is effective, this gives

$$\begin{aligned} h_2(f(V)) &\geq \deg(V)^{-1}\big(V' \cdot (dp_1^* L_2 - C_1\phi^* D) \cdot p_1^* L_1^{m-1}\big) \\ &= dh_2(V) - C_1 \deg(V)^{-1}\big(V' \cdot \phi^* D \cdot p_1^* L_1^{m-1}\big). \end{aligned}$$

By the projection formula, the last intersection number equals

$$(\overline{V} \cdot \pi_1^* D \cdot L_1^{m-1}) = \deg(V)(D \cdot L_X^{m-1}),$$

hence

$$h_2(f(V)) \geq dh_2(V) - C_2$$

where $C_2 := C_1 \deg L_X^m$. It follows that

$$h_2(f(V)) - \frac{C_2}{d-1} \geq d\left(h_2(V) - \frac{C_2}{d-1}\right).$$

If $h_2(V) > \frac{C_2}{d-1}$, then

$$h_2(f^n(V)) \geq d^n\left(h_2(V) - \frac{C_2}{d-1}\right) + \frac{C_2}{d-1}.$$

By (7.1), we get

$$\hat{h}^{L_X}_{\mathrm{geom}}(V) \geq h_2(V) - \frac{C_2}{d-1}.$$

It follows that

$$h_2(V) \leq \frac{C_2}{d-1} + \hat{h}^{L_X}_{\mathrm{geom}}(V).$$

In other words,

$$(\overline{V} \cdot L_2 \cdot L_1^{m-1}) \leq \left(\frac{C_2}{d-1} + \hat{h}^{L_X}_{\mathrm{geom}}(V) \right) \deg V.$$

By the Khovanskii–Teissier inequality [Laz04, Theorem 1.6.1], applied after resolving the irreducible variety $\overline{V}$, the sequence $(\overline{V} \cdot L_2^i \cdot L_1^{m-i}), i = 0, \ldots, m$ is log-concave. So for every $i = 1, \ldots, m$, we have

$$(\overline{V} \cdot L_2^i \cdot L_1^{m-i}) \leq \frac{(\overline{V} \cdot L_2 \cdot L_1^{m-1})^i}{(\overline{V} \cdot L_1^m)^{i-1}} \leq \left(\frac{\frac{C_2}{d-1} + \hat{h}^{L_X}_{\mathrm{geom}}(V)}{(L_1^m)} \right)^i (L_1^m) \deg V.$$

Then we have

$$\mathrm{vol}(V) = (\overline{V} \cdot (L_1 + L_2)^m) \leq \left(1 + \frac{\frac{C_2}{d-1} + \hat{h}^{L_X}_{\mathrm{geom}}(V)}{(L_1^m)} \right)^m (L_1^m) \deg V.$$

This concludes the proof. □

A sequence of irreducible horizontal subvariety V_n is called *generic* if for any proper subvariety $H \subset \Lambda \times \mathbb{P}^N$, there are only finitely many V_n contained in H. If a horizontal subvariety V is over $\overline{k}$, we let $O(V)$ be the *Galois orbit* of V under $\mathrm{Gal}(\overline{k}/k)$, which is a (maybe reducible) horizontal subvariety. Whenever a numerical estimate above is applied to $O(V)$, it is applied componentwise to the irreducible Galois conjugates of V and then summed. The following result is a direct consequence of Theorem 6.2 in our setting.

Theorem 7.2 (Relative arithmetic equidistribution)**.** *In the above setting, let $(V_n)_{n\geq 1}$ be a generic sequence of irreducible horizontal subvarieties over $\overline{k}$ which is small for f, and let $\phi : (X \times \mathbb{P}^N)(\mathbb{C}) \to \mathbb{R}$ be a continuous function with compact support. Then*

$$\lim_{n\to+\infty} \int_{X\times\mathbb{P}^N(\mathbb{C})} \phi \, \frac{[O(V_n)] \wedge \omega_1^m}{\deg(O(V_n))} = \int_{X\times\mathbb{P}^N(\mathbb{C})} \phi \, T_f^N \wedge \omega_1^m.$$

7.2. **Genus and gonality of small curves.** Come back to the setting of Theorem 7.2. Let $(V_n)_{n\geq 1}$ be a generic sequence of irreducible horizontal subvarieties over $\overline{k}$ which is small for f.

Define

$$S_n := \frac{[O(V_n)]}{\deg(O(V_n))},$$

which is a positive closed current of bidimension (m,m) on $X\times\mathbb{P}^N(\mathbb{C})$. For every $s\in X(\mathbb{C})$, let R_s be the current of integration on the fiber $\pi_1^{-1}(s)$. For a positive closed current S of bidimension (m,m) on $\overline{X}\times\mathbb{P}^N(\mathbb{C})$, the slicing S_θ (once it exists) is a positive closed current of bidimension $(1,1)$ supported in $(\pi')^{-1}(\theta)=\pi^{-1}(\theta)\times\mathbb{P}^N(\mathbb{C})$. See Definition 2.9 for the meaning of admissibility of $S_\theta\wedge R_s$ when $s\in\pi^{-1}(\theta)$ and $\pi^{-1}(\theta)$ is smooth.

For a positive closed bidimension (m,m) current S on $\overline{X}\times\mathbb{P}^N(\mathbb{C})$, the slicing S_θ with respect to π' (once it exists) is a positive closed bidimension $(1,1)$ current supported in $(\pi')^{-1}(\theta)=\pi^{-1}(\theta)\times\mathbb{P}^N(\mathbb{C})$.

Proposition 7.3. *In the above setting, by passing to a subsequence, S_n converges in the sense of current to a positive closed bidimension (m,m) current on $X\times\mathbb{P}^N(\mathbb{C})$.*

Proof. Applying Proposition 7.1 to each irreducible component of the Galois orbit $O(V_n)$ and summing, the mass of $S_n\wedge\omega^m$ is uniformly bounded from above. The compactness of the space of positive closed currents with uniformly bounded mass implies the result. □

Proposition 7.4. *Let S be such a limit as in Proposition 7.3. Let σ be a smooth positive bidegree $(m-1,m-1)$ form on $X(\mathbb{C})$. Then*

$$S\wedge T_f\wedge\pi_1^*(\sigma)=0$$

on $X\times\mathbb{P}^N(\mathbb{C})$.

Proof. Since the Moriwaki height is invariant under Galois conjugation, $\hat{h}_f(V_n)\to 0$ implies that $\hat{h}_f(O(V_n))\to 0$. By Proposition 6.1, we also have $\hat{h}_{\mathrm{geom}}(O(V_n))\to 0$. By Corollary 6.9 we have $S_n\wedge T_f\wedge\omega_1^{m-1}\to 0$. Since T_f has continuous local potential and $S_n\to S$, we have

$$S\wedge T_f\wedge\omega_1^{m-1}=0.$$

Let Ω be any relatively compact open subset of $X(\mathbb{C})$. Since ω_X is Kähler, there exists a constant $C=C(\Omega,\sigma)>0$ such that $\sigma\leq C\omega_X^{m-1}$ on Ω. Hence $\pi_1^*(\sigma)\leq C\omega_1^{m-1}$ on $\Omega\times\mathbb{P}^N(\mathbb{C})$. Thus $S\wedge T_f\wedge\omega_1^{m-1}=0$ implies $S\wedge T_f\wedge\pi_1^*(\sigma)=0$ on $\Omega\times\mathbb{P}^N(\mathbb{C})$. Since Ω is arbitrary, the conclusion holds. □

We refer to Definition 2.9 for the meaning of admissibility of $S_\theta \wedge R_s$ when $s \in \pi^{-1}(\theta)$ and $\pi^{-1}(\theta)$ is smooth.

Proposition 7.5. *Let S be such a limit as in Proposition 7.3. If $\pi^{-1}(\theta)$ is smooth and the slicing S_θ with respect to π' is well defined at θ, then*
(1) $S_\theta \wedge T_f = 0$ on $X \times \mathbb{P}^N(\mathbb{C})$;
(2) Assume $S_\theta \wedge R_s$ is admissible for $s \in \pi^{-1}(\theta)$. Then

$$S_\theta \wedge R_s = \mu_s,$$

on $X \times \mathbb{P}^N(\mathbb{C})$, where μ_s is the maximal entropy measure of f_s.

Proof. (1) Since the slicing S_θ with respect to π' is well defined at θ, we can pick a sequence of smooth probability measure ν_n on $B(\mathbb{C})$ such that $\nu_n \to \delta_\theta$ in the weak-* sense, and $S \wedge (\pi')^*\nu_n \to S_\theta$ in the sense of currents. Since $\sigma_n := \pi^*\nu_n$ is a smooth positive closed bidegree $(m-1, m-1)$ form on $X(\mathbb{C})$, by Proposition 7.4 we have

$$S \wedge (\pi')^*\nu_n \wedge T_f = S \wedge T_f \wedge (\pi')^*\nu_n = S \wedge T_f \wedge \pi_1^*\sigma_n = 0.$$

Since $S \wedge (\pi')^*\nu_n \to S_\theta$ and T_f has continuous local potential, let $n \to +\infty$ we get $S_\theta \wedge T_f = 0$.

(2) By passing to a coordinate change, in a local chart $\mathbb{D}^m \subset X$, where $\mathbb{D}$ is the unit disk in the complex plane we can assume that $\pi|_{\mathbb{D}^m}$ is the projection

$$\mathbb{D}^m \to \mathbb{D}^{m-1}$$

to the first $m-1$ coordinates. Let $s = 0 \in \mathbb{D}^m$, $\theta = 0 \in \mathbb{D}^{m-1}$. Then $\pi^{-1}(\theta) \cong \mathbb{D}$, and $(\pi')^{-1}(0) \cong \mathbb{D} \times \mathbb{P}^N(\mathbb{C})$. Define $u(t, z) := \log|t|$. Then u is a p.s.h. function on $(\pi')^{-1}(0) = \mathbb{D} \times \mathbb{P}^N(\mathbb{C})$ and we have $R_0 = dd^c u$. Using standard convolution of $\log|t|$ with some radial function ρ_l, we obtain a sequence of smooth p.s.h. functions ϕ_l on $\mathbb{D}$ that decrease to $\log|t|$. Let $u_l(t, z) := \phi_l(t)$, then we obtain a sequence of smooth p.s.h. functions u_l on $(\pi')^{-1}(0)$ that decrease to u. For fixed $l \geq 1$, choose Φ_l to be a bidegree $(m-1, m-1)$ smooth form on $\mathbb{D}^m$ such that the restriction of Φ_l to the fiber $\pi^{-1}(0)$ is equal to $dd^c\phi_l$. Let $\Omega_l := \pi_1^*\Phi_l$, which is a bidegree $(m-1, m-1)$ smooth form on $\mathbb{D}^m \times \mathbb{P}^N(\mathbb{C})$ such that the restriction of Ω_l to the fiber $(\pi')^{-1}(0)$ is equal to $dd^c u_l$. Moreover $\Omega_l = \beta_l \omega_1^{m-1}$, where β_l is a non-negative smooth function.

Apply Theorem 7.2 to test functions of the form $\phi = \beta_l\tilde{\phi}$ such that $\tilde{\phi}$ is smooth and has compact support in $\mathbb{D}^m \times \mathbb{P}^N(\mathbb{C})$, we get

$$S_n \wedge \Omega_l \to T_f^N \wedge \Omega_l,$$

as $n \to +\infty$, for every fixed $l \geq 1$.

Since $S_n \to S$ in the sense of current as $n \to +\infty$ and since Ω_l is a smooth form, we have $S_n \wedge \Omega_l \to S \wedge \Omega_l$ as $n \to +\infty$. This implies

$$S \wedge \Omega_l = T_f^N \wedge \Omega_l \tag{7.2}$$

for every $l \geq 1$.

Since T_f has continuous local potential, the slicing $T_{f,\theta}$ is well defined with respect to π' for every $\theta \in B$, which is exactly the relative Green current for f restricted on $\pi^{-1}(t) \times \mathbb{P}^N(\mathbb{C})$, viewed as a one-parameter holomorphic family. By our assumption, the slicing S_θ exists. Since the slicing of the smooth form Ω_l at the fiber $(\pi')^{-1}(\theta)$ is $dd^c u_l$, by (7.2) we have

$$S_\theta \wedge dd^c u_l = T_{f,\theta}^N \wedge dd^c u_l \quad \text{on } (\pi')^{-1}(\theta)\ , \tag{7.3}$$

for every $l \geq 1$.

Since u_l decreases to u, and since by our assumption $dd^c u \wedge S_\theta$ is admissible, by [DS10, Proposition A.30], we have

$$S_\theta \wedge dd^c u_l \to S_\theta \wedge dd^c u \quad \text{on } (\pi')^{-1}(\theta)\ , \tag{7.4}$$

when $l \to +\infty$.

Since $T_{f,\theta}$ has continuous local potential, $dd^c u \wedge T_{f,\theta}^N$ is also admissible, using [DS10, Proposition A.30] again, we have

$$T_{f,\theta}^N \wedge dd^c u_l \to T_{f,\theta}^N \wedge dd^c u \quad \text{on } (\pi')^{-1}(\theta)\ , \tag{7.5}$$

when $l \to +\infty$.

Recall that $R_s = dd^c u$ and $\mu_s = T_{f,\theta}^N \wedge R_s$. Combining (7.3), (7.4), and (7.5) we get the desired result. This completes the proof. □

For $\alpha \in B(\mathbb{C})$, let $O(V_n)_\alpha$ be the specialization of $O(V_n)$ at α, which is a pure dimension one Zariski closed subvariety supported in $(\pi')^{-1}(\alpha)$. Define

$$S_{n,\alpha} := \frac{[O(V_n)_\alpha]}{\deg(O(V_n))}.$$

Proposition 7.6. *In the setting of Proposition 7.5. For each $n \geq 1$, there exists $l_n \geq 1$ and a finite set $E_n \subset B(\mathbb{C})$ consisting of fully transcendental points, such that*

$$\frac{1}{\#E_n} \sum_{\alpha \in E_n} S_{l_n,\alpha} \to S_\theta$$

as $n \to +\infty$.

Proof. Since S_θ is well defined, we can pick a sequence of smooth probability measure ν_n on $B(\mathbb{C})$ such that $\nu_n \to \delta_\theta$ in the weak-* sense, and $S \wedge (\pi')^*\nu_n \to S_\theta$. Thus it suffices to show that if ν is a fixed smooth probability measure on $B(\mathbb{C})$, for each $n \geq 1$, there exist $l_n \geq 1$ and a finite set $E_n \subset B(\mathbb{C})$ containing fully transcendental points, such that

$$\frac{1}{\#E_n} \sum_{\alpha \in E_n} S_{l_n,\alpha} \to S \wedge (\pi')^*\nu$$

as $n \to +\infty$.

Since $S_n \to S$ when $n \to +\infty$, we have $S_n \wedge (\pi')^*\nu \to S \wedge (\pi')^*\nu$. In applying the following argument to the sequence ν_n, we first choose l_n large enough so that $S_{l_n} \wedge (\pi')^*\nu_n$ is close to $S \wedge (\pi')^*\nu_n$, and then approximate ν_n by discrete measures supported on fully transcendental points. Thus it suffices to show that if ν is a fixed smooth probability measure on $B(\mathbb{C})$ and $l \geq 1$ is a fixed integer, for each $n \geq 1$, there exists a finite set $E_n \subset B(\mathbb{C})$ containing fully transcendental points, such that

$$\frac{1}{\#E_n} \sum_{\alpha \in E_n} S_{l,\alpha} \to S_l \wedge (\pi')^*\nu = \int S_{l,\alpha}\, d\nu(\alpha)$$

as $n \to +\infty$.

Since π' is flat, $S_{l,\alpha}$ varies continuously with respect to α in the sense of currents, see [Chi89, Page 152-153]. We can approximate ν by a sequence of discrete measures $\frac{1}{\#E_n}\sum_{\alpha\in E_n}\delta_\alpha$, where E_n is a finite set. Since the set of fully transcendental points has full Lebesgue measure, we can further choose E_n such that every point in E_n is fully transcendental. Thus we have

$$\frac{1}{\#E_n} \sum_{\alpha \in E_n} S_{l,\alpha} = \int S_{l,\alpha}\, d\left(\frac{1}{\#E_n} \sum_{\alpha \in E_n} \delta_\alpha\right) \to \int S_{l,\alpha}\, d\nu(\alpha)$$

as $n \to +\infty$, which finishes the proof. □

7.3. **Proof of Theorem 1.4 and 1.2.**

Proof of Theorem 1.4. Let f be a one-parameter algebraic family of endomorphisms of $\mathbb{P}^N$ of degree $d \geq 2$ over $\mathbb{C}$ parametrized by a smooth quasi-projective curve Λ satisfying Assumption A. As in the beginning of Section 7, we view f as the family of endomorphisms of $\mathbb{P}^N$ over a number field k, parametrized by a smooth irreducible quasi-projective variety X of dimension m. Let $\theta_0 \in B(\mathbb{C})$ such that the specialization

of $X_{\mathbb{C}}$ at θ_0 is Λ. Let V_n be the horizontal subvariety in $X \times \mathbb{P}^N$ induced by Γ_n. Then V_n is a generic sequence. Let

$$S_n := \frac{[O(V_n)]}{\deg(O(V_n))},\ S'_n := \frac{[O(V_n)]}{\mathrm{vol}(O(V_n))}.$$

Since (V_n) is small, the geometric heights of all irreducible components of $O(V_n)$ are bounded. Applying Proposition 7.1 componentwise and summing, the ratios $\mathrm{vol}(O(V_n))/\deg(O(V_n))$ are bounded above and below by positive constants. By Proposition 7.6, by passing to a subsequence, $S_n \to S$, $S'_n \to S'$, and there exists $c > 0$ such that $S = cS'$.

For each $\theta \in B(\mathbb{C})$, let f_θ be the one-parameter algebraic family parametrized by $\pi^{-1}(\theta)$. Let T_{bif} be the bifurcation current of f, which is a positive closed bidegree $(1,1)$ current on $X(\mathbb{C})$ with continuous local potential, see Section 4.1 for the definition. By our assumption $\mathrm{Bif}(f_{\theta_0}) \neq \emptyset$, which is equivalent to say

$$T_{\mathrm{bif}} \wedge [\pi^{-1}(\theta_0)] > 0.$$

Since T_{bif} has continuous local potential, there exists a neighborhood $U \subset B(\mathbb{C})$ of θ_0 such that for every $\theta \in U$,

$$T_{\mathrm{bif}} \wedge [\pi^{-1}(\theta)] > 0,$$

hence $\mathrm{Bif}(f_\theta) \neq \emptyset$.

By our assumption f is periodically generic. The periodic subvarieties and the non-resonance and diagonalizability conditions are defined over the finitely generated ground field; hence a fully-transcendental specialization cannot fall into the corresponding exceptional algebraic conditions. For every $\theta \in B(\mathbb{C})$ that is fully-transcendental, f_θ is periodically generic. Thus for every $\theta \in U$ fully transcendental, f_θ satisfies Assumption A. We pick such a point θ such that the slicing of S with respect to π' at θ is well defined. We fix this θ.

For $\alpha \in B(\mathbb{C})$, let $O(V_n)_\alpha$ be the specialization of the Galois orbit $O(V_n)$ at α, which is a pure dimension one Zariski closed subvariety supported in $(\pi')^{-1}(\alpha)$. For each $n \geq 1$, let E_n be the finite set given in Proposition 7.6, we have

$$\frac{1}{\#E_n} \sum_{\alpha \in E_n} S_{l_n,\alpha} \to S_\theta. \tag{7.6}$$

Let $R := (\pi_1)^* T_{\mathrm{bif}}$. Let $\mu_{\mathrm{bif},\theta} := T_{\mathrm{bif}} \wedge [\pi^{-1}(\theta)]$, which is a non-vanishing positive measure on $\pi^{-1}(\theta)$ since $\mathrm{Bif}(f_\theta) \neq \emptyset$. Let $\Omega \subset \pi^{-1}(\theta) \times \mathbb{P}^N(\mathbb{C})$ be an open subset, and let T be any positive closed bidimension $(1,1)$ current on Ω. Since R has continuous local potential,

$T \wedge R$ is admissible. For $s \in X(\mathbb{C})$, recall R_s is the current of integration on the fiber $\pi_1^{-1}(s) \cong \mathbb{P}^N(\mathbb{C})$. We need the following

Lemma 7.7. *Assume $R \wedge T > 0$. Then there exists $s \in \operatorname{supp} \mu_{\mathrm{bif},\theta}$ such that $T \wedge R_s$ is admissible and $T \wedge R_s > 0$.*

Proof. By Proposition 2.11, $T \wedge R_s$ is admissible for $\mu_{\mathrm{bif},\theta}$-a.e. $s \in \pi^{-1}(\theta)$, moreover

$$0 < T \wedge R = \int T \wedge R_s \, d\mu_{\mathrm{bif},\theta}(s).$$

Thus there exists a positive $\mu_{\mathrm{bif},\theta}$-measure subset such that $T \wedge R_s > 0$ when s is contained in this set. In particular we can choose a point s such that $s \in \operatorname{supp} \mu_{\mathrm{bif},\theta}$. □

By Proposition 7.5, we have

(i) $S_\theta \wedge T_f = 0$;
(ii) $S_\theta \wedge R_s = \mu_s$ for $\mu_{\mathrm{bif},\theta}$-a.e. $s \in \pi^{-1}(\theta)$, where μ_s is the maximal entropy measure of f_s.

We first show $\operatorname{genus}(\Gamma_n)/\deg(\Gamma_n) \to +\infty$**.** Assume by contradiction there is a constant $C > 0$ such that $\operatorname{genus}(\Gamma_n) \leq C \deg(\Gamma_n)$. Since genus and degree are invariant under Galois conjugation, for every fully transcendental α,

$$\operatorname{genus}(O(V_n)_\alpha) \leq C \deg(O(V_n)_\alpha).$$

Since $\operatorname{vol}(O(V_n)_\alpha)$ is a constant with respect to α, applying Proposition 7.1 componentwise to the irreducible components of $O(V_n)_\alpha$, there exists $C' > 0$ independent of α and n such that

$$\operatorname{genus}(O(V_n)_\alpha) \leq C' \operatorname{vol}(O(V_n)_\alpha).$$

The sequence of algebraic curves $\cup_{\alpha \in E_n} O(V_{l_n})_\alpha$ satisfies the genus-volume relation in Definition 3.1. By (7.6), we can apply Theorem 3.6 to S_θ and the test $(1,1)$-current R. There is a uniformly woven current $S_r \leq S_\theta$ on an open subset $\Omega \subset \pi^{-1}(\theta) \times \mathbb{P}^N(\mathbb{C})$ such that $S_r \wedge R > 0$. By Lemma 7.7, there exists $s \in \operatorname{Bif}(f_\theta) := \operatorname{supp} \mu_{\mathrm{bif},\theta}$ such that

(i) $S_r \wedge R_s > 0$.
(ii) $S_r \wedge T_f = 0$;
(iii) $S_r \wedge R_t \leq \mu_t$ for every $t \in \pi^{-1}(\theta)$ such that $S_r \wedge R_t$ is admissible.

By Theorem 4.10, $s \in \operatorname{Stab}(f_\theta)$, which is a contradiction.

Finally, we show $\operatorname{gonal}(\Gamma_n) \to +\infty$**.** Assume by contradiction there is a constant $C > 0$ such that $\operatorname{gonal}(\Gamma_n) \leq C$. After passing to a subsequence, we may assume that $\operatorname{gonal}(\Gamma_n) = l$ for every $n \geq 1$. Since gonality is invariant under Galois conjugation, for every fully

transcendental α, every irreducible component of $O(V_n)_\alpha$ has gonality l. By (7.6), we can apply Theorem 3.9 to S_θ and the test $(1,1)$-current R. Let $p : (\pi^{-1}(\theta) \times \mathbb{P}^N)^l \to \pi^{-1}(\theta) \times \mathbb{P}^N$ be the projection to the first coordinate. By Theorem 3.9, there exists a positive closed current S' of bidimension $(1,1)$ on $(\pi^{-1}(\theta) \times \mathbb{P}^N(\mathbb{C}))^l$ such that $p_* S' = S_\theta$, and a uniformly woven current $S'_r \le S'$ on an open subset $\Omega \subset (\pi^{-1}(\theta) \times \mathbb{P}^N(\mathbb{C}))^l$ such that $S'_r \wedge p^* R > 0$. By Proposition 2.7, after shrinking Ω slightly, we may assume $S'_r = \int_{\mathcal{V}^*_K} [\Gamma_a]\, d\nu(a)$, where $K := \overline{\Omega}$ is compact. See Definition 2.5 for the definition of $\mathcal{V}^*_K$. For each $a \in \mathcal{V}^*_K$, Γ_a is an irreducible closed analytic curve in Ω.

Let $a \in \operatorname{supp} \nu$ such that $[\Gamma_a] \wedge p^* R > 0$. Let U be a small ball such that $[\Gamma_a] \wedge p^* R > 0$ in U and $p : \Gamma_b \cap U \to p(U)$ is proper for every b in a small neighborhood $\mathcal{W}$ of a. Let ν' by the restriction of ν on $\mathcal{W}$, let $S'_{\mathcal{W}} := \int_{\mathcal{V}^*_K} [\Gamma_a]\, d\nu'(a)$, which is a uniformly woven current on Ω. Let $\tilde{S}$ be the restriction of $S'_{\mathcal{W}}$ on U, which is a uniformly woven current on U. Then $p_* \tilde{S}$ is a uniformly woven current on $p(U)$ such that $R \wedge p_* \tilde{S} = p^* R \wedge \tilde{S} > 0$. Since $\tilde{S} \le S'$, we have $p_* \tilde{S} \le p_* S' = S_\theta$.

By Lemma 7.7, there exists $s \in \mathrm{Bif}(f_\theta) := \operatorname{supp} \mu_{\mathrm{bif},\theta}$ such that

(i) $p_* \tilde{S} \wedge R_s > 0$.
(ii) $p_* \tilde{S} \wedge T_f = 0$;
(iii) $p_* \tilde{S} \wedge R_t \le \mu_t$ for every $t \in \pi^{-1}(\theta)$ such that $p_* \tilde{S} \wedge R_t$ is admissible.

By Theorem 4.10, $s \in \mathrm{Stab}(f_\theta)$, which is a contradiction. This completes the proof. □

Proof of Theorem 1.2. By McMullen's rigidity theorem [McM87], see also [JX23], if f is not the flexible Lattès family, then the bifurcation set $\mathrm{Bif}(f)$ is non-empty. Then f satisfies Assumption A. Since on $\Lambda \times \mathbb{P}^1$, a sequence of distinct horizontal curves is automatically a generic sequence, the conclusion $\mathrm{gonal}(\Gamma_n) \to +\infty$ and $\mathrm{genus}(\Gamma_n)/\deg(\Gamma_n) \to +\infty$ hold by Theorem 1.4. □

8. Dynamical uniform boundedness: proof of Theorem 1.9, 1.10, 1.14, and 1.15

We first recall:

Lemma 8.1 (Frey)**.** *Let C be an irreducible curve over a number field K, and let $B \ge 1$ be an integer. If the smooth projective model of C contains infinitely many points $x \in C(\overline{K})$ with $[K(x) : K] \le B$, then*

$$\mathrm{gonal}(C) \le 2B.$$

This follows from [Fre94, Proposition 2]. We will use the contrapositive form: if $\mathrm{gonal}(C) > 2B$, then C has only finitely many points of degree at most B over K.

Proof of Theorem 1.9. If f is the flexible Lattès family, the theorem follows from Ingram's uniform boundedness result for the corresponding one-parameter Lattès family [Ing11]. We may therefore assume that f is not the flexible Lattès family.

Let K be a number field over which Λ, f, and the marked point a are defined. Let $\Gamma_a \subset \Lambda \times \mathbb{P}^1$ be the horizontal curve which is the graph of a. For every $m \geq 0$, set

$$Y_m := f^{-m}(\Gamma_a) \subset \Lambda \times \mathbb{P}^1.$$

Let $\mathcal{E}_m$ be the union of the irreducible components Γ of Y_m which are not contained in $\cup_{j<m} Y_j$. We call such components exact preimage curves of depth m.

Fix any polarization $\overline{H} \in \widehat{\mathrm{Pic}}(K/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$. If Γ is an irreducible component of $\mathcal{E}_m$, then $f^m(\Gamma) = \Gamma_a$, and hence

$$\hat{h}_f^{\overline{H}}(\Gamma) = d^{-m} \hat{h}_f^{\overline{H}}(\Gamma_a).$$

In particular, any sequence $\Gamma_m \subset \mathcal{E}_m$ with $m \to \infty$ is small for f. By Theorem 1.2, the gonalities of any distinct sequence of such exact preimage curves tend to infinity.

Fix $D \geq 1$. Put

$$B := D^2.$$

If $t, z \in \overline{\mathbb{Q}}$ satisfy $[\mathbb{Q}(t) : \mathbb{Q}] \leq D$ and $[\mathbb{Q}(z) : \mathbb{Q}] \leq D$, then the point $(t, z) \in \Lambda \times \mathbb{P}^1$ has degree at most B over K. Indeed, $[K(t, z) : K] \leq [\mathbb{Q}(t, z) : \mathbb{Q}] \leq D^2$.

Assume by contradiction that the theorem is false for this D. For $m \geq 0$, let $\mathcal{S}_m$ be the set of points $(t, z) \in Y_m(\overline{K})$ such that

(i) $[K(t, z) : K] \leq B$;
(ii) m is the minimal non-negative integer such that $f_t^m(z) = a_t$.

Thus the sets $\mathcal{S}_m$ parametrize iterated preimages of exact depth m and bounded degree. Notice that $\mathcal{S}_m \subset \mathcal{E}_m(\overline{K})$.

For every $M \geq 0$, the number of points z satisfying $f_t^m(z) = a_t$ for some $m < M$ is bounded above by $1 + d + \cdots + d^{M-1}$, independently of t. Since the desired uniform bound is assumed to fail, it follows that for every $M \geq 0$, the union

$$\bigcup_{m \geq M} \mathcal{S}_m$$

is infinite.

Fix $M \geq 0$. For each $m \geq M$, the map f^{m-M} sends $\mathcal{S}_m$ to $\mathcal{S}_M$: the degree bound is preserved because $f^{m-M}(t,z)$ is defined over $K(t,z)$, and the exactness of the depth is preserved by minimality. We claim that the image of

$$\bigcup_{m\geq M} \mathcal{S}_m \longrightarrow \mathcal{S}_M$$

is infinite. Indeed, fix a point $(t,y) \in \mathcal{S}_M$. The points in its fiber are iterated preimages of y under the single endomorphism $f_t : \mathbb{P}^1 \to \mathbb{P}^1$, and all have degree at most B over K. By Northcott's theorem for the canonical height of f_t, there are only finitely many such points. Since $\bigcup_{m\geq M} \mathcal{S}_m$ is infinite, the image in $\mathcal{S}_M$ must be infinite.

Consequently $\mathcal{S}_M$ is infinite for every M. Since $\mathcal{E}_M$ has only finitely many irreducible components, there is an irreducible component Γ_M of $\mathcal{E}_M$ whose smooth projective model contains infinitely many points of degree at most B over K. By Lemma 8.1,

$$\operatorname{gonal}(\Gamma_M) \leq 2B.$$

Moreover, the components Γ_M obtained for distinct M are distinct by the definition of $\mathcal{E}_M$. For every polarization $\overline{H}$, we have

$$\hat{h}_f^{\overline{H}}(\Gamma_M) = d^{-M}\hat{h}_f^{\overline{H}}(\Gamma_a) \to 0.$$

This gives a sequence of distinct exact preimage curves with bounded gonality which is small for f, contradicting Theorem 1.2. This proves the theorem. □

Proof of Theorem 1.10. Let K be a number field over which Λ, f, and a are defined, and set $F := K(\Lambda)$. We work on the base change $\mathbb{P}^N_{\overline{F}}$ of the generic fiber. The graph Γ_a induces a point $a_F \in \mathbb{P}^N_F(F) \subset \mathbb{P}^N_{\overline{F}}(\overline{F})$. We identify irreducible horizontal curves in $\Lambda \times \mathbb{P}^N$ over $\overline{K}$ with the corresponding closed points of $\mathbb{P}^N_{\overline{F}}$.

For every $m \geq 0$, set

$$Y_m := f^{-m}(\Gamma_a).$$

Let $\mathcal{E}_m$ be the union of the irreducible components Γ of Y_m which are not contained in $\cup_{j<m} Y_j$; we call them exact preimage curves of depth m. Equivalently, after the above identification, the irreducible components of $\mathcal{E}_m$ are the closed points $x \in \mathbb{P}^N_{\overline{F}}$ whose exact hitting time to a_F is m.

If Γ is an irreducible component of $\mathcal{E}_m$, then $f^m(\Gamma) = \Gamma_a$. Hence for every polarization $\overline{H} \in \widehat{\operatorname{Pic}}(K/\mathbb{Z})_{\mathrm{nef},\mathbb{Q}}$,

$$\hat{h}_f^{\overline{H}}(\Gamma) = d^{-m}\hat{h}_f^{\overline{H}}(\Gamma_a).$$

In particular, any sequence of exact preimage curves whose depths tend to infinity is small for f.

Fix $D \geq 1$ and put $B := D^2$. For $m \geq 1$, let $\mathcal{B}_m$ be the finite set of irreducible components Γ of $\mathcal{E}_m$ such that the smooth projective model of Γ contains infinitely many points of degree at most B over K. We call the elements of $\mathcal{B}_m$ bad curves of depth m. By Lemma 8.1, every $\Gamma \in \mathcal{B}_m$ satisfies

$$\operatorname{gonal}(\Gamma) \leq 2B.$$

Moreover, if $\Gamma \in \mathcal{B}_m$ and $m \geq 2$, then $f(\Gamma) \in \mathcal{B}_{m-1}$. Indeed, the image $f(\Gamma)$ is an exact preimage curve of depth $m-1$, and the image of infinitely many degree at most B points again contains infinitely many degree at most B points.

We claim that $\mathcal{B}_m = \emptyset$ for all sufficiently large m. Assume the contrary. Since each $\mathcal{B}_m$ is finite and $f(\mathcal{B}_m) \subset \mathcal{B}_{m-1}$ for $m \geq 2$, we may apply König's lemma [Die10, Lemma 8.1.2] to the locally finite rooted tree whose level m is $\mathcal{B}_m$ and whose edges are induced by the map $f : \mathcal{B}_m \to \mathcal{B}_{m-1}$. We get an infinite path, namely a sequence

$$x_m \in \mathcal{B}_m, \qquad m \geq 1,$$

such that $f(x_m) = x_{m-1}$ for all $m \geq 2$. Put $x_0 := a_F$.

We endow $\mathbb{P}^N_{\overline{F}}$ with the constructible topology. Recall that this topology is generated by constructible subsets; it is compact and Hausdorff, and every Zariski closed subset is open and closed for it [Sta19, Tag 0901]. Since $\overline{F}$ is countable, this topology is second countable, hence compact metrizable. Consequently, the space of probability measures on it is sequentially compact. Consider the Birkhoff averages

$$\mu_n := \frac{1}{n} \sum_{i=0}^{n-1} \delta_{x_i}.$$

After passing to a subsequence, assume μ_n converges weakly to a probability measure μ. Since

$$f_*\mu_n - \mu_n = \frac{1}{n}(\delta_{x_0} - \delta_{x_n}),$$

we have $f_*\mu = \mu$.

Identifying a scheme-theoretic point with the corresponding irreducible closed subset of the generic fiber, we apply [Xie23, Theorem 1.12 and Lemma 5.2] to $\mathbb{P}^N_{\overline{F}}$ with the constructible topology. The measure μ is a convex combination of probability measures supported on periodic orbits of scheme-theoretic points. Let ξ be a point in the support of μ, and let $r \geq 1$ be its period. Set

$$Z := \overline{\{\xi\}} \cup \overline{\{f(\xi)\}} \cup \cdots \cup \overline{\{f^{r-1}(\xi)\}},$$

where the closures are Zariski closures in $\mathbb{P}^N_{\overline{F}}$. Then Z is a Zariski closed subset of $\mathbb{P}^N_{\overline{F}}$ satisfying $f(Z) = Z$.

Since Z is open and closed for the constructible topology and $\mu(Z) > 0$, we have $\mu_n(Z) > 0$ for all large n. Hence $x_i \in Z$ for some $i \geq 1$. Since $f(Z) = Z$ and $f^i(x_i) = a_F$, we get $a_F \in Z$. Therefore the whole forward orbit of a_F is contained in Z. By the assumption that the forward orbit of Γ_a is Zariski dense in $\Lambda \times \mathbb{P}^N$, the forward orbit of a_F is Zariski dense in $\mathbb{P}^N_{\overline{F}}$. Thus $Z = \mathbb{P}^N_{\overline{F}}$. This implies that ξ is the generic point of $\mathbb{P}^N_{\overline{F}}$.

It follows that every weak limit of the measures μ_n is the Dirac mass at the generic point of $\mathbb{P}^N_{\overline{F}}$. Consequently, for every proper subvariety $H \subsetneq \mathbb{P}^N_{\overline{F}}$ over $\overline{F}$, we have

$$\mu_n(H) \to 0.$$

Using the standard diagonal extraction argument over the countable set of proper subvarieties defined over $\overline{F}$, we can choose a subsequence of the bad curves x_{n_j} which is generic in $\mathbb{P}^N_{\overline{F}}$. This gives a generic sequence of exact preimage curves with depths tending to infinity, small for f, and with gonality bounded above by $2B$, contradicting Theorem 1.4. This proves the claim.

We now finish the proof of the uniform boundedness statement. Assume by contradiction that the conclusion fails for this D. For $m \geq 0$, let $\mathcal{S}_m$ be the set of points $(t, z) \in Y_m(\overline{K})$ such that

(i) $[K(t, z) : K] \leq B$;
(ii) m is the minimal non-negative integer such that $f_t^m(z) = a_t$.

Then $\mathcal{S}_m$ parametrizes bounded-degree preimages of exact depth m.

Since the number of points of depth $< M$ in a fiber is at most $1 + d^N + \cdots + d^{N(M-1)}$, the failure of the uniform bound implies that for every $M \geq 0$ the union

$$\bigcup_{m \geq M} \mathcal{S}_m$$

is infinite. Fix M so large that $\mathcal{B}_M = \emptyset$. For $m \geq M$, the map f^{m-M} sends $\mathcal{S}_m$ to $\mathcal{S}_M$: the degree bound is preserved because the image is defined over $K(t, z)$, and exactness of the depth is preserved by minimality.

The image of $\cup_{m \geq M} \mathcal{S}_m$ in $\mathcal{S}_M$ is infinite. Indeed, the fiber over a point $(t, y) \in \mathcal{S}_M$ consists of iterated preimages of y under the fixed endomorphism $f_t : \mathbb{P}^N \to \mathbb{P}^N$, all of degree at most B over K; by Northcott's theorem for the canonical height of f_t, this fiber is finite. Hence $\mathcal{S}_M$ is infinite. Since $\mathcal{E}_M$ has only finitely many irreducible components,

one component of $\mathcal{E}_M$ contains infinitely many points of degree at most B over K, i.e. $\mathcal{B}_M \neq \emptyset$. This contradicts the choice of M and proves the theorem. □

Proof of Theorem 1.14. Fix an integer $D \geq 1$. Let $\pi : \mathrm{End}_{d,1} \to \mathcal{M}_{d,1}$ be the canonical projection. Choose an irreducible algebraic curve $\Lambda \subset \mathrm{End}_{d,1}$ such that $\pi(\Lambda)$ contains a non-empty Zariski open subset of S. After normalizing Λ, the restriction of π induces a generically finite morphism

$$\rho : \Lambda \to S.$$

Let $q_0 := \deg \rho$. Let

$$F : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$$

be the corresponding one-parameter algebraic family.

By Levy's uniform bound for stabilizers [Lev11, Theorem 3.1], there is a constant M_d, depending only on d, such that for every degree d rational map g on $\mathbb{P}^1$, the group $\mathrm{Aut}(g)$ has order at most M_d. Since the set of conjugacies between two conjugate rational maps is a torsor under such an automorphism group, M_d is also an upper bound for the number of conjugacies between two conjugate degree d rational maps on $\mathbb{P}^1$. Set

$$q := q_0 M_d.$$

Set

$$D_S := qD.$$

By Theorem 1.2, and by Nguyen-Saito's theorem in the flexible Lattès case [NS96], the gonalities of distinct dynatomic curves of F tend to infinity. Hence the set

$$\mathcal{D}_{S,D} := \left\{ \Gamma \subset \Lambda \times \mathbb{P}^1 : \begin{array}{l} \Gamma \text{ is a dynatomic curve of } F, \\ \mathrm{gonal}(\Gamma) \leq D_S \end{array} \right\}$$

is finite.

Now let f be a non-isotrivial endomorphism of $\mathbb{P}^1$ over $k = \mathbb{C}(B)$ which lies over S. If $\mathrm{Preper}\,(f, D) = \emptyset$, there is nothing to prove. Otherwise, by Fact 1, the existence of a point in $\mathrm{Preper}\,(f, D)$ implies $\mathrm{gonal}(B) \leq D$.

Let B_1 be the normalization of an irreducible component of $B \times_S \Lambda$ dominating B. Then $\deg(B_1 \to B) \leq q_0$, and we get a morphism $\beta_1 : B_1 \to \Lambda$. The two families over B_1, namely the pullback of the family associated to f and the pullback $\beta_1^* F$, have the same moduli map. They are therefore conjugate over the geometric generic point of B_1.

To choose such a conjugacy over a function field, consider the relative conjugacy scheme over B_1 whose generic fiber is

$$\left\{\phi \in \mathrm{PGL}_2 : \phi \circ f_{B_1} \circ \phi^{-1} = \beta_1^* F\right\}.$$

This generic fiber is non-empty and finite, of cardinality at most M_d. Taking the normalization of an irreducible component dominating B_1, we obtain a finite cover $B' \to B_1$ of degree at most M_d. Hence $\deg(B' \to B) \leq q$. After this base change, the family associated to f is conjugate to the pullback of F by the induced morphism

$$\beta : B' \to \Lambda.$$

Conjugacy does not change preperiodicity, so we may use this model after the base change.

Let $x \in \mathrm{Preper}\,(f, D)$. Let C be the smooth projective curve with function field $k(x)$. Then $\mathrm{gonal}(C) \leq D$. Let C' be an irreducible component of the normalization of $C \times_B B'$ which dominates C. Since $\deg(C' \to C) \leq q$, Fact 1 gives

$$\mathrm{gonal}(C') \leq q\,\mathrm{gonal}(C) \leq D_S.$$

The point x, after the base change $B' \to B$, determines a horizontal curve in $B' \times \mathbb{P}^1$. Its image under the morphism $\beta \times \mathrm{id} : B' \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ is contained in a dynatomic curve, denoted by Γ_x, of F. Since C' dominates Γ_x, Fact 1 gives

$$\mathrm{gonal}(\Gamma_x) \leq \mathrm{gonal}(C') \leq D_S.$$

Thus $\Gamma_x \in \mathcal{D}_{S,D}$.

It remains to bound the number of possible points x. For each $\Gamma \in \mathcal{D}_{S,D}$, the pullback $(\beta \times \mathrm{id})^{-1}(\Gamma)$ has degree at most $\deg(\Gamma/\Lambda)$ over B'. Hence it contributes at most $\deg(\Gamma/\Lambda)$ points to the generic fiber over $\overline{k} = \overline{\mathbb{C}(B')}$. Therefore

$$\#\mathrm{Preper}\,(f, D) \leq \sum_{\Gamma \in \mathcal{D}_{S,D}} \deg(\Gamma/\Lambda).$$

The right-hand side depends only on D and S. This proves that the GUBC holds along S when $N = 1$. □

The proof of Theorem 1.15 follows the same finite base change and conjugacy reduction as the proof above. The main difference is that, in higher dimension, one has to allow an exceptional subvariety. This exceptional subvariety is produced by taking the Zariski closure of all low-gonality dynatomic curves in the fixed family over Λ; Theorem 1.4 ensures that this closure is proper.

Proof of Theorem 1.15. Fix an integer $D \geq 1$. Let $\pi : \mathrm{End}_{d,N} \to \mathcal{M}_{d,N}$ be the canonical projection. Choose an irreducible algebraic curve $\Lambda \subset \mathrm{End}_{d,N}$ such that $\pi(\Lambda)$ contains a non-empty Zariski open subset of S. After normalizing Λ, the restriction of π induces a generically finite morphism

$$\rho : \Lambda \to S.$$

Let $q_0 := \deg \rho$, and let

$$F : \Lambda \times \mathbb{P}^N \to \Lambda \times \mathbb{P}^N$$

be the corresponding one-parameter algebraic family. Since Assumption A is invariant under conjugacy and finite base change, the family F satisfies Assumption A.

By Levy's uniform bound for stabilizers [Lev11, Theorem 3.1], there is a constant $M_{d,N}$, depending only on d and N, such that for every degree d endomorphism g of $\mathbb{P}^N$, the group $\mathrm{Aut}(g)$ has order at most $M_{d,N}$. Since the set of conjugacies between two conjugate endomorphisms is a torsor under such an automorphism group, $M_{d,N}$ is also an upper bound for the number of conjugacies between two conjugate degree d endomorphisms of $\mathbb{P}^N$. Set

$$q := q_0 M_{d,N}, \qquad D_S := qD.$$

Let $W_{S,D}$ be the Zariski closure in $\Lambda \times \mathbb{P}^N$ of the union of all dynatomic curves Γ of F such that

$$\mathrm{gonal}(\Gamma) \leq D_S.$$

We claim that $W_{S,D} \subsetneq \Lambda \times \mathbb{P}^N$. Indeed, if $W_{S,D} = \Lambda \times \mathbb{P}^N$, then by a standard diagonal extraction argument one can choose a generic sequence of distinct dynatomic curves Γ_n of F with $\mathrm{gonal}(\Gamma_n) \leq D_S$ for all n. Such a sequence is small for F, contradicting Theorem 1.4. Thus $W_{S,D}$ is a proper subvariety.

Now let f be a non-isotrivial endomorphism of $\mathbb{P}^N$ over $k = \mathbb{C}(B)$ which lies over S. Let B_1 be the normalization of an irreducible component of $B \times_S \Lambda$ dominating B. Then $\deg(B_1 \to B) \leq q_0$, and we get a morphism $\beta_1 : B_1 \to \Lambda$. As in the proof of Theorem 1.14, by considering the relative conjugacy scheme between the pullback of f to B_1 and $\beta_1^* F$, after a further finite cover $B' \to B_1$ of degree at most $M_{d,N}$, the family associated to f becomes conjugate to the pullback of F by the induced morphism

$$\beta : B' \to \Lambda.$$

In particular, $\deg(B' \to B) \leq q$.

We define $V_{f,D} \subset \mathbb{P}^N_{\overline{k}}$ as follows. Pull back $W_{S,D}$ to $B' \times \mathbb{P}^N$ by $\beta \times \mathrm{id}$, take its generic fiber over B', and then transport it back by the above conjugacy. Since $W_{S,D}$ is a proper subvariety of $\Lambda \times \mathbb{P}^N$ and β is dominant, $V_{f,D}$ is a proper subvariety of $\mathbb{P}^N_{\overline{k}}$.

We show that

$$\mathrm{Preper}\,(f, D) \subset V_{f,D}(\overline{k}).$$

Let $x \in \mathrm{Preper}\,(f, D)$. Let C be the smooth projective curve with function field $k(x)$; then $\mathrm{gonal}(C) \leq D$. Let C' be an irreducible component of the normalization of $C \times_B B'$ which dominates C. Since $\deg(C' \to C) \leq q$, Fact 1 gives

$$\mathrm{gonal}(C') \leq q\,\mathrm{gonal}(C) \leq D_S.$$

After the base change $B' \to B$ and the above conjugacy, the point x determines a horizontal curve in $B' \times \mathbb{P}^N$. Its image in $\Lambda \times \mathbb{P}^N$ is contained in a dynatomic curve Γ_x of F. Since C' dominates Γ_x, Fact 1 gives

$$\mathrm{gonal}(\Gamma_x) \leq D_S.$$

Hence $\Gamma_x \subset W_{S,D}$, and therefore $x \in V_{f,D}(\overline{k})$. This proves the inclusion. Thus the required inequality holds with $C(D, S) = 0$, and the GUBC holds along S. □

9. The Assumption A: proof of Theorem 1.6 and Proposition 1.7

Proof of Theorem 1.6. We first show that if f_t is a Lattès map and f is not entirely contained in the Lattès locus, then the bifurcation set $\mathrm{Bif}(f) \neq \emptyset$. Let $L(t)$ be the sum of Lyapunov exponents of μ_t, where μ_t is the maximal entropy measure of f_t. Then $\mathrm{Bif}(f) = \emptyset$ if and only if L is a harmonic function on $\Lambda(\mathbb{C})$. By [BD05], $L(t) \geq \frac{N \log d}{2}$, where the equality holds if and only if f_t is a Lattès map. Since f_t is a Lattès map and f is not contained in the Lattès locus, the function L is non-constant and achieves its minimum at t. By the minimum principle of harmonic functions, L is not harmonic on $\Lambda(\mathbb{C})$. This implies $\mathrm{Bif}(f) \neq \emptyset$.

It remains to show that if f_t is split Lattès, then f is periodically generic. It suffices to show that the density of n-periodic points of f_t that are not resonant and diagonalizable is equal to one as $n \to +\infty$. Let $g : (\mathbb{P}^1)^N \to (\mathbb{P}^1)^N$ be the endomorphism such that g and f_t are semi-conjugate by a finite morphism $\pi : (\mathbb{P}^1)^N \to \mathbb{P}^N$. The endomorphism g is split, $g = (h, \ldots, h)$, where $h : \mathbb{P}^1 \to \mathbb{P}^1$ is a one-dimensional Lattès map. Let C be the ramification locus of π and let

$B := \pi(C) \subset \mathbb{P}^N$ be the branch locus, which is contained in a hypersurface. Let P_n be the set of n-periodic points of f_t whose orbit has no intersection with B. Let $Z := \cup_{n\geq 0} f_t^n(B)$, which is a pluripolar set. Then P_n is also the set of n-periodic points of f_t whose orbit has no intersection with Z. For each point $x \in P_n$, there exists a n-periodic point $y \in (\mathbb{P}^1)^N$ of g such that the differential $df_t^n(x)$ and $dg^n(y)$ are conjugate. Since the differential of all periodic points y of g are diagonalizable and non-resonant, $df_t^n(x)$ is diagonalizable and non-resonant. Finally, since the periodic points of f_t are equidistributed with respect to the maximal entropy measure μ_{f_t} [BD99], and μ_{f_t} puts no mass on pluripolar sets [DS10, Proposition 1.18], we have $\lim_{n\to+\infty} \#P_n/d^{nN} = 1$. Hence f is periodically generic, and the proof is complete. □

Proof of Proposition 1.7. Let $C \subset \mathcal{M}_{d,N}$ be the image of the moduli map of the family. Since C passes through a very general point of $\mathcal{M}_{d,N}$, it is not contained in the proper closed subset Z in Proposition 6.10. Hence Proposition 6.10 implies that the bifurcation locus of the family is non-empty. This proves the first condition in Assumption A.

It remains to verify periodic genericity when $N \geq 3$. Fix a rational number $0 < \eta < 1$. For each $n \geq 1$, consider the locus of maps in $\mathcal{M}_{d,N}$ for which the total number, counted with multiplicities, of fixed points of f^n that are diagonalizable and non-resonant is smaller than ηd^{Nn}. After passing to the finite cover which labels the fixed points of f^n, the multipliers are algebraic functions. This locus is contained in a countable union of algebraic subsets: non-diagonalizability is algebraic, and each resonance relation

$$w_1^{m_1} \cdots w_N^{m_N} = w_j, \qquad m_1 + \cdots + m_N \geq 2,$$

is algebraic after adding the periodic point and multiplier variables and then eliminating them. These algebraic subsets are proper for infinitely many n. Indeed, by Theorem 1.6 and its proof, there exists a one-parameter family satisfying Assumption A, and at the split Lattès parameter used there the proportion of fixed points of f^n that are diagonalizable and non-resonant tends to one as $n \to +\infty$. Thus the exceptional locus obtained by taking the union over n and over all resonance relations is a countable union of proper algebraic subsets.

Hence, outside a countable union of proper algebraic subsets of $\mathcal{M}_{d,N}$, there are infinitely many n for which at least ηd^{Nn} fixed points of f^n are diagonalizable and non-resonant. Since C passes through a very general point, we can choose a parameter $t \in \Lambda(\mathbb{C})$ with this property. Every irreducible component of the n-periodic locus passing through

one of these good fixed points belongs to the set $\mathcal{P}_n$ in the definition of periodic genericity. Therefore

$$\limsup_{n\to+\infty} d^{-Nn} \sum_{V\in\mathcal{P}_n} \deg V > 0.$$

Hence the family is periodically generic. This proves Assumption A. □

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