DAO FOR CURVES

ZHUCHAO JI AND JUNYI XIE

Abstract. We prove the Dynamical André-Oort (DAO) conjecture proposed by Baker and DeMarco for families of rational maps parameterized by an algebraic curve. In fact, we prove a stronger result, which is a Bogomolov type generalization of DAO for curves.

Contents

Dao that can be daoed is not the true Dao; Name that can be named is not the true Name.

Lao Zi, *Dao De Jing*, 400 BC

1. Introduction

1.1. **Statement of the main results.** A rational map on $\mathbb{P}^1$ over $\mathbb{C}$ is called *postcritically finite* (PCF) if its critical orbits are finite. PCF maps play a fundamental role in complex dynamics and arithmetic dynamics, since the dynamical behavior of critical points usually reflect the general dynamical behavior of a rational map. A consequence of Thurston's rigidity theorem [DH93] shows that PCF maps are defined

Date: September 22, 2023.

over $\overline{\mathbb{Q}}$ in the moduli space of rational maps of fixed degree, except for the well-understood one-parameter family of flexible Lattès maps. Moreover, PCF maps are Zariski dense [DM18, Theorem A], and form a set of bounded Weil height after excluding the flexible Lattès family [BIJL14, Theorem 1.1]. It was suggested by Silverman [Sil12] that PCF points in the moduli space of rational maps play a role analogous to that played by CM points in Shimura varieties.

One may consider PCF maps as "special points" in the moduli space. It is natural to ask what is the distribution of these special points. This leads to the following conjecture [DeM18, Conjecture 1.1] concerning PCF points and critical orbit relations, which is the curve case of the Dynamical André-Oort conjecture proposed by Baker and DeMarco in [BDM13].

Conjecture 1.1 (Dynamical André-Oort conjecture for curves)**.** Let $(f_t)_{t\in\Lambda}$ be a non-isotrivial algebraic family of rational maps with degree $d \geq 2$, parametrized by an algebraic curve Λ over $\mathbb{C}$. Then the following are equivalent:

(i) There are infinitely many $t \in \Lambda$ such that f_t is PCF;
(ii) The family has at most one independent critical orbit.

We need to explain the meaning of "independent critical orbit". Set $k := \mathbb{C}(\Lambda)$. The geometric generic fiber of the family $(f_t)_{t\in\Lambda}$ is a rational function $f_{\overline{k}} : \mathbb{P}^1_{\overline{k}} \to \mathbb{P}^1_{\overline{k}}$. Following DeMarco [DeM18], a pair $a, b \in \mathbb{P}^1(\overline{k})$ is called *dynamically related* if there is an algebraic curve $V \subseteq \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ such that $(a, b) \in V$ and V is preperiodic by the product map $f_{\overline{k}} \times f_{\overline{k}} : \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}} \to \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$. In other words, a pair of points $a, b \in \mathbb{P}^1(\overline{k})$ are dynamically related if and only if the orbit of $(a, b) \in \mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$ is not Zariski dense in $\mathbb{P}^1_{\overline{k}} \times \mathbb{P}^1_{\overline{k}}$.

We say that a family (f_t) has at most one independent critical orbit if any pair of critical points is dynamically related.

In [BDM13], Baker and DeMarco actually proposed a more general conjecture, considering also the higher dimensional parameter spaces, where almost nothing is known. Conjecture 1.1 and its variations also appear in Ghioca-Hsia-Tucker [GHT15], and in DeMarco [DeM16], [DM18],[DeM18]. One of the motivations of Conjecture 1.1 comes from the analogy between PCF points and CM points and the André-Oort Conjecture in arithmetic geometry, which was recently fully solved by Pila, Shankar and Tsimerman [PST21]. We note that Conjecture 1.1 and the André-Oort Conjecture both fit the principle of unlikely intersections, however, there is no overlap between them.

In this paper, we confirm the Dynamical André-Oort conjecture for curves.

Theorem 1.2. *Conjecture 1.1 is true.*

We indeed prove a Bogomolov type generalization of Theorem 1.2 (c.f. Theorem 1.3) and get Theorem 1.2 as a simple consequence.

In a forthcoming paper [Ji23a], using a variant of Theorem 1.2, we show that a general rational map on $\mathbb{P}^1$ of degree $d \geq 2$ is uniquely determined by its multiplier spectrum. This affirmatively answers a question of Poonen [Sil12, Question 2.43].

1.2. **A Bogomolov type generalization of DAO.** Let Λ be an algebraic curve over $\overline{\mathbb{Q}}$ and let $(f_t)_{t\in\Lambda}$ be a non-isotrivial algebraic family of rational maps with degree $d \geq 2$ over $\overline{\mathbb{Q}}$. The critical height $h_{\mathrm{crit}} : \Lambda(\overline{\mathbb{Q}}) \to \mathbb{R}$ is given by

$$t \in \Lambda(\overline{\mathbb{Q}}) \mapsto \widehat{h}_{f_t}(\mathcal{C}_{f_t}),$$

where $\mathcal{C}_{f_t}$ is the critical locus of f_t and $\widehat{h}_{f_t}$ is the canonical height for f_t with respect to. the line bundle $\mathcal{O}(1)$. It is clear that a parameter $t \in \Lambda(\overline{\mathbb{Q}})$ is PCF if and only if $h_{\mathrm{crit}}(t) = 0$.

In the following result, we prove a Bogomolov type generalization of Theorem 1.2 in the sense that we may replace the PCF parameters to small height parameters.

Theorem 1.3. *Let Λ be an algebraic curve over $\overline{\mathbb{Q}}$ and let $(f_t)_{t\in\Lambda}$ be a non-isotrivial algebraic family of rational maps with degree $d \geq 2$ over $\overline{\mathbb{Q}}$. Then the following are equivalent:*

(i) *There are infinitely many $t \in \Lambda$ such that f_t is PCF;*
(ii) *The family has at most one independent critical orbit;*
(iii) *For every $\varepsilon > 0$, the set $\{t \in \Lambda(\overline{\mathbb{Q}}) | \ h_{\mathrm{crit}}(t) < \varepsilon\}$ is infinite.*

Theorem 1.2 is a direct consequence of Theorem 1.3.

Remark 1.4. Keep the notations of Theorem 1.3. Indeed our proof of Theorem 1.3 shows a stronger result. Assume further that f has exactly $2d-2$ marked critical points $c_1, \dots, c_{2d-2}$. Then we may replace (iii) by the following weaker assumption:

(iii') For every $i, j \in \{1, \dots, 2d-2\}$ and every $\varepsilon > 0$, the set $\{t \in \Lambda(\overline{\mathbb{Q}}) | \ \hat{h}_{f_t}(c_i) + \hat{h}_{f_t}(c_j) < \varepsilon\}$ is infinite.

1.3. **Previous results.** In [BDM13], Baker-DeMarco proved Conjecture 1.1 for families of polynomials parameterized by the affine line $\mathbb{A}^1$ with coefficients that were polynomial in t.

Since the fundamental work of Baker-DeMarco [BDM13], plenty of works have been devoted to proving special cases of Conjecture 1.1 and its variations. Most progress is made in the setting that the families are given by polynomials. See Ghioca-Hsia-Tucker [GHT13], Ghioca-Krieger-Nguyen [GKN16], Ghioca-Krieger-Nguyen-Ye [GKNY17], Favre-Gauthier [FG18] [FG22], and Ghioca-Ye [GY18]. Among these results, a remarkable work of Favre and Gauthier [FG22] confirmed Conjecture 1.1 for families of polynomials.

In the case that the family is not given by polynomials, DeMarco-Wang-Ye [DMWY15] proved Conjecture 1.1 for some dynamical meaningful algebraic curves in the moduli space of quadratic rational maps. Ghioca-Hsia-Tucker [GHT15] proved a weak version of Conjecture 1.1 for families of rational maps given by $f_t(z) = g(z) + t$, $t \in \mathbb{C}$, where $g \in \overline{\mathbb{Q}}(z)$ is of $\deg g \geq 3$ with a super-attracting fixed point at ∞.

For the Bogomolov type generalization, as far as we know, Theorem 1.3 is the first result.

1.4. **Strategy in the previous works for families of polynomials.** Before giving a sketch of our proof of Theorem 1.3 (hence Theorem 1.2), we first recall the previous strategy for Conjecture 1.1. As mentioned in Section 1.3, except for a few special cases, all progress is made in the setting that the families are given by polynomials. These progresses roughly follow the line of arguments devised in the original paper of Baker and DeMarco [BDM13]. This strategy culminates in the proof of Conjecture 1.1 for all one-dimensional families of polynomials by Favre and Gauthier [FG22]. The strategy is as follows:

The direction that (ii) implies (i) is not hard. So we only need to show that (i) implies (ii). It is easy to reduce to the case where f is defined over some number field K and has exactly $2d-2$ marked critical points $c_1, \dots, c_{2d-2}$ counted with multiplicity.

The first step is to show that the bifurcation measure μ_{f,c_i} for $c_i, i = 1, \dots, 2d-2$ are proportional to the bifurcation measure μ_{bif} via some arithmetic equidistribution theorems. The application of various equidistribution theorems is one of the most successful ideas in arithmetic dynamics, which backs to the works of Ullmo [Ull98] and Zhang [Zha98], in where they solved the Bogomolov Conjecture. It was first introduced to study Conjecture 1.1 in Baker-DeMarco's fundamental work [BDM13].

When the family f is given by polynomials, for $i = 1, \dots, 2d-2$, there is a canonical Green function g_{f,c_i} on $\Lambda(\mathbb{C})$ such that $g_{f,c_i} \geq 0, \Delta g_{f,c_i} = \mu_{f,c_i}$ and $g_{f,c_i}|_{\mathrm{supp}\ \mu_{f,c_i}} = 0$. The second step is to show that these Green functions g_{f,c_i} are proportional. This step is easy when $\Lambda = \mathbb{A}^1$, but hard in general.

The third step is to construct an algebraic relation between c_i and c_j via the Böttcher coordinates. One may express the Green functions g_{f,c_i} using Böttcher coordinates. From the fact that g_{f,c_i} are proportional, one gets an analytic relation between c_i and c_j. In the end, one shows that this analytic relation is indeed algebraic. This step is highly non-trivial. In [BDM13], it was obtained by an explicit computation using properties of Böttcher coordinates. In [FG22], it relies further on an algebraization theorem for adelic series proved by the second-named author [Xie15].

In practice, the argument could be more complicated. For example, in each step, one needs to work on all places of K, not only on one archimedean place.

The second and the third steps strongly rely on the additional assumption that the families are given by polynomials for several reasons: the canonical Green functions, the Böttcher coordinates, the algebraization theorem, etc.

1.5. **Sketch of our proofs.** Theorem 1.2 is implied by Theorem 1.3. Indeed, by Thurston's rigidity theorem for PCF maps [DH93], we easily reduce to the case where the family $f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ over Λ is defined over $\overline{\mathbb{Q}}$. Then Theorem 1.2 becomes the equivalence of (i) and (ii) in Theorem 1.3.

For Theorem 1.3, the direction that (ii) implies (i) was proved by DeMarco [DeM16, Section 6.4] and the direction that (i) implies (iii) is trivial. We only need to show that (iii) implies (ii). We may assume that f has exactly $2d-2$ marked critical points $c_1, \dots, c_{2d-2}$ counted with multiplicity. Assume for the sake of contradiction that c_1 and c_2 are not dynamically related.

Structure of the proof. There are four steps in our proof. As in the previous works, our first step is to show the equidistribution of parameters of small height. We then get an additional condition that for every c_i, μ_{f,c_i} is proportional to the bifurcation measure μ_{bif} and we only need to get a contradiction under this additional assumption. We do this in the next three steps using a new strategy. Our basic idea is to work on general points with respect to μ_{bif}, which is motivated by the previous work [JX23b] of the authors on the local rigidity of Julia sets.

In Step 2, we show a selection of conditions are satisfied for μ_{bif}-a.e. point $t \in \Lambda(\mathbb{C})$, such that for a good parameter satisfies these conditions, we can construct the similarities in Step 3, moreover we can finally get a contradiction in Step 4.

In Step 3, we construct similarities between the phase space and the parameter space. Such similarities are known in some cases for prerepelling parameters (which is a countable set). The novelty of our result is to get the similarities at μ_{bif}-a.e. point $t \in \Lambda(\mathbb{C})$. This leads to many difficulties coming from the non-uniformly hyperbolic phenomenon.

In Step 4, we construct local symmetries of maximal entropy measures from the similarities constructed in Step 3, and a contradiction comes from these symmetries and an arithmetic condition that we selected in Step 2.

Step 1: Equidistribution. Dujardin and Favre [DF08, Theorem 2.5] (and DeMarco [DeM16]) showed that the bifurcation measure μ_{f,c_i} for c_i is non-zero if and only if c_i is not preperiodic. In this case, c_i is called active. Since c_1 and c_2 are not dynamically related, both of them are active. The equidistribution theorem for small points of Yuan and Zhang [YZ21, Theorem 6.2.3] implies that for every active c_i, μ_{f,c_i} is proportional to the bifurcation measure μ_{bif}. This result is based on their recent theory of adelic line bundles on quasi-projective varieties. Before Yuan-Zhang' theory, this step was usually non-trivial in the previous works as in [BDM13] and [FG22]. The quidistribution theorem is reviewed in Section 2.

Step 2: Parameter exclusion. In this step, we show a selection of conditions are satisfied for μ_{bif}-a.e. point $t \in \Lambda(\mathbb{C})$, such that for a good parameter satisfies these conditions, we can construct the similarities in Step 3, moreover, we can finally get a contradiction in Step 4.

Since aside from the flexible Lattès locus, the exceptional maps [1] are isolated in the moduli space, it is easy to show that

(1) Let $(f_t)_{t\in\Lambda}$ be an algebraic family of rational maps over an algebraic curve Λ, then μ_{bif}-a.e. point $t \in \Lambda(\mathbb{C})$ is non-exceptional.

Let $\mathrm{Corr}(\mathbb{P}^1)_*^{f_t}$ be the set of $f_t \times f_t$-invariant Zariski closed subsets $\Gamma_t \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ of pure dimension 1 such that both $\pi_1|_{\Gamma_t}$ and $\pi_2|_{\Gamma_t}$ are finite.

[1]As in [JX23a, Section 1.1], we call g *exceptional* if it is a Lattès map or semi-conjugates to a monomial map.

Let $\mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f_*$ be the set of $f\times_\Lambda f$-invariant Zariski closed subsets $\Gamma\subseteq \Lambda\times(\mathbb{P}^1\times\mathbb{P}^1)$ which is flat over Λ and whose generic fiber is in $\mathrm{Corr}(\mathbb{P}^1_\eta)^{f_\eta}_*$, where η is the generic point of Λ. In general, a correspondence $\Gamma_t\in \mathrm{Corr}(\mathbb{P}^1)^{f_t}_*$ may not be contained in any correspondence in $\mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f_*$. On the other hand, it is the case if t is *transcendental* in the sense of [XY23] (c.f. Proposition 3.11). Moreover, $\mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f_*$ is countable (c.f. Proposition 3.8). It is easy to show that

(2) Let $(f_t)_{t\in\Lambda}$ be an algebraic family of rational maps over an algebraic curve Λ, then μ_{bif}-a.e. point $t\in\Lambda(\mathbb{C})$ is transcendental.

For every $\Gamma\in\mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f_*$, we introduce a condition $\mathrm{FS}(\Gamma_t)$ with respect to the pair c_1,c_2 for every $t\in\Lambda(\mathbb{C})$. Roughly speaking, this condition means that in most of the time $n\geq 0$, $(f_t^n(c_1(t)),f_t^n(c_2(t)))$ is not too close to Γ. We show that

(3) Let $(f_t)_{t\in\Lambda}$ be an algebraic family of rational maps over an algebraic curve Λ, assume moreover that (iii) in Theorem 1.3 is satisfied, then μ_{bif}-a.e. point $t\in\Lambda(\mathbb{C})$ satisfies the $\mathrm{FS}(\Gamma_t)$ condition for every $\Gamma\in\mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f_*$.

To prove (3), we introduce another condition $\mathrm{AS}(\Gamma_t)$ which implies $\mathrm{FS}(\Gamma_t)$. To show that $\mathrm{AS}(\Gamma_t)$ is satisfied for μ_{bif}-a.e. point, we consider integrations with respect to μ_{bif} having arithmetic meaning, and the aim is to show that these integrations are bounded. We bound these integrations via the arithmetic intersection theory on quasi-projecitve varieties [YZ21]. This was done in Section 3.

Next, we consider some typical non-uniformly hyperbolic conditions. A result of De Thélin-Gauthier-Vigny [DTGV21] shows that

(4) Let $(f_t)_{t\in\Lambda}$ be an algebraic family of rational maps over an algebraic curve Λ and let a be a marked point, then $\mu_{f,a}$-a.e. parameters satisfy the Parametric Collet-Eckmann and Marked Collet-Eckmann condition.

We introduce a condition $\mathrm{PR}(s)$ for $s>1/2$ with respect to a marked point a. It means that the orbit of a could at most polynomially (with power s) close to the critical locus. We show that

(5) Let $(f_t)_{t\in\Lambda}$ be an algebraic family of rational maps over an algebraic curve Λ and let a be a marked point, then for every $s>1/2$, $\mu_{f,a}$-a.e. parameters satisfy the $\mathrm{PR}(s)$ condition.

This was done in Section 4.

The proof of Condition (6) is a combination of Condition (4), Siegel's linearization theorem [Mil11, Theorem 11.4] and the fact that the set of Liouville numbers has Hausdorff dimension 0 [Mil11, Lemma C.7],

(6) Let $(f_t)_{t\in\Lambda}$ be an algebraic family of rational maps over an algebraic curve Λ, assume moreover that for every active marked critical point c_i, μ_{f,c_i} is proportional to the bifurcation measure $\mu_{\rm bif}$, then $\mu_{\rm bif}$-a.e. point $t \in \Lambda(\mathbb{C})$ satisfies the Collet-Eckmann condition.

Condition (1), (2) and (6) are relatively easy to show, and Condition (4) can be easily deduced by the work of De Thélin-Gauthier-Vigny [DTGV21]. The major part of Step 2 is the proof of Condition (3) and (5). The proof of Condition (5) requires pluripotential theory. The proof of Condition (3) requires both pluripotential theory and arithmetic intersection theory.

Step 3: Similarity between the phase space and the parameter space. To get a contradiction, we show that the conditions (1),(2),(4),(5),(6) imply the opposite of (3). Our idea is to get similarity between the bifurcation measure $\mu_{\rm bif}$ on the parameter space and the maximal entropy measure μ_{f_t} on the phase space. This can be thought of as a generalization of Tan's work [Tan90], in where she got such a similarity at Misiurewicz points in Mandelbrot set, and as a generalization of Gauthier's [Gau22, Section 3.1] and Favre-Gauthier's works [FG22, Section 4.1.4], in where they got such a similarity at properly prerepelling parameters. We show that when $t \in \Lambda(\mathbb{C})$ satisfies (4),(5),(6), for each c_i active, there is a subset $A \subseteq \mathbb{Z}_{\geq 0}$ of large lower density and a sequence of positive real numbers $(\rho_n)_{n\in A}$ tending to zero, such that we can construct a family of renormalization maps $h_n : \mathbb{D} \to \mathbb{P}^1, n \in A$, defined by first shrinking the parameter disk $\mathbb{D}$ to a small disk of radius ρ_n, then use f^n to iterate the graph of c_i over this small disk and projects to the phase space $\mathbb{P}^1(\mathbb{C})$. We show that this family is normal and no subsequences of $h_n, n \in A$ tending to a constant map (c.f. Theorem 5.2).

Comparing with the previous results, we get similarity between phase space and parameter space not only for prerepelling parameters (which is a countable set) but also for parameters satisfying Topological Collet-Eckamnn condition and Polynomial Recurrence condition (which is a set of full $\mu_{\rm bif}$ measure). We believe that our result has an independent interest in complex dynamics.

In the previous works for prerepelling parameters, the key point is that the orbits of the marked points have a uniform distance from the critical locus for all but finitely many terms. This is not true in our case. For this reason, we introduce the following new strategy.

Our proof is divided into two parts. In the first part, we work only on the phase space. We select the "good time set" $A \subseteq \mathbb{Z}_{\geq 0}$ of large lower density. For each good time $n \in A$, we construct n maps from certain fixed simply connected domain to $\mathbb{P}^1(\mathbb{C})$. Roughly speaking, the goodness of n means that the above maps have a uniformly bounded number of critical points. Then we need to study the distortions of such maps which are non-injective in general (c.f. Section 6). To describe the distortion of perhaps non-injective holomorphic maps, we introduce the concepts of upper and proper lower radius (c.f. Section 5). Comparing with the usual lower radius of the image, the advantage of the proper one is the stability under small perturbations.

In the second part, we use a binding argument to get the renormalization maps from the maps defined above and show that this family is normal and no subsequence of $h_n, n \in A$ tending to a constant map (c.f. Section 7). In particular, we decide the rescaling factors $\rho_n, n \in A$ in this process.

We also show that μ_{f_0} can be read from μ_{bif} via the family $h_n, n \in A$ (c.f. Proposition 7.1) and $\rho_n, n \in A$ can be read from μ_{bif} up to equivalence (c.f. Proposition 7.2). Step 3 was done in Sections 5, 6 and 7.

Step 4: Conclusion via local symmetries of maximal entropy measures. We have constructed a family of renormalization maps $h_n : \mathbb{D} \to \mathbb{P}^1, n \in A$. Since A has a large lower density, after taking an intersection, we may assume that the set A for c_1 and c_2 are the same. After suitable adjustments of $\mathcal{H}_a := \{h_{a,n}, n \in A\}$ and $\mathcal{H}_b := \{h_{b,n}, n \in A\}$, we show that they form an asymptotic symmetry, which basically means that every limit of $h_{a,n} \times h_{b,n}$ produces a symmetry of μ_{f_t}. Applying an argument based on [JX23b, Theorem 1.7], we show that (3) is not true. This concludes the proof, see Section 8. Since we may assume that f_t is Collet-Eckmann, in this last step we may replace [JX23b, Theorem 1.7] by combing [DFG22, Theorem A] with [DFG22, Corollary 3.2].

Acknowledgement. We would like to thank Xinyi Yuan for discussions on the theory of adelic line bundles on quasi-projective varieties and for his help for the proofs of Proposition 3.19 and Proposition 3.20. We also would like to thank Thomas Gauthier and Laura DeMarco for helpful comments on the first version of this paper.

The first-named author would like to thank Beijing International Center for Mathematical Research in Peking University for the invitation. The second-named author Junyi Xie is supported by NSFC Grant (No.12271007).

2. Equidistribution of small parameters

2.1. **Family of rational maps.** For $d \geq 1$, let $\mathrm{Rat}_d(\mathbb{C})$ be the space of degree d endomorphisms on $\mathbb{P}^1(\mathbb{C})$. It is a smooth quasi-projective variety of dimension $2d+1$ [Sil12]. The group $\mathrm{PGL}_2(\mathbb{C}) = \mathrm{Aut}(\mathbb{P}^1(\mathbb{C}))$ acts on $\mathrm{Rat}_d(\mathbb{C})$ by conjugacy. The geometric quotient

$$\mathcal{M}_d(\mathbb{C}) := \mathrm{Rat}_d(\mathbb{C})/\mathrm{PGL}_2(\mathbb{C})$$

is the (coarse) *moduli space* of endomorphisms of degree d [Sil12]. The moduli space $\mathcal{M}_d(\mathbb{C}) = \mathrm{Spec}\,(\mathcal{O}(\mathrm{Rat}_d(\mathbb{C})))^{\mathrm{PGL}_2(\mathbb{C})}$ is an affine variety of dimension $2d-2$ [Sil07, Theorem 4.36(c)]. Let $\Psi : \mathrm{Rat}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ be the quotient morphism. One note that, $\mathrm{Rat}_d(\mathbb{C})$, $\mathcal{M}_d(\mathbb{C})$ and Ψ are defined over $\mathbb{Q}$.

Definition 2.1. A (one-dimensional) *holomorphic family of rational maps* is a holomorphic map

$$f : \Lambda \times \mathbb{P}^1(\mathbb{C}) \to \Lambda \times \mathbb{P}^1(\mathbb{C}), \quad (t, z) \mapsto (t, f_t(z)), \tag{2.1}$$

where Λ is a Riemann surface and $f_t : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a rational map of degree $d \geq 2$.

A holomorphic family f is called *algebraic* if Λ is a smooth algebraic curve over $\mathbb{C}$ and the morphism $f : \Lambda \times \mathbb{P}^1(\mathbb{C}) \to \Lambda \times \mathbb{P}^1(\mathbb{C})$ is algebraic. Moreover, we say that f is an algebraic family over a subfield K of $\mathbb{C}$ if both Λ and $f : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ are defined over K. In other words, give an algebraic family f on a smooth algebraic curve Λ over $\mathbb{C}$ is equivalent to give an algebraic morphism $\phi_f : t \mapsto f_t \in \mathrm{Rat}_d$. Moreover f is defined over K if Λ and ϕ_f are defined over K. We say that f is *non-isotrivial* if $\Psi \circ \phi_f$ is not a constant map.

Let $\pi_1 : \Lambda \times \mathbb{P}^1(\mathbb{C}) \to \Lambda$ and $\pi_2 : \Lambda \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be the canonical projections. Let $\omega_{\mathbb{P}^1}$ be the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$, and let ω_Λ be a fixed Kähler form on Λ with $\int_\Lambda \omega_\Lambda = 1$. Let $\omega_1 := \pi_1^*(\omega_\Lambda)$ and $\omega_2 := \pi_2^*(\omega_{\mathbb{P}^1})$. The *relative Green current*[2] of f is defined by

$$T_f := \lim_{n \to +\infty} d^{-n}(f^n)^*(\omega_2) = \omega_2 + dd^c g,$$

where g is a Hölder continuous quasi-p.s.h. function [DS10, Lemma 1.19].

For every $t_0 \in \Lambda$, we have $T_f \wedge [t = t_0] = \mu_{f_{t_0}}$, where $\mu_{f_{t_0}}$ is the maximal entropy measure of f_{t_0}.

[2] In [GV19], it is called "fibered Green current".

A *marked point* a is a holomorphic map $a : \Lambda \to \mathbb{P}^1(\mathbb{C})$. The *bifurcation measure* of the pair (f, a) is defined by

$$\mu_{f,a} := (\pi_1)_*(T_f \wedge [\Gamma_a]) = a^* T_f,$$

where Γ_a is the graph of a. When the family f is algebraic, the marked point a is said to be *algebraic* if the map $a : \Lambda \to \mathbb{P}^1(\mathbb{C})$ is algebraic. When a is algebraic, $\mu_{f,a}$ has finite mass. See Gauthier-Vigny [GV19, Proposition 13 (1)], where they proved the finiteness of the mass of bifurcation currents' for general algebraic families of polarized dynamical systems.

To simplify the notation, for an algebraic family f, a marked point a is always assumed to be algebraic in the whole paper.

A marked point a is called *active* if $\mu_{f,a}$ does not vanish. Otherwise, it is called *passive*.

A *marked critical point* c is a marked point such that $c(t)$ is a critical point of f_t for each $t \in \Lambda$. Let f be an algebraic family of rational maps as in (2.1) with marked critical points $(c_i)_{1 \leq i \leq 2d-2}$. We define the *bifurcation measure* of f by

$$\mu_{\mathrm{bif}} := \sum_{i=1}^{2d-2} \mu_{f,c_i}.$$

Since each c_i is algebraic, μ_{bif} has finite mass.

The following theorem is useful.

Theorem 2.2 (DeMarco, [DeM16])**.** *Let f be a holomorphic family of rational maps as in (2.1) and a be a marked point. Then the following are equivalent:*

(i) *a is passive;*
(ii) *a is stable, i.e. the family of maps $\{t \to f_t^n(a(t))\}_{n \geq 1}$ forms a normal family.*

If moreover, f is a non-isotrivial algebraic family, then the above two conditions are equivalent to that a is preperiodic.

When a is a marked critical point, the last statement was proved in an earlier article [DF08, Theorem 2.5].

2.2. **Equidistribution.** The following deep result about equidistribution of preperiodic points is a direct consequence of Yuan-Zhang [YZ21, Theorem 6.2.3]. Note that for an algebraic family of rational maps as in (2.1) defined over a number field K, the canonical height of a preperiodic point is equal to 0. See also Gauthier [Gau21, Theorem 3].

Theorem 2.3. *Let f be an algebraic family of rational maps as in (2.1) and a be an active marked point, all defined over a number field K. Let $t_n \in \Lambda(\overline{\mathbb{Q}})$ be an infinite sequence of distinct points such that $\widehat{h}_{f_t}(a(t_n)) \to 0$ as $n \to 0$. Then we have*

$$\frac{1}{|\mathrm{Gal}(t_n)|} \sum_{t \in \mathrm{Gal}(t_n)} \delta_t \to \frac{\mu_{f,a}}{\mu_{f,a}(\Lambda)},$$

when $n \to +\infty$, where $\mathrm{Gal}(t_n)$ is the Galois orbit of t_n.

The following is an important corollary of the equidistribution of parameters of small heights.

Corollary 2.4. *Let f be an algebraic family of rational maps as in (2.1) with marked critical points $(c_i)_{1 \leq i \leq 2d-2}$. Assume that f is defined over $\overline{\mathbb{Q}}$ and there is an infinite sequence $t_n, n \geq 0$ of distinct points in $\Lambda(\overline{\mathbb{Q}})$ such that $h_{\mathrm{crit}}(t_n) \to 0$, then for c_i, c_j being active marked critical points, we have*

$$\frac{\mu_{f,c_i}}{\mu_{f,c_i}(\Lambda)} = \frac{\mu_{f,c_j}}{\mu_{f,c_j}(\Lambda)}.$$

Proof. We may assume that f has at least two active marked critical points. Otherwise, the statement is trivial. This implies that f is not isotrivial and $\phi_f(\Lambda)$ is not contained in the locus of flexible Lattès maps.

As $h_{\mathrm{crit}}(t_n) \to 0$, both $\widehat{h}_{f_{t_n}}(c_i(t_n))$ and $\widehat{h}_{f_{t_n}}(c_j(t_n))$ tend to zero. By Theorem 2.3, both $\frac{\mu_{f,c_i}}{\mu_{f,c_i}(\Lambda)}$ and $\frac{\mu_{f,c_j}}{\mu_{f,c_j}(\Lambda)}$ equal to $\frac{1}{|\mathrm{Gal}(t_n)|} \sum_{t \in \mathrm{Gal}(t_n)} \delta_t$, which concludes the proof. □

3. Frequently separated parameters

3.1. **Frequently separated condition.** Let (M, d) be a metric space and $g : M \to M$ be a self-map.

Definition 3.1. Let A be a subset of $\mathbb{Z}_{\geq 0}$. The *asymptotic lower/upper density* of A is defined by

$$\underline{d}(A) := \liminf_{n \to \infty} |A \cap [0, n-1]|/n,$$

and

$$\overline{d}(A) := \limsup_{n \to \infty} |A \cap [0, n-1]|/n.$$

If $\underline{d}(A) = \overline{d}(A)$, we set $d(A) := \underline{d}(A) = \overline{d}(A)$ and call it the *asymptotic density* of A.

We still denote by d the distance in on $M \times M$ by

$$d((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), (y_1, y_2)\}.$$

Let Σ be a non-empty subset of $M \times M$.

Remark 3.2. We view Σ as a correspondence on M. The most typical example is the diagonal. For $x \in M$, we denote by $\Sigma(x) := \pi_2(\pi_1^{-1}(x))$, where π_1, π_2 are the first and the second projections. When $\pi^{-1}(x) \neq \emptyset$, for every $y \in M$, we have $d((x, y), \Sigma) \geq d(y, \Sigma(x))$.

Definition 3.3. A pair of points $x, y \in M$ is called

(i) *Frequently separated* $\mathrm{FS}(\Sigma)$ for Σ, if for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\overline{d}(\{n \geq 0 |\, d((g^n(x), g^n(y)), \Sigma) \geq \delta\}) > 1 - \varepsilon.$$

(ii) *Average separated* $\mathrm{AS}(\Sigma)$ for Σ, if

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \max\{-\log d((g^n(x), g^n(y)), \Sigma), 0\} < +\infty.$$

The above conditions depend only on the equivalence class of distance functions on M and $M \times M$. It is clear that if $\Sigma \subseteq \Sigma'$, then $\mathrm{AS}(\Sigma')$ (resp. $\mathrm{FS}(\Sigma')$) implies $\mathrm{AS}(\Sigma)$ (resp. $\mathrm{FS}(\Sigma)$).

Remark 3.4. When Σ is the diagonal Δ, the $\mathrm{FS}(\Delta)$ condition means that in most of the time $n \geq 0$, the orbits of x and y are δ-separated for some small $\delta > 0$. The proportion of such time could tend to 1 when δ tends to 0.

Lemma 3.5. *The* $\mathrm{AS}(\Sigma)$ *condition implies the* $\mathrm{FS}(\Sigma)$ *condition.*

Proof. Assume that the $\mathrm{AS}(\Sigma)$ condition holds. Set

$$\phi_n := \max\{-\log d((g^n(x), g^n(y)), \Sigma), 0\}$$

and

$$s_n := \frac{1}{n} \sum_{i=0}^{n-1} \phi_n.$$

The $\mathrm{AS}(\Sigma)$ condition shows that there is $A \geq 0$ and a sequence $n_j, j \geq 0$ tending to $+\infty$ such that $s_n \leq A$. For $\varepsilon > 0$, we have

$$\frac{\#\{i = 0, \ldots, n_j - 1 |\ \phi_i \geq A/\varepsilon\}}{n_j} \leq \varepsilon.$$

Set $\delta := e^{-A/\varepsilon}$, we get

$$\frac{\#\{i=0,\dots,n_j-1|\ d((g^i(x),g^i(y)),\Sigma)\geq\delta\}}{n_j}\geq 1-\varepsilon,$$

which concludes the proof. □

3.2. **Correspondences for rational maps.** Let $\mathbf{k}$ be a field. Let $g:\mathbb{P}^1_{\mathbf{k}}\to\mathbb{P}^1_{\mathbf{k}}$ be an endomorphism of degree $d\geq 2$.

A *correspondence* of $\mathbb{P}^1_{\mathbf{k}}$ is a non-empty Zariski closed subset of $\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}$. We denote by $\mathrm{Corr}(\mathbb{P}^1_{\mathbf{k}})^g$ the set of correspondences of $\mathbb{P}^1_{\mathbf{k}}\times\mathbb{P}^1_{\mathbf{k}}$ which are invariant under $g\times g$ and $\mathrm{Corr}(\mathbb{P}^1_{\mathbf{k}})^g_*$ the subset of $\mathrm{Corr}(\mathbb{P}^1_{\mathbf{k}})^g$ consisting of those Γ which are of pure dimension 1 and such that both $\pi_1|_\Gamma$ and $\pi_2|_\Gamma$ are finite.

Lemma 3.6. *Let K be a subfield of $\mathbf{k}$ such that g is defined over K. Then for every $\Gamma\in\mathrm{Corr}(\mathbb{P}^1_{\mathbf{k}})^g_*$, there is $\Gamma_K\in\mathrm{Corr}(\mathbb{P}^1_K)^g_*$ such that $\Gamma\subseteq\Gamma_K\otimes_K\mathbf{k}$.*

Proof. When $\mathbf{k}=\overline{K}$, we may define Γ_K to be the image of Γ under the natural morphism $(\mathbb{P}^1\times\mathbb{P}^1)_{\mathbf{k}}\to(\mathbb{P}^1\times\mathbb{P}^1)_K$. Now we may assume that both $\mathbf{k}$ and K are algebraically closed. Then every preperiodic point is defined over K.

For every $\Gamma\in\mathrm{Corr}(\mathbb{P}^1_{\mathbf{k}})^g_*$, we have

$$P_\Gamma:=\Gamma\cap\pi^{-1}(\mathrm{Preper}\,(g))=\Gamma\cap\pi_2^{-1}(\mathrm{Preper}\,(g))\subseteq\mathrm{Preper}\,(g)\times\mathrm{Preper}\,(g),$$

where π_1,π_2 are the first and the second projections. Since $\mathrm{Preper}\,(g)$ is Zariski dense in $\mathbb{P}^1_{\mathbf{k}}$ and $\pi_1|_\Gamma$ is surjective, P_Γ is Zariski dense in Γ. Since every points in P_Γ are defined over K, Γ is defined over K, which concludes the proof. □

Lemma 3.7. *The set $\mathrm{Corr}(\mathbb{P}^1_{\mathbf{k}})^g$ is countable.*

Proof. We may assume that $\mathbf{k}$ is algebraically closed. There is an algebraically closed subfield K of $\mathbf{k}$, such that K has finite transcendence degree and g is defined over K. Since $\mathrm{Corr}(\mathbb{P}^1_K)^g_*$ is countable, we conclude the proof by Lemma 3.6 and the fact that $\mathrm{Corr}(\mathbb{P}^1_{\mathbf{k}})^g\setminus\mathrm{Corr}(\mathbb{P}^1_{\mathbf{k}})^g_*$ is countable. □

Let $f:\Lambda\times\mathbb{P}^1\to\Lambda\times\mathbb{P}^1$ be an algebraic family of rational maps as in (2.1). A *flat family of correspondences* over Λ is a closed subset $\Gamma\subseteq(\mathbb{P}^1\times\mathbb{P}^1)\times\Lambda$ which is flat over Λ. Let η be the generic point of Λ. The map $\Gamma\mapsto\Gamma_\eta$ gives a bijection between flat family of correspondences over Λ and correspondences of the generic fiber $\mathbb{P}^1_\eta$. Moreover Γ is $f\times_\Lambda f$-invariant if and only if Γ_η is $f_\eta\times f_\eta$-invariant. Hence, by Lemma 3.7, we get the following result.

Corollary 3.8. *The set* $\mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f$ *of* $f \times_\Lambda f$*-invariant flat family of correspondences over* Λ *is countable.*

Denote by $\mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f_*$ the set of $\Gamma \in \mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f$ whose generic fiber is in $\mathrm{Corr}^\flat(\mathbb{P}^1_\eta)^{f_\eta}_*$.

Transcendental points. The notion of transcendental points was introduced in [XY23]. Let $a_i, i = 1, \ldots .m$ be marked points. Let L be an algebraically closed subfield of $\mathbb{C}$ such that Λ, f and $a_i, i = 1, \ldots, m$ are defined over L. Then there is a variety Λ_0 over L, a morphism $F : \Lambda_0 \times \mathbb{P}^1 \to \mathbb{P}^1$ and marked points $a'_1, \ldots, a'_m$ such that $\Lambda = \Lambda_0 \otimes_L \mathbb{C}$, $f = F \otimes_L \mathbb{C}$ and $a_i = a'_i \otimes_L \mathbb{C}$. A point $b \in \Lambda(\mathbb{C}) = \Lambda_0(\mathbb{C})$ is called a *transcendental point* for Λ_0/L if the image of $b : \mathrm{Spec}\,\mathbb{C} \to \Lambda_0$ is the generic point of Λ_0. In other words, $b \in \Lambda(\mathbb{C}) = \Lambda_0(\mathbb{C})$ is transcendental if and only if it is not in $\Lambda_0(L)$.

Remark 3.9. We can always assume L to have a finite transcendence degree. In this case, L is countable, hence $\Lambda_0(L)$ is countable. So all but countably many points in $\Lambda(\mathbb{C})$ are transcendental for Λ_0/L. In particular, if f is defined over $\overline{\mathbb{Q}}$, and a_i are marked critical points, we can take L to be $\overline{\mathbb{Q}}$.

Remark 3.10. Since μ_{f,a_i} has continuous potential, it does not have atoms. So if L have finite transcendence degree, the for μ_{f,a_i}-a.e. $t \in \Lambda(\mathbb{C})$, t is transcendental with respect to Λ_0/L.

Proposition 3.11. *Let* $b \in \Lambda(\mathbb{C})$ *be a transcendental point with respect to* Λ_0/L*. Then for every* $\Gamma_b \in \mathrm{Corr}(\mathbb{P}^1)^{f_b}_*$*, there is* $\Gamma' \in \mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f_*$ *such that* $\Gamma'_b := \Gamma' \cap \pi_1^{-1}(b)$ *contains* Γ_b*. Moreover, for every* $n \geq 0$ *and* $i, j \in \{1, \ldots, m\}$*, if* $(f^n_b(a_i(b)), f^n_b(a_j(b))) \in \Gamma_b$*, then* $(f^n(a_i), f^n(a_j)) \in \Gamma'$*.*

Proof. Since the image of $b : \mathrm{Spec}\,\mathbb{C} \to \Lambda_0$ is the generic point of Λ_0, f_b and $(a_i)_b, i = 1, \ldots, m$ are the base change of $F_K, (a'_i)_K, i = 1, \ldots, m$ via the natural morphism

$$K := L(\Lambda_0) \hookrightarrow \mathbb{C}$$

defined by b. By Lemma 3.6, for every $\Gamma_b \in \mathrm{Corr}(\mathbb{P}^1)^{f_b}_*$, there is $\Gamma'_K \in \mathrm{Corr}(\mathbb{P}^1_K)^{F_K}_*$ such that $\Gamma_b \subseteq \Gamma'_K \otimes_K \mathbb{C}$. Via the natural bijection between $\mathrm{Corr}(\mathbb{P}^1_K)^{F_K}_*$ and $\mathrm{Corr}^\flat(\mathbb{P}^1_{\Lambda_0})^F_*$, Γ'_K is the generic fiber of $\Gamma'_L \in \mathrm{Corr}^\flat(\mathbb{P}^1_{\Lambda_0})^F_*$. Set $\Gamma' := \Gamma'_L \otimes_L \mathbb{C} \in \mathrm{Corr}^\flat(\mathbb{P}^1_\Lambda)^f_*$. Then we get $\Gamma'_K \otimes_K \mathbb{C} = \Gamma'_b$. For every $n \geq 0$ and $i, j \in \{1, \ldots, m\}$, if $(f^n_b(a_i(b)), f^n_b(a_j(b))) \in \Gamma_b$, then $(F^n_K((a'_i)_K), F^n_K((a'_j)_K)) \in \Gamma'_K$, hence $(f^n(a_i), f^n(a_j)) \in \Gamma'$. This concludes the proof. □

3.3. The $\mathrm{AS}(\Sigma)$ condition for families of rational maps.

Theorem 3.12. *Let f be an algebraic family of rational maps as in (2.1). Let a, b be active marked points. Let V be a Zariski closed subset of $\Lambda \times (\mathbb{P}^1 \times \mathbb{P}^1)$. Assume that f, a, b and V are defined over $\overline{\mathbb{Q}}$; for every $n \geq 0$, the image Γ_n of $p_n := (f^n(a), f^n(b)) : \Lambda \to \Lambda \times (\mathbb{P}^1 \times \mathbb{P}^1)$ is not contained in V; and there is a sequence of distinct points $t_i \in \Lambda(\mathbb{C}), i \geq 0$ such that both $\widehat{h}_{f_{t_i}}(a(t_i))$ and $\widehat{h}_{f_{t_i}}(b(t_i))$ tend to zero as $i \to \infty$. Then for $\mu_{f,a}$-a.e. t in $\Lambda(\mathbb{C})$, the pair $a(t), b(t)$ satisfies the $\mathrm{AS}(V_t)$ condition for f_t.*

Remark 3.13. By Theorem 2.2 and [GV19, Proposition 13], $\mu_{f,a}$ and $\mu_{f,b}$ are of finite non-zero mass. Moreover, by Theorem 2.3, $\mu_{f,a}$ and $\mu_{f,b}$ are proportional.

Combing Corollary 3.8, Remark 3.9, Remark 3.10 with Proposition 3.11, we get the following result.

Corollary 3.14. *Let f, a, b as in Theorem 3.12. Assume further that a, b are not dynamically related. Then for $\mu_{f,a}$-a.e. t in $\Lambda(\mathbb{C})$, we have that for every $\Gamma_t \in \mathrm{Corr}(\mathbb{P}^1)_*^{f_t}$, the pair $a(t), b(t)$ satisfies the $\mathrm{AS}(\Gamma_t)$ condition for f_t.*

Proof of Theorem 3.12. Let $P_i : \Lambda \times (\mathbb{P}^1 \times \mathbb{P}^1) \to \Lambda \times \mathbb{P}^1, i = 1, 2$ be the morphism defined by $(t, x_1, x_2) \mapsto (t, x_i)$. Let $\pi_1 := \Lambda \times (\mathbb{P}^1 \times \mathbb{P}^1) \to \Lambda$ be the first projection. Let $T := P_1^* T_f + P_2^* T_f$. Set $g := f \times_\Lambda f$. Since $\mu_{f,a}$ does not have atomic point, we may assume that V is of dimension 2 and is flat over Λ. Let B be a smooth projective curve containing Λ as an open subset. Then $W := \overline{V}$ is a Cartier divisor of $B \times (\mathbb{P}^1 \times \mathbb{P}^1)$.

Let g_W be a Green function on for W i.e. g_W is a continuous function on $B \times (\mathbb{P}^1 \times \mathbb{P}^1)(\mathbb{C})$ such that for every point $y \in W(\mathbb{C})$, there is an open neighborhood U of y such that $g_W = -\log |h| + O(1)$ where W is defined by $h = 0$ in U. We may assume that $g_W \geq 0$. The following lemma is the key to the proof.

Lemma 3.15. *There is $C > 0$ such that for every $n \geq 0$,*

$$\int g_W \, T \wedge [\Gamma_n] \leq C d^n.$$

Since the proof of Lemma 3.15 is based on the arithmetic intersection theory, we postpone it to the end of this section. We indeed prove a more general result which implies Lemma 3.15 as a direct consequence.

For every $t \in \Lambda(\mathbb{C})$, $g_{W_t} := g_W|_{\pi_1^{-1}(t)}$ is a Green function of $W_t = V_t$. Then we have

$$Cd^n \geq \int g_W \, P_1^* T \wedge \Gamma_n = d^n \int g_{W_t}(p_n(t)))\mu_{f,a}.$$

Hence $\int g_{W_t}(p_n(t)))\mu_{f,a} \leq C$. Set $\phi_n(t) := \max\{-\log d(p_n(t), V), 0\}$. There is $C_1 \geq 1$ such that

$$\max\{-\log d(p_n(t), V), 0\} \leq C_1(g_{W_t}(p_n(t))) + 1).$$

So there is $C_2 > 0$ such that for every $n \geq 0$, $\int \phi_n(t)\mu_{f,a} \leq C_2$. Set $s_n := \frac{1}{n}\sum_{i=0}^{n-1} \phi_n$, we have $\int s_n \mu_{f,a} \leq C_2$. According to Fatou's lemma,

$$\int (\liminf_{n\to\infty} s_n)\mu_{f,a} \leq \liminf_{n\to\infty} \int s_n \mu_{f,a} \leq C_2.$$

So for $\mu_{f,a}-a.e.\ t \in \Lambda(\mathbb{C})$, we have $\liminf_{n\to\infty} s_n(t) < +\infty$ which concludes the proof. □

Chern-Levine-Nirenberg inequality. Recall the Chern-Levine-Nirenberg inequality from pluripotential theory, see [DS10, Theorem A.31], [CLN69] and [Dem, (3.3)].

Theorem 3.16. *Let (X, ω) be a Hermitian manifold. Let S be a positive closed current of bi-dimension $(1, 1)$ on X. Let u be a locally bounded p.s.h. function on X and K a compact subset of X. Then there is a constant $c = c(X, K) > 0$ such that if v is p.s.h. on X, then*

$$\int_K |v| \, dd^c u \wedge S \leq c \left(\int_X |v| \, S \wedge \omega \right) \|u\|_{L^\infty(X)}.$$

Recall that a function v is called quasi-p.s.h. if there is $C > 0$ such that $C\omega + dd^c v \geq 0$. Then we have the following two variants of the Chern-Levine-Nirenberg inequality.

Corollary 3.17. *Let (X, ω) be a Kähler manifold. Let S be a positive closed current of bi-dimension $(1, 1)$ on X. Let v be a quasi-p.s.h. function on X. Then there is a constant $c'' = c''(X, K, v) > 0$ such that if u be a locally bounded p.s.h. function on X, then*

$$\int_K |v| \, dd^c u \wedge S \leq c' \|u\|_{L^\infty(X)}.$$

Proof. Assume that $C\omega + dd^c v \geq 0$. For every $x \in K$, there is a compact neighborhood Z_x of x and an open neighborhood U_x of Z_x such that $\omega|_{U_x} = dd^c w_x$ where w_x is a bounded continuous p.s.h. functions. There is a constant $B_x \geq 0$, such that $-B_x \leq w_x \leq B_x$ on U_x. There is a finite set $F \subseteq K$ such that $K \subseteq \cup_{x\in F} Z_x$. Set $v'_x := v + Cw_x + B_x$

and $v''_x := v + Cw_x - B_x$. Both v'_x and v_x are locally bounded p.s.h. on U_x and $v'_x \leq v \leq v''_x$. By Theorem 3.16, we have

$$\begin{aligned}\int_K |v|\, dd^c u \wedge S &\leq \sum_{x\in F} \int_{K\cap Z_x} |v|\, dd^c u \wedge S \\ &\leq \sum_{x\in F} \int_{K\cap Z_x} (|v'_x| + |v''_x|)\, dd^c u \wedge S \\ &\leq \left(\sum_{x\in F} c_{K\cap Z_x, U_x} \int_{U_x} (|v'_x| + |v''_x|)\, \omega \wedge S \right) \|u\|_{L^\infty(X)},\end{aligned}$$

which concludes the proof. □

Since every closed positive $(1,1)$-current having continuous potential locally takes form $dd^c u$ for some continuous p.s.h. function u, the proof of the following result is similar to the proof of Corollary 3.17.

Corollary 3.18. *Let (X, ω) be a Hermitian manifold. Let S be a positive closed current of bi-dimension $(1,1)$ on X. Let T be a closed positive $(1,1)$-current having continuous potential. Then there is a constant $c' = c'(X, K, T) > 0$ such that if v is a p.s.h. on X, then*

$$\int_K |v|\, T \wedge S \leq c' \left(\int_X |v|\, S \wedge \omega \right).$$

Integrations via arithmetic intersection theory. This section is based on the theory of adelic line bundles on quasi-projective varieties developed by Yuan and Zhang in [YZ21]. We follow their notations and terminologies.

Let S be a smooth curve over a number field K. Let S' be a smooth projective curve containing S as an open subset. Let $\pi' : X' \to S'$ be a projective model of $\pi : X \to S$. Set $D_S := S' \setminus S$ and $\overline{D_S} = (D_S, g_{D_s})$ be an ample arithmetic divisor associated with D_S, where g_{D_s} is a continuous Green function. Let $D := \pi'^{-1}(D_S)$ and $\overline{D} := \pi^* \overline{D_S}$. Write $\overline{D} = (D, g_D)$ where $g_D = \pi^* g_{D_s}$.

Let (X, g, L) be a *polarized dynamical system* over S i.e.

(1) $\pi : X \to S$ is a projective and flat scheme over S;
(2) $g : X \to X$ is an endomorphism over S;
(3) $L \in \mathrm{Pic}(X)$ is a line bundle, relatively ample over S, such that $g^* L = qL$ for some integer $q > 1$.

In [YZ21, Theorem 6.1.1], by Tate's limiting argument, Yuan and Zhang constructed an adelic line bundle $\overline{L}_g \in \widehat{\mathrm{Pic}}(X)$ extending L such that $g^* \overline{L}_g = q \overline{L}_g$. Moreover, it is strongly nef in the following sense:

There is a sequence of nef adelic line bundles $\overline{L_n}, n \geq 1$ which are defined over some projective models $\pi_n : X'_n \to S'$ of $\pi : X \to S$, and positive numbers $\varepsilon_n, n \geq 0$ tending to 0, such that $\overline{L_g} - \overline{L_n}$ is represented by an arithmetic divisor $\overline{D_n} = (D_n, g_{D_n}) \in \widehat{\mathrm{Pic}}(X)_{int}$ with

$$-\varepsilon_n \overline{D} \leq \overline{D_n} \leq \varepsilon_n \overline{D}.$$

We may assume that for every $n \geq 0$, X'_n dominates X'. For every archimedean place $v \in \mathcal{M}_K$, $c_1(\overline{L}_g)_v$ is a positive $(1,1)$-current on X_v^{an} having following properties

(i) $c_1(\overline{L}_g)_v$ has continuous potentials;
(ii) for every $t \in \Lambda$, $c_1(\overline{L}_g)_v|_{X_t}$ is a Green current for g_t;
(iii) $\int c_1(\overline{L}_g)_v^{\dim X} = 0$.

Let $p : S \to X$ be a section of π and let Γ_p be its image. Let $\overline{A}$ be an ample adelic line bundle defined on X', let $A \in \mathrm{Pic}(X')$ be the line bundle associated with $\overline{A}$.

Proposition 3.19. *Let s be a small section of $A \in \mathrm{Pic}(X')$ i.e. for every $v \in M_K$ and $x \in X_v'^{\mathrm{an}}$, $|s(x)|_v \leq 1$. Assume that $\Gamma_p \not\subseteq \mathrm{div}(s)$. Then for every archimedean place $v \in \mathcal{M}_K$, we have*

$$\int g_{\mathrm{div}(s),v}\, c_1(\overline{L}_g)_v \wedge [\Gamma_p] \leq \overline{A} \cdot \overline{L}_g \cdot \Gamma_p,$$

where $g_{\mathrm{div}(s),v}(x) = -\log |s(x)|_v$ is the Green function.

Proof of Proposition 3.19. Let Y be any compact subset of X_v^{an}. Pick U_Y an open neighborhood of Y such that $U_Y \subset\subset X_v^{\mathrm{an}}$.

Since $g_{\mathrm{div}(s),v}$ is a quasi-p.s.h. function, by Corollary 3.17, there is a constant $c = c(Y, U_Y, g_{\mathrm{div}(s),v})$ such that for every $n \geq 0$, we have

$$\int_Y g_{\mathrm{div}(s),v} dd^c g_{D_n,v} \wedge [\Gamma_p] \leq c \|g_{D_n,v}\|_{L^\infty(U_Y)} \leq \varepsilon_n c \|g_{D,v}\|_{L^\infty(U_Y)}.$$

Then for every $n \geq 0$, we have

$$\begin{aligned} &\int_Y g_{\mathrm{div}(s),v} c_1(\overline{L}_g)_v \wedge [\Gamma_p] \\ &= \int_Y g_{\mathrm{div}(s),v} c_1(\overline{L_n})_v \wedge [\Gamma_p] + \int_Y g_{\mathrm{div}(s),v} dd^c g_{D_n,v} \wedge [\Gamma_p] \\ &\leq \int_Y g_{\mathrm{div}(s),v} c_1(\overline{L_n})_v \wedge [\Gamma_p] + \varepsilon_n c \|g_{D,v}\|_{L^\infty(U_Y)}. \end{aligned} \tag{3.1}$$

Since X_n' dominates X', we may view s as a section on X_n. Let $\Gamma_{p,n}$ be the Zariski closure of Γ_p in X_n. By [CLT09, Theorem 1.4] (see also [YZ17, Page 1161]), we have

$$\overline{A}\cdot\overline{L_n}\cdot\Gamma_{p,n}=\overline{L_n}\cdot(\Gamma_{p,n}\cdot\mathrm{div}s)+\sum_{w\in\mathcal{M}_k}\int g_{\mathrm{div}(s),w}c_1(\overline{L_n})_w\wedge[\Gamma_p].$$

Since each term on the right hand side is positive, we get

$$\int_Y g_{\mathrm{div}(s),v}c_1(\overline{L_n})_v\wedge[\Gamma_p]\leq\int g_{\mathrm{div}(s),v}c_1(\overline{L_n})_v\wedge[\Gamma_p]\leq\overline{A}\cdot\overline{L_n}\cdot\Gamma_{p,n}. \tag{3.2}$$

The definition of $\overline{A}\cdot\overline{L}_g\cdot\Gamma_p$ shows that

$$\overline{A}\cdot\overline{L}_g\cdot\Gamma_p=\lim_{n\to\infty}\overline{A}\cdot\overline{L_n}\cdot\Gamma_{p,n}. \tag{3.3}$$

As $\varepsilon_n\to 0$, we conclude the proof by (3.1), (3.2) and (3.3). □

Set $\Gamma_n:=\Gamma_{g^n(p)}$.

Proposition 3.20. *Assume that* $\overline{L_g}^2\cdot\Gamma_p=0$*, then for every ample adelic line bundle* $\overline{A}$ *defined on* X'*, there is* $C>0$ *such that for every* $n\geq 0$*,*

$$\overline{A}\cdot\overline{L_g}\cdot\Gamma_n\leq Cq^n,$$

where q *is the integer in the definition (3) of the polarized dynamical system* (X,g,L)*.*

Proof. For every $n\geq 0$, by projection formula, we have

$$\overline{L_g}^2\cdot\Gamma_n=\overline{L_g}^2\cdot g_*^n(\Gamma_p)=g^{*n}(\overline{L_g})^2\cdot\Gamma_p=q^{2n}\overline{L_g}^2\cdot\Gamma_p=0. \tag{3.4}$$

As L is relatively ample over S, after replacing X' by some projective model X'' which dominates X' and X_0', L by some multiple of it and $\overline{A}$ by $\overline{A}+\overline{A'}$ for some ample adelic line bundle $\overline{A'}$ on X'', we may assume that L extends to a line bundle on X'. We now view L and L_0 as line bundles on X'.

There is an integer $m_1>0$ and an ample line bundle M on S' such that $E:=m_1L+M-A$ is ample. Pick a suitable metric on E, we get an ample adelic line bundle $\overline{E}$ extending E. After replacing $\overline{A}$ by $\overline{A}+\overline{E}$, we may assume that $A=m_1L+M$ on X'. Since m_1L_0-A is trivial on the generic fiber of $\pi':X'\to S'$, $m_1\overline{L_0}-\overline{A}$ is represented by an arithmetic divisor $\overline{F}$ on X' such that $\pi(\mathrm{supp}\,F)\neq S'$. Then there is an ample arithmetic divisor $\overline{Q}$ on S' such that

$$-\pi^*\overline{Q}\leq\overline{F}\leq\pi^*\overline{Q}.$$

Set $R := \varepsilon_0\overline{D_S} + \overline{Q}$. By (3.4), we get

$$\begin{aligned}\overline{A}\cdot\overline{L}_g\cdot\Gamma_n =&(m_1\overline{L_g} - m_1\overline{D_0} - \overline{F})\cdot\overline{L}_g\cdot\Gamma_n\\ \leq& m_1\overline{L_g}^2\cdot\Gamma_n + \pi^*\overline{R}\cdot\overline{L}_g\cdot\Gamma_n\\ =&\pi^*\overline{R}\cdot\overline{L}_g\cdot\Gamma_n\\ =&\pi^*\overline{R}\cdot\overline{L}_g\cdot g^n_*(\Gamma_p)\\ =&(g^n)^*\pi^*\overline{R}\cdot(g^n)^*\overline{L}_g\cdot\Gamma_p\\ =&q^n(\pi\circ g^n)^*\overline{R}\cdot\overline{L}_g\cdot\Gamma_p\\ =&q^n(\pi^*\overline{R}\cdot\overline{L}_g\cdot\Gamma_p),\end{aligned}$$

which concludes the proof. □

Combining Proposition 3.19 and Proposition 3.20, we get the following result.

Corollary 3.21. *Let V be a Zariski closed subset of X'. Let $v \in \mathcal{M}_K$ be an archimedean place. Let g_V be a Green function of V. Assume that $\overline{L_g}^2\cdot\Gamma_p = 0$. Then there is $C > 0$ such that, for every $n \geq 0$, if Γ_n is not contained in V, we have*

$$\int g_V\, c_1(\overline{L}_g)_v\wedge[\Gamma_n] \leq Cq^n,$$

where q is the integer in the definition (3) of the polarized dynamical system (X, g, L).

Proof. Set $N := \{n \geq 0 |\ \Gamma_n \not\subseteq V\}$. Let I_V be the ideal sheaf of V in X'. We fix an ample adelic line bundle $\overline{A}$ defined on X', and we denote $A \in \mathrm{Pic}(X')$ to be the line bundle associated with $\overline{A}$. By replacing $\overline{A}$ by a suitable multiple, we may assume that $A\otimes I_V$ is generated by global sections. Then there are sections $s_1, \ldots, s_m$ of A such that $\cap_{i=1}^m \mathrm{div}(s_i) = V$. For every $n \in N$, there is $i_n \in \{1, \ldots, m\}$ such that $\Gamma_n \not\subseteq \mathrm{div}(s_{i_n})$. After modifying the metric of A, we may assume that for every $i = 1, \ldots, m$, s_i is small for $\overline{A}$ and $g_{\mathrm{div}(s_i),v} \geq g_V$. By Proposition 3.19 and Proposition 3.20, there is $C > 0$ such that for every $n \in N$, we have

$$\int g_V\, c_1(\overline{L}_g)_v\wedge[\Gamma_n] \leq \int g_{\mathrm{div}(s_{i_n})}\, c_1(\overline{L}_g)_v\wedge[\Gamma_n] \leq \overline{A}\cdot\overline{L}_g\cdot\Gamma_n \leq Cq^n,$$

which concludes the proof. □

Proof of Lemma 3.15. Since Λ, f, a, b and V are defined over $\overline{\mathbb{Q}}$, there is a number field K, a smooth curve S over K and a polarized endomorphism $h : \mathbb{P}^1_S \to \mathbb{P}^1_S$ over S of degree d, a closed subset W of X and a section $p : S \to X := (\mathbb{P}^1 \times \mathbb{P}^1) \times S$ of $\pi : X \to S$ such that Λ, f, $(a, b) : \Lambda \to (\mathbb{P}^1 \times \mathbb{P}^1) \times \Lambda$ and V are the base change of S, h and p via an embedding $K \hookrightarrow \mathbb{C}$ defined by an archimedean place $v \in \mathcal{M}_K$. Let S' be a smooth projective curve containing S as a Zariski closed subset. We still denote by W by its Zariski closure in $X' := (\mathbb{P}^1 \times \mathbb{P}^1) \times S'$. Let $L := \pi_1^* O(1) + \pi_2^* O(1)$ where $\pi_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ is the i-th projection. Set $g := h \times_{S'} h$. Let Γ_n be the image of $g^n(p)$. Then we have $T = c_1(\overline{L_g})_v$. Let $\widetilde{L_g} \in \widetilde{\mathrm{Pic}(X)}$ be the geometric part of $\overline{L_g}$. By [YZ21, Lemma 5.4.4], we have

$$\widetilde{L_v} \cdot \Gamma_0 = \int c_1(\overline{L_g})_v \wedge \Gamma_0 = \int T \wedge \Gamma_0 = (\mu_{f,a} + \mu_{f,b})(\Lambda) > 0.$$

As Γ_0 contains an infinite sequence $(a(t_i), b(t_i)), i \geq 0$ of distinct points with heights tend to 0, the fundamental inequality [YZ21, Theorem 5.3.2] shows that $\overline{L_v}^2 \cdot \Gamma_0 = 0$. We conclude the proof by Corollary 3.21. □

4. Typical non-uniformly hyperbolic conditions

In this section, we show that certain non-uniformly hyperbolic conditions are typical for the bifurcation measure. We begin with some definitions. Let f be a holomorphic family of rational maps as in (2.1) and a be a marked point. In this section, the distance and the norm of the derivatives are computed with respect to the metrics induced by ω_Λ and $\omega_{\mathbb{P}^1}$.

For every $n \geq 0$, let $\xi_{a,n} : \Lambda \to \mathbb{P}^1(\mathbb{C})$ denote the map $\xi_{a,n}(t) := f_t^n(a(t))$.

Definition 4.1. A parameter $t_0 \in \Lambda$ is called

(i) *Marked Collet-Eckmann* $\mathrm{CE}^*(\lambda)$ for some $\lambda > 1$, if there exists $C > 0$ and $N > 0$ such that
$$|df_{t_0}^n(f_{t_0}^N(a(t_0)))| \geq C\lambda^n$$
for every $n \geq 0$.

(ii) *Parametric Collet-Eckmann* $\mathrm{PCE}(\lambda)$ for some $\lambda > 1$, if there exists $C > 0$ such that
$$\left|\frac{d\xi_{a,n}}{dt}(t_0)\right| \geq C\lambda^n$$
for every $n \geq 0$.

(iii) *Polynomial Recurrence* $\mathrm{PR}(s)$ for some $s > 1/2$, if there exists an integer $N > 0$ such that
$$d(f_{t_0}^n(a(t_0)), \mathcal{C}_{t_0}) \geq n^{-s}$$

for every $n \geq N$. Where $\mathcal{C}_{t_0}$ is the critical set of f_{t_0}

Lemma 4.2. *Let f be a holomorphic family of rational maps as in (2.1) and a be a marked point. Assume $t \in \operatorname{supp} \mu_{f,a}$, then there exists $N > 0$ such that the f_t-orbit of $f_t^N(a(t))$ does not intersect $\mathcal{C}_t$.*

Proof. Assume by contradiction that there is no such N. Since the cardinality of $\mathcal{C}_t$ is finite, there exists $c \in \mathcal{C}_t$ such that c is f_t-periodic and $a(t)$ is a preimage of c. This implies $a(t)$ is contained in the super-attracting basin of the f_t-orbit of c. Since the attracting basin is stable under perturbation, it implies that a is stable at t, contradicts to $t \in \operatorname{supp} \mu_{f,a}$ by Theorem 2.2. □

The following result was essentially due to De Thélin-Gauthier-Vigny.

Theorem 4.3 (De Thélin-Gauthier-Vigny, [DTGV21])**.** *Let f be an algebraic family of rational maps as in (2.1) and a be a marked point. Then we have:*

(i) *For every $1 < \lambda < d^{1/2}$, the condition* PCE(λ) *is typical with respect to $\mu_{f,a}$;*
(ii) *Assume t satisfies* PCE(λ_0) *for some $\lambda_0 > 1$, then t satisfies* CE$^*(\lambda)$ *for every $1 < \lambda < \lambda_0$.*

Proof. The statement (i) was proved in [DTGV21, Theorem 3]. In [DTGV21, Proposition 9], it is proved that if t satisfies PCE(λ_0) for some $\lambda_0 > 1$, then t satisfies CE$^*(\lambda)$ for every $1 < \lambda < \lambda_0$, provided that the f_t-orbit of $a(t)$ does not intersect $\mathcal{C}_t$. The condition PCE(λ_0) implies $t \in \operatorname{supp} \mu_{f,a}$, by Lemma 4.2, we get that (ii) holds. □

4.1. **A transversality statement.** Let f be a holomorphic family of rational maps as in (2.1) and a be a marked point. A direct computation gives the following equality, relating $df_{t_0}^n(a(t_0))$ and $d\xi_{a,n}/dt(t_0)$. A proof can be found in [AGMV19, Lemma 4.4]. We set $F(t,z) := f_t(z)$.

Lemma 4.4. *Let $t_0 \in \Lambda$ such that the f_{t_0}-orbit of $a(t_0)$ does not intersect $\mathcal{C}_{t_0}$. Then for every $n \geq 1$, we have*

$$\frac{d\xi_{a,n}}{dt}(t_0) = df_{t_0}^n(a(t_0)) \left(\frac{da}{dt}(t_0) + \sum_{k=0}^{n-1} \frac{\frac{\partial F}{\partial t}(t_0, f_{t_0}^k(a(t_0)))}{df_{t_0}^{k+1}(a(t_0))} \right).$$

The following transversality statement holds for PCE parameters.

Lemma 4.5 (Transversality condition)**.** *Let $t_0 \in \Lambda$ such that the f_{t_0}-orbit of $a(t_0)$ does not intersect $\mathcal{C}_{t_0}$, and t_0 satisfies* PCE(λ) *for some $\lambda > 1$. Then there exists a non-zero $\gamma \in \mathbb{C}$ such that*

$$\frac{da}{dt}(t_0) + \sum_{k=0}^{\infty} \frac{\frac{\partial F}{\partial t}(t_0, f_{t_0}^k(a(t_0)))}{df_{t_0}^{k+1}(a(t_0))} = \gamma.$$

Proof. By Theorem 4.3 (ii), t_0 satisfies CE$^*(\lambda_1)$ for every $1 < \lambda_1 < \lambda$. Since $|\partial F/\partial t(t_0, f_{t_0}^k(a(t_0))|$ is uniformly bounded by a constant $M > 0$, the power seris in Lemma 4.5 converges. It remains to show that it converges to a non-zero number.

Let $\chi > 0$ be the lower Lyapunov exponent of f_{t_0} at $a(t_0)$. Let $\varepsilon > 0$ small such that $\varepsilon < \min(\log \lambda/10, \chi)$. By the definition of lower Lyapunov exponent, there exists a constant $C_1(\varepsilon) > 0$ and a sequence of positive integers $n_j \to +\infty$ such that

$$|df_{t_0}^{n_j}(a(t_0))| \leq C_1 e^{(\chi+\varepsilon)n_j}. \tag{4.1}$$

There is also a constant $C_2(\varepsilon) > 0$ such that for every $n \geq 1$,

$$\begin{aligned} \left|\sum_{k=n+1}^{\infty} \frac{\frac{\partial F}{\partial t}(t_0, f_{t_0}^k(a(t_0)))}{df_{t_0}^{k+1}(a(t_0))}\right| &\leq \sum_{k=n+1}^{\infty} \left|\frac{\frac{\partial F}{\partial t}(t_0, f_{t_0}^k(a(t_0)))}{df_{t_0}^{k+1}(a(t_0))}\right| \\ &\leq \sum_{k=n+1}^{\infty} \frac{M}{C_2 e^{(\chi-\varepsilon)k}} \\ &:= C_3 e^{-(\chi-\varepsilon)n}, \end{aligned} \tag{4.2}$$

where $C_3(\varepsilon) > 0$ is a constant.

Assume by contradiction that the power seris in Lemma 4.5 converges to 0. By Lemma 4.4, (4.1) and (4.2), for each n_j in (4.1) we have

$$\begin{aligned} \left|\frac{d\xi_{a,n_j}}{dt}(t_0)\right| &= \left|df_{t_0}^{n_j}(a(t_0))\right| \left|\sum_{k=n_j+1}^{\infty} \frac{\frac{\partial F}{\partial t}(t_0, f_{t_0}^k(a(t_0)))}{df_{t_0}^{k+1}(a(t_0))}\right| \\ &\leq C_1 e^{(\chi+\varepsilon)n_j} C_3 e^{-(\chi-\varepsilon)n_j} \\ &= C_1 C_3 e^{2\varepsilon n_j}, \end{aligned}$$

which contradicts to the fact t_0 satisfies PCE(λ). □

4.2. **Polynomial Recurrence parameters are typical.** In this subsection, we prove the following:

Theorem 4.6. *Let f be an algebraic family of rational maps as in (2.1) and a be a marked point. Then for every $s > 1/2$, $\mu_{f,a}$-a.e. point $t_0 \in \Lambda$ satisfies* PR(s).

There is a finite Zariski open cover $\mathcal{W}$ of Λ, such that for every $W \in \mathcal{W}$ and every marked critical point c, there is an algebraic family $g_{W,c} : W \to \mathrm{PGL}_{2,\mathbb{C}}$ such that for every $t \in W(\mathbb{C})$, we have

$$(g_{W,c}(t))(c(t)) = 0 \in \mathbb{P}^1(\mathbb{C}).$$

Since we only need to prove Theorem 4.6 for the restriction of f on each $W \in \mathcal{W}$. We may assume that $\mathcal{W} = \{\Lambda\}$ and write g_c for $g_{\Lambda,c}$.

For every marked critical point c, define an algebraic automorphism $\sigma_c : \Lambda \times \mathbb{P}^1 \to \Lambda \times \mathbb{P}^1$ sending (t, z) to $(t, (g_c(t))(z))$. It is clear that $\sigma_c(\Gamma_c) = \Lambda \times \{0\}$. Recall that $\omega_1 = \pi_1^* \omega_\Lambda$ and $\omega_2 = \pi_2^* \omega_{\mathbb{P}^1}$. Set $\omega := \omega_1 + \omega_2$. For every closed algebraic curve $V \subseteq \Lambda \times \mathbb{P}^1$, if $\pi_2(V)$ is not a single point, then for every $z \in \mathbb{P}^1(\mathbb{C})$, we have $\#(\pi_2|_V)^{-1}(z) \leq \deg_{\omega_2} V$. The equality holds for all but finitely many $z \in \mathbb{P}^1(\mathbb{C})$. There is a constant $C_0 > 0$ such that for every marked critical point c and every closed algebraic curve $V \subseteq \Lambda \times \mathbb{P}^1$, we have

$$\deg_\omega \sigma_c(V) \leq C_0 \deg_\omega V. \tag{4.3}$$

Note that we always have $\deg_\omega V \geq 1$. Let a be a marked point, we have $\deg_{\omega_1} \Gamma_a = 1$, hence $\deg_\omega \Gamma_a = \deg_{\omega_2} \Gamma_a + 1$. Since $\sigma_c(\Gamma_a)$ is the graph of the marked point $\sigma(a) : t \in \Lambda \to (g_c(t))(a(t)) \in \mathbb{P}^1$, we have $\deg_\omega \sigma_c(\Gamma_a) = \deg_{\omega_2} \sigma_c(\Gamma_a) + 1$. By (4.3), we get

$$\deg_{\omega_2} \sigma_c(\Gamma_a) \leq \deg_\omega \sigma_c(\Gamma_a) \leq C_0 \deg_\omega \Gamma_a. \tag{4.4}$$

There is $D > 0$ such that for every marked point a,

$$\deg_\omega \Gamma_{f(a)} \leq d \deg_\omega \Gamma_a + D \tag{4.5}$$

where $d := \deg f \geq 2$. It follows that for every $n \geq 0$,

$$\deg_\omega \Gamma_{f^n(a)} \leq \deg_\omega \Gamma_{f^n(a)} + D \leq d^n(\deg_\omega \Gamma_a + D). \tag{4.6}$$

We set

$$\phi(t, z) := d(z, \mathcal{C}_t)^{-1},$$

where $\mathcal{C}_t$ is the critical set of f_t.

Lemma 4.7. *Let K be compact subset of Λ and $0 < \alpha < 2$. Then there exists a constant $C := C(K, \alpha) > 0$ such that for every marked point a which is not a marked critical point and for every $A > 0$ we have*

$$\int_{K \times \mathbb{P}^1(\mathbb{C})} \min(A, \phi^\alpha)\, T_f \wedge [\Gamma_a] \leq C(A + \deg_\omega \Gamma_a).$$

Proof. Pick an open neighborhood U of K which is relatively compact in Λ. There is a constant $B_1 = B_1(U) > 1$ such that for every marked critical point c and $(t, z) \in \pi_1^{-1}(U)$, we have

$$B_1^{-1} d((g_c(t))(z), 0) \leq d(z, c(t)) \leq B_1 d((g_c(t))(z), 0). \tag{4.7}$$

There is $\delta_0 > 0$ and $B_2 > 1$, such that for every $z \in \mathbb{P}^1(\mathbb{C})$ with $d(z,0) \leq \delta_0$, we have

$$B_2^{-1}|z| \leq d(z,0) \leq B_2|z|. \tag{4.8}$$

Set

$$\Omega_c := \{(t,z) : t \in U \text{ and } d(z,c(t)) < \delta_0\}.$$

Set $B_0 := B(0, B_1\delta_0)$, then we have

$$\sigma_c(\Omega_c) \subseteq U \times B_0.$$

For every marked critical point c, set $\phi_c(t,z) := d(z,c(t))^{-1}$. Then we have $\phi = \max_c \phi_c$ where c is taken over all the $2d-2$ marked critical points. Set $B_3 := B_1B_2$. By (4.7) and (4.8), for every $(t,z) \in \sigma_c(\Omega_c)$,

$$\phi_c \circ \sigma_c^{-1}(t,z) \leq B_3|z|^{-1}.$$

Set $V_c := \sigma_c(\Gamma_a)$. We have

$$\begin{aligned}
&\int_{K\times\mathbb{P}^1(\mathbb{C})} \min(A,\phi^\alpha)\, T_f \wedge [\Gamma_a] \\
=&\int_{K\times\mathbb{P}^1(\mathbb{C})} \max_c \min(A,\phi_c^\alpha)\, T_f \wedge [\Gamma_a] \\
\leq&\sum_c \int_{K\times\mathbb{P}^1(\mathbb{C})} \min(A,\phi_c^\alpha)\, T_f \wedge [\Gamma_a] \\
=&\sum_c \int_{(K\times\mathbb{P}^1(\mathbb{C}))\setminus\Omega_c} \min(A,\phi_c^\alpha)\, T_f \wedge [\Gamma_a] + \\
&\sum_c \int_{\sigma_c(((K\times\mathbb{P}^1(\mathbb{C}))\cap\Omega_c)} \min(A,\phi_c^\alpha\circ\sigma_c^{-1})\, \sigma_{c*}(T_f) \wedge [V_c] \\
\leq&(2d-2)\delta_0^{-\alpha}\int_{K\times\mathbb{P}^1(\mathbb{C})} T_f \wedge [\Gamma_a] + \\
&\sum_c \int_{K\times B_0} \min(A, B_3^\alpha|z|^{-\alpha})\, \sigma_{c*}(T_f) \wedge [V_c].
\end{aligned}$$

Since T_f has continuous potential, by Corollary 3.18, there is a constant c' depending on K such that

$$\int_{K\times\mathbb{P}^1(\mathbb{C})} T_f \wedge [\Gamma_a] \leq c' \deg_\omega \Gamma_a. \tag{4.9}$$

So we only need to show that for every marked critical point c,

$$\int_{\overline{K\times B_0}} \min(A, B_3^\alpha|z|^{-\alpha})\, \sigma_{c*}(T_f) \wedge [V_c] \leq C_1(A + \deg_\omega \Gamma_a)$$

for some constant C_1 depending on K. Since

$$\min(A, B_3^\alpha |z|^{-\alpha}) = e^{-(\max(-\log A, \alpha \log |z| - \alpha \log B_3))}$$

is a p.s.h. function, and $\sigma_{c*}(T_f)$ has continuous potential, by Corollary 3.18, there is a constant C' such that

$$\begin{aligned}&\int_{\overline{K\times B_0}} \min(A, B_3^\alpha |z|^{-\alpha})\, \sigma_{c*}(T_f) \wedge [V_c] \\ \leq & C' \left(\int_{U\times 2B_0} \min(A, B_3^\alpha |z|^{-\alpha})\, \omega \wedge [V_c] \right)\end{aligned}$$

We only need to bound $\int_{U\times 2B_0} \min(A, B_3^\alpha |z|^{-\alpha})\, \omega \wedge [V_c]$. It suffices to show

$$\int_{U\times 2B_0} \min(A, B_3^\alpha |z|^{-\alpha})[V_c] \wedge \omega_1 \leq C_2 A \tag{4.10}$$

and

$$\int_{U\times 2B_0} \min(A, B_3^\alpha |z|^{-\alpha})[V_c] \wedge \omega_2 \leq C_3 \deg_\omega \Gamma_a \tag{4.11}$$

for some constants $C_2 > 0$ and $C_3 := C_3(\alpha) > 0$.

Since V_c is a graph, we have

$$\int_{U\times 2B_0} \min(A, B_3^\alpha |z|^{-\alpha})[V_c] \wedge \omega_1 \leq A \int_{U\times 2B_0} [V_c] \wedge \omega_1 = A \int_U \omega_\Lambda,$$

hence (4.10) is true. On the other hand, since a is not a marked critical point and the function $|z|^{-\alpha}$ is Lebesgue integrable when $0 < \alpha < 2$, we have

$$\begin{aligned}\int_{U\times 2B_0} \min(A, B_3^\alpha |z|^{-\alpha})[V_c] \wedge \omega_2 \leq & B_3^\alpha \int_{U\times 2B_0} |z|^{-\alpha}[V_c] \wedge \omega_2 \\ \leq & B_3^\alpha \int_{2B_0} |z|^{-\alpha}(\pi_{2*}[V_c]) \wedge \omega_{\mathbb{P}^1}.\end{aligned}$$

By (4.4), $\deg_{\omega_2} V_c \leq C_0 \deg_\omega \Gamma_a$, we have

$$\int_{2B_0} |z|^{-\alpha}(\pi_{2*}[V_c]) \wedge \omega_{\mathbb{P}^1} \leq C_0 \deg_\omega \Gamma_a \int_{2B_0} |z|^{-\alpha} \omega_{\mathbb{P}^1},$$

hence (4.11) is true. This finishes the proof. $\square$

Let a be a marked point. For every $n \geq 0$, set $\psi_n(t) := \phi(t, f_t^n(a(t)))$. Recall that $\mu_{f,a} = (\pi_1)_*(T_f \wedge [\Gamma_a])$. We note that $\mu_{f,f^n(a)} = d^n \mu_{f,a}$ for every $n \geq 0$. A direct corollary of Lemma 4.7 and Inequality (4.6) is the following:

Corollary 4.8. *Let K be compact subset of U and $0 < \alpha < 2$. Then there exists a constant $C := C(K, \alpha) > 0$ such that for every $n \geq 0$, we have*

$$\int_K \min(A, \psi_n^\alpha)\, d\mu_{f,a} \leq C(d^{-n}A + \deg_\omega \Gamma_a + D).$$

Lemma 4.9. *Let K be compact subset of U and $0 < \alpha < 2$. Then there exists a constant $C(K, \alpha) > 0$ such that for every $n \geq 0$, we have*

$$\mu_{f,a}(t \in K : \psi_n^\alpha(t) \geq A) \leq C(d^{-n} + A^{-1}(\deg_\omega \Gamma_a + D)).$$

Proof. By Markov inequality and Corollary 4.8 we have

$$\begin{aligned}\mu_{f,a}(t \in K : \psi_n^\alpha \geq A) &= \mu_{f,a}(t \in K : \min(A, \psi_n^\alpha) \geq A)\\ &\leq A^{-1} \int_K \min(A, \psi_n^\alpha)\, d\mu_{f,a}\\ &\leq C(d^{-n} + A^{-1}(\deg_\omega \Gamma_a + D)).\end{aligned}$$

□

Proof of Theorem 4.6. It suffices to prove for every $s > 1/2$ and for every compact subset $K \subseteq \Lambda$, there exists a set $E(K, s) \subseteq K$ satisfying $\mu_{f,a}(K \setminus E) = 0$ such that every $t_0 \in E$ satisfies $\mathrm{PR}(s)$. For every $n \geq 1$, set

$$F_n := \left\{t \in K : d(f_t^n(a(t)), \mathcal{C}_t) \leq n^{-s}\right\}.$$

Pick a constant $\alpha \in (1/s, 2)$. By Lemma 4.9, for n large enough, there exists a constant $C := C(K, \alpha) > 0$ such that

$$\begin{aligned}\mu_{f,a}(F_n) &= \mu_{f,a}(t \in K : \psi_n(t)^\alpha \geq n^{\alpha s})\\ &\leq C(d^{-n} + n^{-\alpha s}(\deg_\omega \Gamma_a + D))\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty} \mu_{f,a}(F_n) < \infty,$$

and we conclude the proof by the Borel-Cantelli lemma. □

Combine Theorem 4.3 with Theorem 4.6, we have:

Proposition 4.10. *Let f be an algebraic family of rational maps as in (2.1) and a be an active marked point. Let $1 < \lambda < d^{1/2}$ and $s > 1/2$, then $\mu_{f,a}$-a.e. $t \in \Lambda$ satisfies $\mathrm{CE}^*(\lambda)$, $\mathrm{PCE}(\lambda)$ and $\mathrm{PR}(s)$, in particular $a(t) \in \mathcal{J}(f_t)$.*

5. Distortion of non-injective maps

Definition 5.1. A rational map $g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is called *Topological Collet-Eckmann* TCE(λ) for some $\lambda > 1$ if there exists $\delta_0 > 0$ such that for each $n \geq 0$ and $z \in \mathcal{J}(g)$, we have

$$\operatorname{diam} W_n \leq \lambda^{-n},$$

where W_n is any connected component of $g^{-n}(B(z, \delta_0))$, and $\mathcal{J}(g)$ is the Julia set of g.

The aim of Section 5, 6 and 7 is to prove the following theorem. Recall that for a holomorphic family of rational maps $f : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{D} \times \mathbb{P}^1$ as in (2.1) and for a marked point a, we let $\xi_{a,n} : \mathbb{D} \to \mathbb{P}^1(\mathbb{C})$ be the map $\xi_{a,n}(t) := f_t^n(a(t))$.

Theorem 5.2. *Let $f : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{D} \times \mathbb{P}^1$ be a holomorphic family of rational maps as in (2.1) and a be a marked point. Assume $0 \in \mathbb{D}$ satisfies*

(i) PCE(λ_0) *for some* $\lambda_0 > 1$*;*
(ii) PR(s) *for some* $s > 0$*;*
(iii) f_0 *is* TCE(λ) *for some* $\lambda > 1$*;*

Then for every $\varepsilon > 0$, there exists a subset $A \subseteq \mathbb{Z}_{\geq 0}$ with $\underline{d}(A) > 1 - \varepsilon$ and $0 < \rho_m < 1, m \in A$ such that the family $\{h_m, m \in A\}$ of maps,

$$h_m : \mathbb{D} \to \mathbb{P}^1(\mathbb{C}),$$
$$t \mapsto \xi_{a,m}(\rho_m t)$$

form a normal family, for which every limit map is non-constant. Moreover, we have $\rho_m \to 0$ as $m \to \infty$.

The proof of Theorem 5.2 is given in Section 7. In Section 5, 6 we do some preparations.

Section 5 is devoted to proving some distortion properties for non-injective holomorphic maps. In this section the distances on $\mathbb{P}^1(\mathbb{C})$ are computed with respect to the spherical metric. Note that under the spherical metric, $\operatorname{diam}(\mathbb{P}^1(\mathbb{C})) = \pi$. For every $a \in \mathbb{P}^1(\mathbb{C})$, the unique point a^- satisfying $d(a, a^-) = \pi$ is the antipodal point of a. We have $0^- = \infty$. When $a \notin \{0, \infty\}$, we have $a^- = -\overline{a}^{-1}$.

5.1. **Upper radius and proper lower radius.** Let Ω be a connected Riemann surface and $x \in \Omega$. Let $h : \Omega \to \mathbb{P}^1(\mathbb{C})$ be a holomorphic map.

We say a connected open neighborhood U of $h(x)$ in $\mathbb{P}^1(\mathbb{C})$ is *properly in the image of* (h, Ω, x) and write $U \subseteq_p h(\Omega, x)$, if there is a connected open neighborhood W of x in Ω such that $h|_W : W \to U$ is proper.

Easy to check the following properties.

Proposition 5.3. (i) *If $U \subseteq_p h(\Omega, x)$, then for every connected open neighborhood V of $h(x)$ contained in U, $V \subseteq_p h(\Omega, x)$.*
(ii) *Let $\Omega' \subseteq \Omega$ be a connected open neighborhood of x, if $U \subseteq_p h(\Omega', x)$, then $U \subseteq_p h(\Omega, x)$.*

The following criterion is useful.

Lemma 5.4. *Assume that Ω is compactly contained in a Riemann surface S. Assume that h is not constant and extends to a neighborhood of $\overline{\Omega}$. Let V be a connected open neighborhood of $h(x)$. If there is a connected open neighborhood W of x and $D > 0$ such that for every $y \in V$, it has exactly D preimages under $h|_W$ counted with multiplicities, then $V \subseteq_p h(\Omega, x)$.*

Proof. We may assume that h extends to S and $W = \Omega$. Set $U := h|_{\Omega}^{-1}(V)$ and let U_0 be the connected component of U containing x. We only need to show that $h|_{U_0} : U_0 \to V$ is proper. So we only need to show that $h|_U$ is proper.

If $h|_U$ is not proper, then there is a compact subset K of V such that

$$h^{-1}(K) \cap \Omega \neq h^{-1}(K) \cap \overline{\Omega}.$$

Pick $z \in \partial\Omega \cap h^{-1}(K)$. Let $z_1, \dots z_s$ be the preimages of $h(z)$ under $h|_{\Omega}$ with multiplicities $m_1, \dots, m_s$. Then $\sum_{i=1}^{s} m_i = D$. Pick open neighborhoods W_i of z_i in Ω and W_0 of z such that $W_i \cap W_j = \emptyset$ for $i \neq j$. Pick $w \in W_0 \cap \Omega$ sufficiently close to z. Then $h(w)$ has exactly m_i preimages in $W_i, i = 1, \dots, s$ counted with multiplicities and has a preimage w in $W_0 \cap \Omega$. So $h(w)$ has at least $\sum_{i=1}^{s} m_i + 1 = D + 1$ preimages in Ω counted with multiplicities. This is a contradiction. □

Definition 5.5. Assume that h is not constant. We define the *upper radius* of (h, Ω, x) to be

$$\rho^*(h, \Omega, x) := \inf \{r \geq 0 : h(\Omega) \subseteq B(h(x), r)\}$$

and the *proper lower radius* of (h, Ω, x) to be

$$\rho_*(h, \Omega, x) := \sup \{r \geq 0 : B(h(x), r) \subseteq_p h(\Omega, x)\}.$$

For convenience, we define $\rho_*(h, \Omega, x) = \rho^*(h, \Omega, x) := 0$ when h is a constant map. It is clear that $\rho^*(h, \Omega, x) \geq \rho_*(h, \Omega, x)$.

The above definition generalizes the usual notion of upper and lower radius for connected open subsets in $\mathbb{P}^1(\mathbb{C})$. Let U be a connected open subset of $\mathbb{P}^1(\mathbb{C})$ and $a \in U$. The *upper radius* of (U, a) is

$$\rho^*(U, a) := \inf \{r \geq 0 : U \subseteq B(a, r)\}.$$

The *lower radius* of (U, a) is

$$\rho_*(U, a) := \sup \{r \geq 0 : B(a, r) \subseteq U\}.$$

Then $\rho^*(U, a) = \rho^*(\mathrm{id}, U, a)$ and $\rho_*(U, a) = \rho_*(\mathrm{id}, U, a)$. If h is not constant, we have

$$\rho^*(h, \Omega, x) = \rho^*(h(\Omega), h(x)) \text{ and } \rho_*(h, \Omega, x) \leq \rho_*(h(\Omega), h(x)).$$

The equality holds if $h : \Omega \to h(\Omega)$ is proper.

Proposition 5.6. *We have the following properties:*

(i) *Let $\Omega' \subseteq \Omega$ be a connected open neighborhood of x. Then*

$$\rho^*(h, \Omega', x) \leq \rho^*(h, \Omega, x) \text{ and } \rho_*(h, \Omega', x) \leq \rho_*(h, \Omega, x).$$

(ii) *Let $\Omega_i, i \geq 0$ be an increasing sequence of connected open neighborhood of x satisfying $\cup_{i\geq 0}\Omega_i = \Omega$. Then*

$$\rho^*(h, \Omega, x) = \sup_{i\geq 0} \rho^*(h, \Omega_i, x) \text{ and } \rho_*(h, \Omega, x) = \sup_{i\geq 0} \rho_*(h, \Omega_i, x).$$

Proof. If h is constant, the proposition is trivial. Now assume that h is not constant. Property (i) and the ρ^* part of (ii) are obvious. We only prove the ρ_* part of (ii). By (i), we have $\rho_*(h, \Omega, x) \geq \sup_{i\geq 0} \rho_*(h, \Omega_i, x)$. For every $r < \rho_*(h, \Omega, x)$, pick $r' \in (r, \rho_*(h, \Omega, x))$. There is an open neighborhood W' of x such that $h|_{W'} : W' \to B(h(x), r')$ is proper. Then $(h|_{W'})^{-1}\left(\overline{B(h(x), r)}\right)$ is compact. There is $i \geq 0$ such that $(h|_{W'})^{-1}\left(\overline{B(h(x), r)}\right) \subseteq \Omega_i$.

Set $W := (h|_{W'})^{-1}(B(h(x), r)) \subseteq \Omega_i$. Since $h|_W : W \to B(h(x), r)$ is proper, $\rho_*(h, \Omega_i, x) \geq r$, which concludes the proof. □

The following lemma shows that the upper and proper lower radii are stable under perturbations.

Lemma 5.7. *Let Ω be a Riemann surface and $x \in \Omega$. For holomorphic maps $g, h : \Omega \to \mathbb{P}^1(\mathbb{C})$, define $\rho(h, g) := \sup_{z\in\Omega} \rho(h(z), g(z))$. Then we have*

$$\rho^*(h, \Omega, x) \leq \rho^*(g, \Omega, x) + 2\rho(h, g). \tag{5.1}$$

Assume further that $\rho^*(g, \Omega, x) + \rho(h, g) < \pi$, then we have

$$\rho_*(h, \Omega, x) \geq \rho_*(g, \Omega, x) - 2\rho(h, g). \tag{5.2}$$

Proof. The first assertion is obvious. We now prove the second assertion. There is a sequence $\Omega_i, i \geq 0$ of open neighborhood of x compactly contained in Ω and having smooth boundary such that $\cup_{i\geq 0}\Omega_i = \Omega$. By (ii) of Proposition 5.6, we only need to prove (5.2) for each Ω_i. So we may assume that Ω is compactly contained in a connected Riemann surface S with a smooth boundary and h extends to a neighborhood of $\overline{\Omega}$. Set $\varepsilon := \rho(h, g)$.

If g is constant, (5.2) is trivial. If h is constant, by (5.1), $\rho^*(g, \Omega, x) \leq 2\varepsilon$. So $\rho_*(g, \Omega, x) \leq 2\varepsilon$ which implies (5.2). Now assume that both g and h are not constant and $\rho_*(g, \Omega, x) > 2\varepsilon$.

Pick any $r \in (2\varepsilon, \rho_*(g, \Omega, x))$, we claim that

$$B(g(x), r - \varepsilon) \subseteq_p h(\Omega, x). \tag{5.3}$$

Since $d(h(x), g(x)) \leq \varepsilon$, $B(h(x), r-2\varepsilon) \subseteq B(g(x), r-\varepsilon)$. Hence $r-2\varepsilon \leq \rho_*(h, \Omega, x)$. Let r tend to $\rho_*(g, \Omega, x)$, then we get (5.2).

We only need to prove the claim. We may assume that $g(x) = 0$. Since $\rho^*(g, \Omega, x) + \rho(h, g) < \pi$, $h(\Omega) \subseteq \mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$. We identify $\mathbb{P}^1(\mathbb{C}) \setminus \{\infty\}$ with $\mathbb{C}$ and view g, h as holomorphic functions on Ω. There is a connected neighborhood W of x such that $g|_W : W \to B(0, r)$ is proper. Then W is compactly contained in Ω and has a piecewisely smooth boundary. There is $D \geq 1$ such that for every $y \in B(0, r)$, it has exactly D preimages under $g|_W$ counted with multiplicity.

For every $t \in [0, 1]$, define $h_t := th + (1 - t)g$. For every $z \in \Omega$, $h(z) \in \overline{B(g(z), \varepsilon)}$. Note that $\overline{B(g(z), \varepsilon)}$ is a disk in $\mathbb{C}$, though its center may not be $g(z)$. It follows that $h_t(z) \in \overline{B(g(z), \varepsilon)}$ for every $t \in [0, 1]$. Hence $d(g, h_t) \leq \varepsilon$ for every $t \in [0, 1]$. Since $g(\partial W) \subseteq \partial B(0, r)$,

$$h_t(\partial W) \subseteq \mathbb{C} \setminus B(0, r - \varepsilon)$$

for every $t \in [0, 1]$. In other words, for every $y \in B(0, r-\varepsilon)$ and $t \in [0, 1]$ there is no zero of $h_t - y$ in ∂W. By argument principle, for every $y \in B(0, r - \varepsilon)$, the number of preimages of y under $h_t|_W$ is constant in t, hence equal to D. Since $h_1 = h$, for every $y \in B(0, r - \varepsilon)$, y has exactly D preimages under $h|_W$. By Lemma 5.4, $B(0, r-\varepsilon) \subseteq_p h(\Omega, x)$, which concludes the proof. □

5.2. **Euclidean coordinates.** It is often easier to do the computation using Euclidean metric rather than the spherical metric. For this reason, we introduce an Euclidean coordinate Z_a at each point $a \in \mathbb{P}^1(\mathbb{C})$. Let z be the standard coordinate on $\mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[z] \subseteq \mathbb{P}^1$. We note that the spherical metric is invariant under the action of $\mathrm{PU}(2, \mathbb{C}) < \mathrm{PGL}\,(2, \mathbb{C})$. For every $a \in \mathbb{P}^1(\mathbb{C})$, pick an element $H_a \in \mathrm{PU}(2, \mathbb{C})$ such that $H_a(a) = 0$. Define $Z_a := 2H_a^* z$. Note that the choice of H_a is unique up to composing a rotation $z \to \gamma z, |\gamma| = 1$ by left. Hence the induced coordinate Z_a is unique up to multiplying some γ with $|\gamma| = 1$. Note that the point defined by $Z_a = \infty$ is $H_a^{-1}(\infty) = a^-$. Via Z_a, we identify $\mathbb{P}^1(\mathbb{C}) \setminus a^-$ with the standard complex plane $\mathbb{C}$ with origin a.

The spherical metric at a is given by

$$ds^2 = \frac{1}{(1 + 1/4Z_a\overline{Z_a})^2} dZ_a \overline{dZ_a} \tag{5.4}$$

We let $B(a,r)$ be the ball centered at a of radius r with respect to the spherical metric and set $\mathbb{D}(a,r) := \{Z_a < r\}$. Then there is a strict increasing function $\tau : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that $B(0,\tau(r)) = \mathbb{D}(0,r)$. Since H_a preserves the spherical metric, for every $a \in \mathbb{P}^1(\mathbb{C})$, $B(a,\tau(r)) = \mathbb{D}(a,r)$. By (5.4), we have

$$\tau(r) = r + O(r^2) \tag{5.5}$$

when $r \to 0$.

Let $g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ be a rational map.

Lemma 5.8. *There are $r_0 > 0$ and $C > 0$ such that the following holds: if a ball $B := B(x,r)$ satisfies $r < r_0$, then we have*

$$\rho^*(g, B, x) \leq |dg(x)|r + Cr^2, \tag{5.6}$$

and

$$\rho_*(g, B, x) \geq |dg(x)|r - Cr^2. \tag{5.7}$$

Note that (5.7) is trivial if $|dg(x)|r - Cr^2 \leq 0$.

Proof. By (5.5), we only need to prove (5.6) and (5.7) for Euclidean metric induced by the local coordinate $Z_a, a \in \mathbb{P}^1(\mathbb{C})$.

For every $a \in \mathbb{P}^1(\mathbb{C})$, define $r_a := \sup\{r \geq 0 |\ g(\mathbb{D}(a,r)) \subseteq \mathbb{D}(g(a), 1)\}$. Since g is continuous, $r_a > 0$ and the maps $a \in \mathbb{P}^1(\mathbb{C}) \mapsto r_a \in \mathbb{R}_{>0}$ is continuous. Since $\mathbb{P}^1(\mathbb{C})$ is compact, $c_0 := \min\{r_a, a \in \mathbb{P}^1(\mathbb{C})\} > 0$ exists. For every $a \in \mathbb{P}^1(\mathbb{C})$, in the local coordinates $Z_a, Z_{g(a)}$, g takes form

$$g(Z) = \sum_{i \geq 1} A_i(a) Z^i. \tag{5.8}$$

By (5.4), $|A_1(a)| = |dg(a)|$. By Cauchy integration formula, we have $|A_i(a)| \leq c_0^{-i}$ for every $a \in \mathbb{P}^1(\mathbb{C})$ and $i \geq 1$. Set $c_1 := c_0/10$. For every $Z \in \mathbb{D}(a, c_1)$, we have

$$|g(Z)| \leq |dg(a)||Z| + \sum_{i \geq 2} c_0^{-i}|Z|^i \leq |dg(a)||Z| + \frac{c_0^{-2}}{9}|Z|^2.$$

So for every $r < c_1$,

$$g(\mathbb{D}(a,r)) \subseteq \mathbb{D}\left(g(a), |dg(a)|r + \frac{c_0^{-2}}{9}r^2\right).$$

It implies (5.6) via (5.5).

Set $D_1 := c_0^{-2}/9$ and $D_2 := 2D_1$. By (5.5), we only need to show that for every $r \in (0, c_1)$, if $|dg(a)|r - D_2r^2 > 0$, then

$$\mathbb{D}(g(a), |dg(a)|r - D_2r^2) \subseteq_p g(\mathbb{D}(a,r), a). \tag{5.9}$$

We write g as in (5.8). For $|Z| = r$ and $|Y| \leq |dg(a)|r - D_2r^2$, we have

$$|(g(Z) - Y) - (A_1(a)Z - Y))| \leq \sum_{i\geq 2} c_0^{-i}r^i \leq D_1r^2$$

and

$$|A_1(a)Z - Y| \geq |dg(a)|r - (|dg(a)|r - D_2r^2) = D_2r^2 > D_1r^2.$$

Since $A_1(a)Z - Y$ has exactly one zero in $\{|Z| < r\}$, by Rouché's theorem $g(Z) - Y$ has exactly zero in $\{|Z| < r\}$. By Lemma 5.4, we get (5.9), which concludes the proof. □

5.3. **Critical points.** Let $\mathcal{C}$ be the critical set of g. For every $c \in \mathcal{C}$, we let l_c be its multiplicity. Then $\sum_{c\in\mathcal{C}} l_c = 2d-2$. Set $l := \max\{l_c |\ c \in \mathcal{C}\}$.

Lemma 5.9. *There exists $r_1 > 0$ and $C_1 > 1$ such that the following holds: if $c \in \mathcal{C}$ and $r < r_1$ then we have*

$$\rho^*(g, B(c,r), c) \leq C_1 r^{l_c}, \tag{5.10}$$

and

$$\rho_*(g, B(c,r), c) \geq C_1^{-1} r^{l_c}. \tag{5.11}$$

Moreover, for every $a \in B(c,r)$, if $g(a) \in B(g(c), C_1^{-1}r^{l_c})$, then

$$B(g(c), C_1^{-1}r^{l_c}) \subseteq_p g(B(c,r), a). \tag{5.12}$$

Proof. By (5.5), to show (5.10) and (5.11), we only need to prove (5.10) and (5.11) for Euclidean metric for the local coordinate $Z_a, a \in \mathbb{P}^1(\mathbb{C})$.

Let c_0, c_1 as in the proof of Lemma 5.8. As in the proof of Lemma 5.8, for every $c \in \mathcal{C}$, in the local coordinates $Z_c, Z_{g(c)}$, g takes form

$$g(Z) = \sum_{i\geq l_c} A_i(c)Z^i, \tag{5.13}$$

where $A_{l_c}(c) \neq 0$ and $|A_i(c)| \leq c_0^{-i}$.

For every $Z \in \mathbb{D}(c, c_1)$, we have

$$|g(Z)| \leq A_{l_c}(c)|Z|^{l_c} + \sum_{i\geq l_c+1} c_0^{-i}|Z|^i \leq \left(A_{l_c}(c) + c_0^{-l_c}/9\right) Z^{l_c}.$$

So for every $r < c_1$,

$$g(\mathbb{D}(c,r)) \subseteq \mathbb{D}(g(c), (A_{l_c}(c) + c_0^{-l_c}/9)Z^{l_c}).$$

Since $\mathcal{C}$ is finite, the above implies (5.6) via (5.5).

Set $A := \min_{c\in\mathcal{C}} |A_{l_c}(c)|$. There is $c' > 0$ such that for every $x \in \mathbb{P}^1(\mathbb{C})$, x has at most l_c preimages counted with multiplicities in $\mathbb{D}(c, c')$. Pick $c_2 := \min\{c_1, \min\{1, c_0\}^l A/10, c'\}$. Set $D_1 := c_0^{-l_c} \frac{c_2/c_1}{1-c_2/c_1}$ and $D_2 := 2D_1$. We may check that

$$|A_{l_c}(c)| - D_2 > A/2. \tag{5.14}$$

Since $\mathcal{C}$ is finite, by (5.5) and (5.14), to prove (5.15) we only need to show that for every $r \in (0, c_2)$,

$$\mathbb{D}(g(c), (|A_{l_c}(c)| - D_2)r^{l_c}) \subseteq_p g(\mathbb{D}(c, r), c). \tag{5.15}$$

We write g as in (5.13). For $|Z| = r$ and $|Y| \leq (|A_{l_c}(c)| - D_2)r^{l_c}$, we have

$$|(g(Z) - Y) - (A_{l_c}(c)Z^{l_c} - Y))| \leq \sum_{i\geq l_c+1} c_0^{-i} r^i \leq D_1 r^{l_c}$$

and

$$|A_{l_c}(c)Z^{l_c} - Y| \geq |A_{l_c}(c)|r^{l_c} - (|A_{l_c}(c)| - D_2)r^{l_c} = D_2 r^{l_c} > D_1 r^{l_c}.$$

Since $A_{l_c}(c)Z^{l_c} - Y$ has exactly l_c zeros in $\{|Z| < r\}$, by Rouché's theorem $g(Z) - Y$ has exactly l_c zeros in $\{|Z| < r\}$. By Lemma 5.4, we get (5.15).

We now prove (5.12). Let $a \in B(c, r)$ with $g(a) \in B(g(c), C_1^{-1} r^{l_c})$. By (5.15), there is an open neighborhood W of c in $B(c, r)$ such that $g|_W : W \to B(g(c), C_1^{-1} r^{l_c})$ is proper. Since $g(c)$ has exactly one preimage c of multiplicity l_c, $g(a)$ has l_c preimages with multiplicity in W. Since $r \leq c'$, $g(a)$ has at most l_c preimages with multiplicity in $B(c, r)$. Hence $g^{-1}(g(a)) \cap B(c, r) = g^{-1}(g(a)) \cap W$. Then $a \in W$, which implies (5.12). $\square$

Set

$$\delta := \min\{d(c_1, c_2)|\ c_1, c_2 \text{ are distint points in } \mathcal{C}\}/3. \tag{5.16}$$

Then for every $a \in \mathbb{P}^1(\mathbb{C})$ and $r \in (0, \delta]$ there is at most one $c \in \mathcal{C} \cap B(a, r)$. For every $a \in \mathbb{P}^1(\mathbb{C})$, define l_a as follows: if $d(a, \mathcal{C}) \geq \delta$, set $l_a := 0$; otherwise, let c_a be the unique critical point with $d(a, c_a) < \delta$ and $l_a := l_c$. This extends our previous definition of l_c for $c \in \mathcal{C}$. Note that, for every $a \in \mathbb{P}^1(\mathbb{C})$ either for every $b \in B(a, \delta/100)$, $l_b = l_a$ or $d(B(a, \delta/100), \mathcal{C}) > \delta/2$. So there is a constant $A_1 > 1$ such that for every $a \in \mathbb{P}^1(\mathbb{C})$ and $b \in B(a, \delta/100)$, we have

$$A_1^{-1} d(x, \mathcal{C})^{l_b - 1} \geq |dg(a)| \geq A_1 d(x, \mathcal{C})^{l_b - 1}. \tag{5.17}$$

The following lemma is a corollary of Lemma 5.8.

Lemma 5.10. *There exists $r_2 > 0$ and $C_2 > 0$ such that the following holds: if two balls $B := B(x,r)$, $B' := B(x,r')$ satisfy $r' < r < r_2$, and $d(x,\mathcal{C})^l > r$, then we have*

$$\frac{\rho_*(g,B',x)}{\rho^*(g,B,x)} \geq \frac{r'}{r} - C_2 r^{1/l} \tag{5.18}$$

and

$$\frac{\rho^*(g,B',x)}{\rho_*(g,B,x)} \leq \frac{r'}{r} + C_2 r^{1/l}. \tag{5.19}$$

Proof. Let r_0, C as in Lemma 5.8. We may further ask that $r_0 < \min\{1, A_1/C\}$. Then under the assumption $r < r_0$ and $d(x,\mathcal{C})^l > r$, for every $u \in (0,r]$, we have $|dg(x)|u - Cu^2 > 0$.

By Lemma 5.8 we have

$$\begin{aligned}\frac{\rho_*(g,B',x)}{\rho^*(g,B,x)} &\geq \frac{|dg(x)|r' - Cr'^2}{|dg(x)|r + Cr^2}\\ &= \frac{r'}{r} - \frac{Cr'(r-r')}{r|dg(x)| + Cr^2}\\ &\geq \frac{r'}{r} - \frac{Cr^2/4}{A_1 r^{2-1/l} + Cr^2}\\ &\geq \frac{r'}{r} - C_2 r^{1/l},\end{aligned}$$

where $C_2 > 0$ is a constant. This implies (5.18). Similarly one can prove (5.19). □

Lemma 5.11. *There is a constant $A_2 > 1$ such that for every point $x \in \mathbb{P}^1(\mathbb{C}) \setminus \mathcal{C}$, $g|_{B(x,d(x,\mathcal{C})/A_2)}$ is injective.*

Proof. For every $c \in \mathcal{C}$, there is an open neighborhood U_c of c such that such that there are isomorphisms $\phi_c : U_c \to \mathbb{D}$ and $\psi_c : g(U_c) \to \mathbb{D}$, such that $\phi_c(c) = 0, \psi(f(c)) = 0$ and $G_c := \psi_c \circ g \circ (\phi_c)^{-1} : z \to z^{l_c}$. Recall that l_c is the multiplicity of c. After shrinking U_c, we may assume that $U_c \subseteq B(c,\delta)$ and on U_c and $g(U_c)$, the spherical metrics are equivalent to the metrics induced by Euclidean metric on $\mathbb{D}$ via ϕ_c and ψ_c respectively. There is $D > 1$ such that for every $c \in \mathcal{C}$ and $x,y \in U_c, z,w \in g(U_c)$, we have

$$D^{-1}|\phi_c(x) - \phi_c(y)| \leq d(x,y) \leq D|\phi_c(x) - \phi_c(y)| \tag{5.20}$$

Set $V_c := \phi_c^{-1}(\mathbb{D}(0,1/2))$. and $K := \mathbb{P}^1(\mathbb{C}) \setminus (\cup_{c\in\mathcal{C}} V_c)$. For every $x \in K$, there is $r_x > 0$ such that $g|_{B(x,r_x)}$ is injective. Since K is compact, there is a finite subset $F \subseteq K$ such that $K \subseteq \cup_{x\in F} B(x, r_x)$. There is $\delta_1 > 0$ such that for every $x \in K$, there is $y \in F$ such that

$$B(x, \delta_1) \subseteq B(y, r_y). \tag{5.21}$$

For every $x \in V_c$, $|\phi_c(x)| < 1/2$, it is clear that $G_c|_{\mathbb{D}(\phi_c(x),|\phi_c(x)|/(100l))}$ is injective. By (5.20), we have

$$B(x, d(x, \mathcal{C})/(100D^2 l)) \subseteq B(x, |\phi_c(x)|/(100Dl))$$

and

$$\phi_c(B(x, |\phi_c(x)|/(100Dl))) \subseteq \mathbb{D}(\phi_c(x), |\phi_c(x)|/(100l)).$$

So $g|_{B(x,d(x,\mathcal{C})/(100D^2 l))}$ is injective. Set $A_2 := \max\{100D^2 l, 2\pi/\delta_1\}$, we conclude the proof by (5.21). □

Set $A_3 := 2A_2$, where A_2 is the constant in Lemma 5.11. By Koebe distortion theorem and (5.17), there is $C_3 > 1$ such that for every $x \in \mathbb{P}^1(\mathbb{C}) \setminus \mathcal{C}$ and $r \leq d(x, \mathcal{C})/A_3$ we have

$$\rho^*(g, B(x, r), x) \leq C_3 d(x, \mathcal{C})^{l_x - 1} r \tag{5.22}$$

and

$$\rho_*(g, B(x, r), x) \geq C_3^{-1} d(x, \mathcal{C})^{l_x - 1} r. \tag{5.23}$$

Without assuming $d(x, \mathcal{C})^l > r$, we also have the following weaker distortion estimates.

Lemma 5.12. *There exists $r_3 > 0$ and $\theta > 1$ such that the following holds: if two balls $B := B(x, r)$, $B' := B(x, r')$ satisfy $r' < r < r_3$, then we have*

$$\frac{\rho_*(g, B', x)}{\rho^*(g, B, x)} \geq \frac{1}{\theta} \frac{(r')^l}{r^l} \tag{5.24}$$

and

$$\frac{\rho^*(g, B', x)}{\rho_*(g, B, x)} \leq \theta \frac{r'}{r}. \tag{5.25}$$

Proof. The proof is based on the following lemma.

Lemma 5.13. *There are $p_0 > 0$ and two constants $\theta_1 > 0$ and $\theta_2 > 0$ such that for $r < p_0$, the following holds:*

(i) *If a ball $B := B(x, r)$ satisfies $r < d(x, \mathcal{C})/2$, then*

$$\rho^*(g, B, x) \leq \theta_1 d(x, \mathcal{C})^{l_x - 1} r \tag{5.26}$$

and

$$\rho_*(g, B, x) \geq \theta_2 d(x, \mathcal{C})^{l_x - 1} r. \tag{5.27}$$

(ii) *If a ball* $B := B(x,r)$ *satisfies* $r \geq d(x,\mathcal{C})/2$, *then*

$$\rho^*(g,B,x) \leq \theta_1 r^{l_x} \tag{5.28}$$

and

$$\rho_*(g,B,x) \geq \theta_2 r^{l_x}. \tag{5.29}$$

Set $\theta := 2\theta_1/\theta_2$. To show (5.24) and (5.25), there are three cases.

Case 1: we have $r < d(x,\mathcal{C})/2$. Then

$$\frac{\rho_*(g,B',x)}{\rho^*(g,B,x)} \geq \frac{\theta_2 d(x,\mathcal{C})^{l_x-1}r'}{\theta_1 d(x,\mathcal{C})^{l_x-1}r} \geq \frac{1}{\theta}\frac{r'}{r},$$

and

$$\frac{\rho^*(g,B',x)}{\rho_*(g,B,x)} \leq \frac{\theta_1 d(x,\mathcal{C})^{l_x-1}r'}{\theta_2 d(x,\mathcal{C})^{l_x-1}r} \leq \theta\frac{r'}{r}.$$

Case 2: we have $r' < d(x,\mathcal{C})/2$ but $r \geq d(x,\mathcal{C})/2$. Then

$$\frac{\rho_*(g,B',x)}{\rho^*(g,B,x)} \geq \frac{\theta_2 d(x,\mathcal{C})^{l_x-1}r'}{\theta_1 d(x,\mathcal{C})^{l_x}} \geq \frac{1}{\theta}\frac{r'}{r},$$

and

$$\frac{\rho^*(g,B',x)}{\rho_*(g,B,x)} \leq \frac{\theta_1 d(x,\mathcal{C})^{l_x-1}r'}{\theta_2 d(x,\mathcal{C})^{l_x}} \leq \theta\frac{r'}{r}.$$

Case 3: we have $r' \geq d(x,\mathcal{C})/2$. Then

$$\frac{\rho_*(g,B',x)}{\rho^*(g,B,x)} \geq \frac{\theta_2 (r')^{l_x}}{\theta_1 r^{l_x}} \geq \frac{1}{\theta}\frac{(r')^l_x}{r^l_x},$$

and

$$\frac{\rho^*(g,B',x)}{\rho_*(g,B,x)} \leq \frac{\theta_1 (r')^{l_x}}{\theta_2 r^{l_x}} \leq \theta\frac{r'}{r}.$$

□

Proof of Lemma 5.13. By (5.17), for $r < \delta/100$ and every $y \in B(x,r)$, we have

$$|dg(y)| \leq A_1 d(y,\mathcal{C})^{l_x-1}.$$

If $r < d(x,\mathcal{C})/2$, then $d(y,\mathcal{C}) \leq 3/2d(x,\mathcal{C})$. So

$$g(B(x,r)) \subseteq B(g(x), A_1(3/2)^{l-1}d(x,\mathcal{C})^{l_x-1}r),$$

which implies (5.26).

Next, we prove (5.27). Since $r < d(x,\mathcal{C})/2$, $r/A_3 < d(x,\mathcal{C})/A_3$. By (5.23), we have

$$\rho_*(g,B(x,r),x)) \geq \rho_*(g,B(x,r/A_3),x)) \geq (C_3A_3)^{-1}d(x,\mathcal{C})^{l_x-1}r.$$

This implies (5.27).

Now we assume that $r \geq d(x, \mathcal{C})/2$. Let r_1, C_1 as in Lemma 5.9. Assume that $r < \min\{r_1, \delta, 1/C_1\}/100$. Then $d(x, \mathcal{C}) < \min\{r_1, \delta, 1/C_1\}/50$. Set $c := c_x$. We have $l_x = l_c$. Recall that l_c is the multiplicity of c. Since $B(x, r) \subseteq B(c, 3r)$, $g(B(x, r)) \subseteq g(B(c, 3r))$. By Lemma 5.9,

$$g(B(x,r)) \subseteq g(B(c,3r)) \subseteq B(g(c), C_1 3^l_c r^{l_c}).$$

Since $d(c, x) \leq 2r$, by Lemma 5.9, we have $d(g(c), g(x)) \leq C_1 2^l_c r^{l_c}$. Then we have

$$g(B(x,r)) \subseteq B(g(c), C_1 3^l_c r^{l_c}) \subseteq B(x, C_1(3^{l_c} + 2^{l_c}) r^{l_c})).$$

Since $C_1(3^{l_c} + 2^{l_c}) r^{l_c}) \leq C_1(3^l + 2^l) r^{l_c})$, we get (5.28).

Finally, we prove (5.29). We first treat the case where $r \leq 10(C_1^2 + 1) d(x, \mathcal{C})$. Since $r \geq d(x, \mathcal{C})/2$ and $A_3 > 2$, $B(x, r)$ contains $B(x, d(x, \mathcal{C})/2)$. By (5.23),

$$\begin{aligned}
\rho_*(g(B(x,r)), g(x)) \geq & \rho_*(g(B(x, d(x,\mathcal{C})/A_3)), g(x)) \\
\geq & (A_3 C_3)^{-1} (d(x,\mathcal{C}))^{l_x} \\
\geq & (A_3 C_3)^{-1} (r/(10(C_1^2+1)))^{l_x},
\end{aligned}$$

which implies (5.29). Now assume that $r > (10(C_1^2 + 1)) d(x, \mathcal{C})$. Note that $d(x, \mathcal{C}) = d(x, c)$. Set $Q := (10(C_1^2 + 1))$. Since $r > Q d(x, \mathcal{C})$,

$$C_1^{-1}(r - d(x,c))^{l_x} \geq C_1^{-1} \left(\frac{Q-1}{Q} \right)^{l_x} r^{l_x}. \tag{5.30}$$

By Lemma 5.9, we have

$$d(g(c), g(x)) \leq C_1 d(c,x)^{l_x} \leq C_1 Q^{-l_x} r^{l_x}.$$

One may check that $C_1 Q^{-l_x} \leq 1/10 C_1^{-1} (\frac{Q-1}{Q})^{l_x}$. We get

$$B(g(x), 9C_1 Q^{-l_x} x^{l_x}) \subseteq B(g(c), C_1^{-1}(r - d(x,c))^{l_x}) \tag{5.31}$$

and

$$g(x) \in B(g(c), C_1^{-1}(r - d(x,c))^{l_x}).$$

Since $r > (10(C_1^2 + 1)) d(x, c) > 10 d(x, c)$, $x \in B(c, r - d(x, c))$. Then by (5.12) of Lemma 5.9, we get

$$B(g(c), C_1^{-1}(r - d(x,c))^{l_x}) \subseteq_p g(B(c, r - d(x,c)), x).$$

Since $B(c, r - d(x, c)) \subseteq B(x, r)$,

$$B(g(c), C_1^{-1}(r - d(x,c))^{l_x}) \subseteq_p g(B(x,r), x).$$

By (5.31), $B(g(x), 9C_1 Q^{-l_x} x^{l_x}) \subseteq_p g(B(x, r), x)$, which concludes the proof. $\square$

6. Bounded distortion for non-uniformly hyperbolic maps

In this section, we show some nice bounded distortion properties of Topological Collet-Eckmann and Polynomial Recurrence maps.

Let $q := \#(\mathcal{C} \cap \mathcal{J}(g))$. Define $d_1(\cdot,\cdot) := \min\{d(\cdot,\cdot), 1\}$. We have $d_1(x,y) \leq d(x,y)$ for every $x, y \in \mathbb{P}^1(\mathbb{C})$. The following lemma is [DPU96, (3.3) in the proof of Lemma 3.4].

Lemma 6.1 (Denker-Przytycki-Urbanski [DPU96])**.** *There exists $Q > 0$ such that for every $x \in \mathcal{J}(g)$ and $n \geq 1$, the following holds:*

$$\sum_{\substack{0 \leq k \leq n-1, \\ \text{except } q \text{ terms}}} -\log d_1(g^k(x), \mathcal{C}) \leq Qn.$$

Let $\delta_0 > 0$. For every fixed $x \in \mathcal{J}(g)$ and $n \geq 1$, for $0 \leq k \leq n$ we define $W_k(n)$ to be the connected component of $g^{k-n}(B(g^n(x), \delta_0))$ containing $g^k(x)$. When n is clear, we write W_m for the simplicity. The following lemma is inspired by Przytycki-Rohde [PR98].

Lemma 6.2. *Assume g is $\mathrm{TCE}(\lambda)$ for some $\lambda > 1$, $\delta_0 > 0$. Let $x \in \mathcal{J}(g)$. Then for every $\varepsilon > 0$, there exists $N > 0$ and a subset $A \subseteq \mathbb{Z}_{\geq 0}$ satisfying $\underline{d}(A) > 1 - \varepsilon$ such the following holds: for every $m \in A$, and $k \notin E_m$ where E_m is a subset of $\{0, \dots, m\}$ containing at most N elements, we have*

$$d_1(g^k(x), \mathcal{C})^l > \operatorname{diam} W_k(m).$$

Proof. For each $k \in \mathbb{Z}_{\geq 0}$, let I_k be the closed interval

$$I_k := \left[k, k + \frac{l}{\log \lambda}(-\log d_1(g^k(x), \mathcal{C}))\right].$$

By Lemma 6.1, for every $n \geq q$ there is a subset $F_n \subseteq \{0, \dots, n-1\}$ with $\#F_n = q$ such that

$$\sum_{0 \leq k \leq n-1, k \notin F_n} |I_k \cap [0, n]| \leq \frac{lQ}{\log \lambda} n.$$

For every $N \geq 1$, set

$$A_N := \{n \in \mathbb{Z}_{\geq 0} : \text{there are at most } N \text{ intervals } I_k \text{ containing } n\}.$$

For every $i \in \{0, \dots, n-1\} \setminus A_N$, i is covered by at least $N+1$ intervals I_k. Since the left endpoint of those I_k are distinct integers, at least N left endpoints are $\leq k - 1$. Hence there are at least $N - q$ intervals among $I_k, k \in \{0, \dots, n-1\} \setminus F_n$ covers the interval $[i-1, i]$. Hence

$$(N - q)(n - \#(A_N \cap [0, n-1])) \leq \frac{lQ}{\log \lambda} n.$$

It follows that

$$\#(A_N \cap [0, n-1]) \geq \left(1 - \frac{lQ}{\log \lambda(N-q)}\right) n.$$

Pick N large enough, we have $\underline{d}(A_N) > 1-\varepsilon$. We set $A := A_N$. We need to show that A satisfies the property we want. It suffices to show for $m \in A$, if $m \notin I_k$ for $0 \leq k \leq m$, then we have $d(g^k(x), \mathcal{C})^l > \operatorname{diam} W_k$. The condition $m \notin I_k$ and the the TCE(λ) property imply that

$$d_1(g^k(x), \mathcal{C})^l > \lambda^{k-m} \geq \operatorname{diam} W_k.$$

This finishes the proof. □

If in addition a point $x \in \mathcal{J}(g)$ satisfies PR(s), we have the following two lemmas.

Lemma 6.3. *Assume g is* TCE(λ) *for some $\lambda > 1$, $\delta_0 > 0$. Let $x \in \mathcal{J}(g)$ satisfy* PR(s) *for some $s > 0$ and x is not a preimage of a critical point. Then the following holds: for every large $n \geq 1$, if some $k \in \{0, \ldots, n-1\}$ sartisfies $d(g^k(x), \mathcal{C})^l \leq \operatorname{diam} W_k(n)$, then $k \geq n - (sl/\log \lambda) \log n$.*

Proof. By PR(s), there is a constant $N_0 > 0$ such that

$$d(g^k(x), \mathcal{C}) > k^{-s} \tag{6.1}$$

for every $k \geq N_0$. There is $N_1 > 0$ such that for every $k = 0, \ldots, N_0$,

$$d(g^k(x), \mathcal{C})^l > \lambda^{-(N_1 - N_0)}. \tag{6.2}$$

Let $n \geq N_1$, and $k \in \{0, \ldots, n\}$. Assume that $d(g^k(x), \mathcal{C})^l \leq \operatorname{diam} W_k$. Since $\operatorname{diam} W_k \leq \lambda^{k-n}$, we have

$$d(g^k(x), \mathcal{C})^l \leq \lambda^{k-n}. \tag{6.3}$$

By (6.2), we get $k \geq N_0 + 1$. Then by (6.1), we get

$$k \geq n - (sl/\log \lambda) \log n,$$

which concludes the proof. □

Lemma 6.4. *Assume that g is* TCE(λ) *for some $\lambda > 1$. Let $\delta_0 > 0$ be small enough. Let $x \in \mathcal{J}(g)$ satisfy* PR(s) *for some $s > 0$ and x is not a preimage of a critical point. Let $\varepsilon > 0$ and let A be the subset defined in Lemma 6.2. Then*

(i) *There is a constant $C > 0$ such that the following holds: for every $m \in A$, we have*

$$\sum_{k=0}^{m} \frac{\operatorname{diam} W_0(m)}{\operatorname{diam} W_k(m)} < C.$$

(ii) *For every $\eta > 0$ there exists $N_0 > 0$ such that the following holds: for every $m \in A$, $m \geq N_0$ we have*

$$\sum_{k=N_0}^{m} \frac{\operatorname{diam} W_0(m)}{\operatorname{diam} W_k(m)} < \eta.$$

Proof. Since x satisfies PR(s), g is TCE(λ) and x is not a preimage of a critical point, by [Ji23b, Lemma A.4], there exists $\lambda_1 > 1$ and $C' > 0$ such that for every $n \geq 1$, $|dg^n(x)| \geq C'\lambda_1^n$.

Since g is TCE(λ), for δ_0 small enough, we may assume that all $W_i(m)$ has diameter at most δ (see (5.16)). So every $W_i(m)$ meets at most one critical point. Hence all $W_i(m)$ are simply connected. For each $i = 0, \dots, m$, the map $g^{m-i}|_{W_i} : W_i \to W_m$ is proper with at most l^N critical points counted with multiplicity. By the Koebe type distortion property for proper holomorphic maps with a bounded number of critical points, see [PR98, Lemma 2.1], there exists a uniform constant $\beta > 0$ such that

$$\rho_*(W_i(m), x) > \beta \operatorname{diam} W_i(m). \tag{6.4}$$

By Lemma 6.3, for $k < m - (sl/\log\lambda)\log m$, $g^k : W_0 \to W_k$ is injective. By Koebe one-quarter theorem, we have

$$\operatorname{diam} W_k \geq |dg^k(x)|\rho_*(W_0, x)/4 \geq C_4\lambda_1^k \operatorname{diam} W_0, \tag{6.5}$$

where $C_4 > 0$ is a constant.

On the other hand there exists a constant $L > 1$ such that for every $0 \leq k \leq m-1$ we have

$$\operatorname{diam} W_k \geq L^{k-m}\delta_0. \tag{6.6}$$

Let $p := \lfloor m - (sl/\log\lambda)\log m \rfloor$. Combine with (6.5) and (6.6) there exists $C_5 > 0$ such that

$$\begin{aligned}
\sum_{k=0}^{m} \frac{\operatorname{diam} W_0}{\operatorname{diam} W_k} &= \sum_{k=0}^{p} \frac{\operatorname{diam} W_0}{\operatorname{diam} W_k} + \sum_{k=p+1}^{m} \frac{\operatorname{diam} W_0}{\operatorname{diam} W_k} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{C_4\lambda_1^k} + \sum_{k=p+1}^{m} \frac{\lambda^{-m}}{L^{k-m}\delta_0} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{C_4\lambda_1^k} + \frac{\lambda^{-m}}{\delta_0(L-1)} L^{m-p} \\
&\leq \sum_{k=0}^{\infty} \frac{1}{C_4\lambda_1^k} + \frac{\lambda^{-m}}{\delta_0(L-1)} L^{(sl/\log\lambda)\log m + 1} \\
&< C_5.
\end{aligned}$$

This proves (i). To show (ii), similarly we have

$$\begin{aligned}\sum_{k=N_0}^{m} \frac{\operatorname{diam} W_0}{\operatorname{diam} W_k} &= \sum_{k=N_0}^{p} \frac{\operatorname{diam} W_0}{\operatorname{diam} W_k} + \sum_{k=p+1}^{m} \frac{\operatorname{diam} W_0}{\operatorname{diam} W_k} \\ &\le \sum_{k=N_0}^{\infty} \frac{1}{C_4 \lambda_1^k} + \sum_{k=p+1}^{m} \frac{\lambda^{-m}}{L^{k-m}\delta_0} \\ &< \eta,\end{aligned}$$

for N_0 large enough. This finishes the proof. □

7. From phase space to parameter space

Proof of Theorem 5.2. By Lemma 4.2, after replacing a by a suitable iterate, we may assume that the orbit of $a(0)$ does not intersect $\mathcal{C}_0$. The PCE(λ_0) condition shows that there exists $\gamma \neq 0$ such the transversality condition in Lemma 4.5 holds. After replacing a by a suitable iterate, we may assume that

$$|d\xi_{a,n}/dt(0)|/|d(g^n)(z)| \in (|\gamma|/2, 2|\gamma|). \tag{7.1}$$

Set $C_6 := \sup_{t_0 \in \mathbb{D}} |d\xi_{a,0}/dt(t_0)|$. Set $g := f_0$ and $z := a(0)$. By Theorem 4.3, 0 is marked Collet-Eckmann parameter, in particular $z \in \mathcal{J}(g)$. Moreover, z is not a preimage of a critical point. For an arbitrary fixed $\varepsilon > 0$, let A be the subset defined in Lemma 6.2. To show A has the properties we want, it suffices to construct $\{\rho_m\}$ such that for every $m \in A$ large, the following holds:

$$h_m(\mathbb{D}) \subseteq B(g^m(z), \delta_0), \tag{7.2}$$

and there exists $\delta_1 > 0$ such that

$$B(g^m(z), \delta_1) \subseteq h_m\left(\frac{1}{2}\mathbb{D}\right). \tag{7.3}$$

Since g is TCE(λ), for δ_0 small enough, we may assume that for every $m \ge 0, i = 0, \ldots, m$,

$$\operatorname{diam}(W_i(m)) \le \min\{\delta, 1/10\} \tag{7.4}$$

where δ is defined in (5.16). Hence all $W_i(m)$ are simply connected.

For $0 \le k \le m$, set $R_k := \rho_*(W_k, g^k(z))$ and $D := B(z, R_0)$. For $0 \le k \le m$, set $R'_k := \rho^*(g^k(D), g^k(z))$. We first show that R_k, R'_k, and $\operatorname{diam} W_k$ are comparable. By (6.4), we have $\beta \operatorname{diam} W_k \le R_k \le \operatorname{diam} W_k$. By [PR98, (2.2) of Lemma 2.1], there is $C_\tau > 0, \tau \in (0, 1/2)$ with $C_\tau \to 0$ as $\tau \to 0$ such that the following holds: For every $r \le \tau R_k$, let $W''(\tau)$ be the connected component of $g^{-k}(B(g^k(z), r))$ containing

z, then we have $\operatorname{diam} W''(\tau) < C_\tau \operatorname{diam} W_0$. Pick $\beta_1 \in (0, 1/2)$ such that $C_{\beta_1} < \beta$. Then $\operatorname{diam} W''(\beta_1) < \beta \operatorname{diam} W_0 \leq R_0$. Hence $W''(\beta_1) \subseteq D$. Note that $g^k(D) \subseteq W_k$, we get

$$\beta\beta_1 \operatorname{diam} W_k \leq \beta_1 R_k \leq \rho_*(g^k(D), g^k(z)) \leq R'_k \leq \operatorname{diam} W_k \tag{7.5}$$

Let $L := 2\sup_{x \in \frac{1}{2}\mathbb{D}\times\mathbb{P}^1(\mathbb{C})} |df(x)|$. For every $t \in \frac{1}{2}\mathbb{D}$, we have

$$d(\xi_{a_{k+1}}(t), g(\xi_{a,k}(t))) = d(f_t(\xi_{a,k}(t)), g(\xi_{a,k}(t))) \leq L|t| \tag{7.6}$$

Let $E_m \subseteq \{0, 1, \ldots, m-1\}$ be the exceptional set as in Lemma 6.2. We have $\#E_m \leq N$. Set $r_k := \rho^*(\xi_{a,k}(\rho_m \mathbb{D}), g^k(z))$ and $B_k := B(g^k(z), r_k)$. By (7.6), if $\rho_m < 1/2$, we have

$$r_{k+1} \leq \rho^*(g, B_k, g^k(z)) + L\rho_m, \tag{7.7}$$

Hence

$$\begin{aligned} \frac{r_{k+1}}{R_{k+1}} \leq& \frac{\rho^*(g, B_k, g^k(z))}{\rho_*(W_{k+1}, g^{k+1}(z))} + \frac{L\rho_m}{R_{k+1}} \\ \leq& \frac{\rho^*(g, B_k, g^k(z))}{\rho_*(g, B(g^k(z), R_k), g^k(z))} + \frac{L\rho_m}{R_{k+1}}. \end{aligned} \tag{7.8}$$

For $0 \leq k \leq m-1$ such that $k \notin E_m$, if

$$\frac{r_k}{R_k} < 1, \tag{7.9}$$

then by (5.19) of Lemma 5.10 and (7.8) we have

$$\frac{r_{k+1}}{R_{k+1}} \leq \frac{r_k}{R_k} + C_2 R_k^{1/l} + \frac{L\rho_m}{R_{k+1}} \tag{7.10}$$

If $k \in E_m$, and (7.9) holds, by (5.25) of Lemma 5.12 and (7.8) we have

$$\frac{r_{k+1}}{R_{k+1}} \leq \theta\frac{r_k}{R_k} + \frac{L\rho_m}{R_{k+1}}. \tag{7.11}$$

If (7.9) holds for every $i = 0, \ldots, k-1$, by (7.10) and (7.11) we have for every $0 \leq i \leq m$

$$\begin{aligned} \frac{r_i}{R_i} &\leq \theta^N \sum_{k=0}^{i-1}\left(C_2 R_k^{1/l} + \frac{L\rho_m}{R_{k+1}}\right) + \theta^N \frac{r_0}{R_0} \\ &\leq \theta^N \sum_{k=0}^{m-1}\left(C_2 R_k^{1/l} + \frac{L\rho_m}{R_{k+1}}\right) + \theta^N \frac{r_0}{R_0} \\ &\leq \theta^N \sum_{k=0}^{m-1}\left(C_2 R_k^{1/l} + \frac{L\rho_m}{R_{k+1}}\right) + \theta^N \frac{C_6\rho_m}{R_0}. \end{aligned} \tag{7.12}$$

Since $R_k \leq \lambda^{k-m}$, shrink δ_0 if necessary we may assume that

$$\theta^N \sum_{k=0}^{m-1} C_2 R_k^{1/l} < \beta\beta_1/4. \tag{7.13}$$

By Lemma 6.4 (i), there exists a constant $\alpha \in (0, 1/(10L + 10\pi))$ such that

$$\theta^N \frac{C_6 \alpha \operatorname{diam} W_0(m)}{R_0} + \theta^N \sum_{k=0}^{m-1} \frac{L\alpha \operatorname{diam} W_0(m)}{R_{k+1}} < \beta\beta_1/4. \tag{7.14}$$

We define $\rho_m := \alpha \operatorname{diam} W_0(m)$, then $\rho_m < 1/2$. Hence (7.7) holds. Moreover, by the TCE(λ) condition, $\rho_m \to 0$ as $m \to \infty$. Since $r_0 \leq C_6 \times \rho_m = C_6 \alpha \operatorname{diam} W_0$, by (7.14) $r_0/R_0 < \beta\beta_1/4 < 1$. Apply (7.9), (7.12), (7.13) and (7.14) inductively, we get

$$r_i/R_i < \beta\beta_1/2 \tag{7.15}$$

for every $i = 0, \ldots, m$, which implies (7.2). By (7.4), for every $m \geq 0$,

$$\rho_m L \leq 1/10. \tag{7.16}$$

It remains to show for our choice $\rho_m := \alpha \operatorname{diam} W_0(m)$, there exists $\delta_1 > 0$ such that (7.3) holds. For $0 \leq k \leq m$ we set $r'_k := \rho_*(\xi_{a,k}, (\rho_m/2)\mathbb{D}, 0)$ and $B'_k := B(g^k(z), r'_k)$. By (7.15) and (7.5), we get

$$r'_k \leq r_k < \beta\beta_1 R_i \leq \beta\beta_1 \operatorname{diam} W_i \leq R'_k. \tag{7.17}$$

We need to show that there exists $\delta_1 > 0$ such that $r'_m \geq \delta_1$.

Combining (7.1) with Koebe distortion theorem, there exists $\alpha_0 \in (0, 1/2)$ such that

$$\frac{r'_k}{R'_k} > 2\alpha_0, \tag{7.18}$$

provided that $\xi_{a,k}$ is injective when restricted on $\rho_m\mathbb{D}$ and g^k is injective on $2D$.

Set $\alpha_1 := \alpha_0^{l^N}/(2\theta^{Nl^N})$. Combing Lemma 6.4 (ii) which (7.5), shrink δ_0 if necessary we can choose N_0 large enough such that for $N_0 \leq k \leq m-1$, we have

$$\frac{\alpha_1^l}{\theta} > 2\left(C_2 {R'_k}^{1/l} + \frac{L\rho_m}{R'_{k+1}}\right). \tag{7.19}$$

Since

$B'_k \subseteq_p \xi_{a,k}(\rho_m/2)\mathbb{D}, 0)$ and $B(g^{k+1}(z), \rho_*(g, B'_k, g^k(z))) \subseteq_p g(B'_k, g^k(z))$, we have $B(g^{k+1}(z), \rho_*(g, B'_k, g^k(z))) \subseteq_p (g\circ \xi_k)((\rho_m/2)\mathbb{D}, 0)$. Hence

$$\rho_*(g, B'_k, g^k(z)) \leq \rho_*(g\circ\xi_k, (\rho_m/2)\mathbb{D}, 0).$$

By (7.4) and (7.16), we may apply (5.2) of Lemma 5.7 and get

$$r'_{k+1} \geq \rho_*(g, B'_k, g^k(z)) - L\rho_m. \tag{7.20}$$

Hence we have

$$\begin{aligned}\frac{r'_{k+1}}{R'_{k+1}} \geq & \frac{\rho_*(g, B'_k, g^k(z))}{\rho^*(g, g^k(D), g^k(z))} - \frac{L\rho_m}{R'_{k+1}} \\ \geq & \frac{\rho_*(g, B'_k, g^k(z))}{\rho^*(g, B(g^k(z), R'_k), g^k(z))} - \frac{L\rho_m}{R'_{k+1}}\end{aligned} \tag{7.21}$$

For $0 \leq k \leq m-1$ such that $k \notin E_m$, by (7.17), (5.18) of Lemma 5.10 and (7.21), we have

$$\frac{r'_{k+1}}{R'_{k+1}} \geq \frac{r'_k}{R'_k} - C_2(R'_k)^{1/l} - \frac{L\rho_m}{R'_{k+1}}.$$

If in addition $k \geq N_0$ and $r'_k/R'_k > \alpha_1$, the above inequality and (7.19) implies

$$\log\frac{r'_{k+1}}{R'_{k+1}} \geq \log\frac{r'_k}{R'_k} - \frac{2}{\alpha_1}\left(C(R'_k)^{1/l} + \frac{L\rho_m}{R'_{k+1}}\right), \tag{7.22}$$

here we use the inequality $\log(1-a) \geq -2a$ for $0 < a \leq 1/2$. If $k \in E_m$, by (7.17), (5.24) of Lemma 5.12 and (7.21), we have

$$\frac{r'_{k+1}}{R'_{k+1}} \geq \frac{1}{\theta}\frac{(r'_k)^l}{(R'_k)^l} - \frac{L\rho_m}{R'_{k+1}}.$$

If in addition $k \geq N_0$ and $r'_k/R'_k > \alpha_1$, by (7.19) the above inequality implies

$$\log\frac{r'_{k+1}}{R'_{k+1}} \geq l\log\frac{r'_k}{R'_k} - \frac{2\theta}{\alpha_1^l}\frac{L\rho_m}{R'_{k+1}} - \log\theta, \tag{7.23}$$

again we are using $\log(1-a) \geq -2a$ for $0 < a \leq 1/2$. By 7.5, Lemma 6.4 (ii) and the PCE(λ_0) condition, shrink δ_0 if necessary, we choose N_0 large enough such that

$$l^N \sum_{k=N_0}^{m-1}\left(\frac{2}{\alpha_1}C{R'_k}^{1/l} + \frac{2\theta}{\alpha_1^l}\frac{L\rho_m}{R'_{k+1}}\right) \leq \log 2.$$

Since $\rho_m \to 0$ as $m \to \infty$, for m sufficient large, by (7.1) and the PCE(λ_0) condition, ξ_{a,N_0} is injective when restricted on $\rho_m\mathbb{D}$ and g^{N_0}

is injective on $2D$. By (7.18),

$$\frac{r'_{N_0}}{R'_{N_0}} > 2\alpha_0 > \alpha_1. \tag{7.24}$$

We show that $\frac{r'_p}{R'_p} > \alpha_1$ for every $p = N_0 + 1, \dots, m$ by induction. By (7.24) and the induction hypothesis, we assume that $\frac{r'_{p-1}}{R'_{p-1}} > \alpha_1$. By (7.22) and (7.23), we have

$$\begin{aligned}\log \frac{r'_p}{R'_p} &\geq -l^N \sum_{k=N_0}^{p} \left(\frac{2}{\alpha_1} C {R'_k}^{1/l} + \frac{2\theta}{\alpha_1^l} \frac{L\rho_m}{R'_{k+1}} \right) + l^N \left(\log \frac{r'_{N_0}}{R'_{N_0}} - N \log \theta \right) \\ &> l^N (\log \alpha_0 - N \log \theta) - \log 2 \\ &= \log \alpha_1.\end{aligned}$$

Since $W_m = B(g^m(z), \delta_0)$, we have $R_m = \delta_0$. By (7.5), we get $\frac{r'_m}{\delta_0} \geq \beta \frac{r'_m}{R'_m} \geq \beta\alpha_1$. Set $\delta_1 := \beta\alpha_1\delta_0$, then $B(g^m(z), \delta_1) \subseteq h_m\left(\frac{1}{2}\mathbb{D}\right)$, which concludes the proof. □

7.1. **Construction of invariant correspondences.**

Read maximal entropy measure from bifurcation measure. We say that two sequences of positive real numbers $\rho_n, n \geq 0$ and $\rho'_n, n \geq 0$ are *equivalent* if $\log(\rho_n/\rho'_n), n \geq 0$ is bounded.

Let μ_{f_0} be the maximal entropy measure of f_0. For $\rho \in (0, 1]$, let $[\rho] : \mathbb{D} \to \mathbb{D}$ be the map $t \mapsto \rho t$.

The following two results show that under the assumption of Theorem 5.2, μ_{f_0} can be read from $\mu_{f,a}$ via a suitable rescaling process. Moreover, the scales are determined by $\mu_{f,a}$ itself up to equivalence.

Recall that for a holomorphic family of rational maps $f : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{D} \times \mathbb{P}^1$ as in (2.1) and for a marked point a, we let $\xi_{a,n} : \mathbb{D} \to \mathbb{P}^1(\mathbb{C})$ be the map $\xi_{a,n}(t) := f_t^n(a(t))$.

Proposition 7.1. *Let $f : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{D} \times \mathbb{P}^1$ be a holomorphic family of rational maps as in (2.1) and a be a marked point. Let $n_j, j \geq 0$ be an infinite subsequence of $n \geq 0$. Let $\rho_{n_j}, j \geq 0$ be a sequence of positive real number tending to 0. Define $h_{n_j} := \xi_{a,n_j} \circ [\rho_{n_j}]$. Assume that $h_{n_j} \to h$ locally uniformly. Then we have*

$$d^{n_j} [\rho_{n_j}]^* \mu_{f,a} \to h^*(\mu_{f_0}) \tag{7.25}$$

where the convergence is the weak convergence of measures. Moreover, if $a(0) \in \mathcal{J}(f_0)$ and h is non-constant, then $0 \in \operatorname{supp} h^*(\mu_{f_0})$.

Proof. Define $H_{n_j} : \mathbb{D} \to \mathbb{D} \times \mathbb{P}^1(\mathbb{C})$ by $H_n(t) := (\rho_n t, h_n(\rho_n t))$ and $H : \mathbb{D} \to \mathbb{D} \times \mathbb{P}^1(\mathbb{C})$ by $H(t) := (0, h(\rho_n t))$. Since $H_{n_j} = f^{n_j}(a) \circ [\rho_{n_j}]$, we have

$$H_{n_j}^*(T_f) = [\rho_{n_j}]^*((f^{n_j}(a))^* T_f) = d^{n_j}[\rho_{n_j}]^* \mu_{f,a}. \tag{7.26}$$

Since $H_{n_j} \to H$ locally uniformly, $H_{n_j}^*(T_f) \to H^*(T_f)$. Since $H^*(T_f) = h^*(T_f \wedge [t = 0]) = h^*(\mu_{f_0})$, we get (7.25) by (7.26). Assume that $a(0) \in \mathcal{J}(f_0)$ and h is not constant. Since $h(0) = \lim_{j\to\infty} f_0^{n_j}(a(0)) \in \mathcal{J}(f_0)$, $0 \in \mathrm{supp}\ h^*(\mu_{f_0})$. This concludes the proof. □

Proposition 7.2. *Let μ be a Borel measure on $\mathbb{D}$. Let $\rho_n, \rho'_n, n \geq 0$ be two sequences of real numbers in $(0, 1]$. Let $d_n, n \geq 0$ be a sequence of positive real numbers. Let μ_1, μ_2 be Borel measures on $\mathbb{D}$ having positive mass. Assume that $\mu_1(\{0\}) = \mu_2(\{0\}) = 0$. If $d_n[\rho_n]^*\mu \to \mu_1$ and $d_n[\rho'_n]^*\mu \to \mu_2$, then $\rho_{n_j}, j \geq 0$ and $\rho'_{n_j}, j \geq 0$ are equivalent.*

Proof. Assume by contradiction that it is not the case, then without loss of generality, by passing to a subsequence we may assume that $r_n := \frac{\rho'_n}{\rho_n} \to 0$ as $n \to \infty$. Set $D_n := \mathbb{D}(0, r_n)$. We have

$$d_n[\rho_n]^*\mu(D_n) = d_n[\rho'_n]^*\mu(\mathbb{D}). \tag{7.27}$$

By the property of convergence of measures we have

$$\liminf_{n\to\infty} d_n[\rho'_n]^*\mu(\mathbb{D}) \geq \mu_2(\mathbb{D}) > 0.$$

So for $n \gg 0$, $d_n[\rho'_n]^*\mu(\mathbb{D}) \geq \mu_2(\mathbb{D})/2$. By (7.27), for $n \gg 0$,

$$d_n[\rho_n]^*\mu(\overline{D_n}) \geq d_n[\rho_n]^*\mu(D_n) \geq \mu_2(\mathbb{D})/2.$$

Since $r_n \to 0$ as $n \to 0$, for every $r \in (0, 1)$, we have

$$\mu_1(\mathbb{D}(0, r)) \geq \limsup_{n\to\infty} d_n[\rho_n]^*\mu(\overline{D_n}) \geq \mu_2(\mathbb{D})/2.$$

So $\mu_1(\{0\}) \geq \mu_2(\mathbb{D})/2$ which contradicts to our assumption. □

Asymptotic symmetry. Let X be a complex manifold and Let $\mathcal{H} := \{h_i, i \in A\} \subseteq \mathrm{Hol}\ (\mathbb{D}, X)$ be a family holomorphic maps from $\mathbb{D}$ to X. Let $\lim \mathcal{H}$ be the set of $h \in \mathrm{Hol}\ (\mathbb{D}, X)$, for which there is an infinity sequence of distinct $i_n \in A$ such that $h_{i_n} \to h$ as $n \to \infty$. It is clear that $\lim \mathcal{H}$ is closed in $\mathrm{Hol}\ (\mathbb{D}, X)$ and is contained in $\overline{\mathcal{H}}$. The family $\mathcal{H}$ is normal if and only if $\lim \mathcal{H}$ is compact.

Remark 7.3. Note that $\mathrm{Hol}\ (\mathbb{D}, X)$ is a metric space. So by Lindelöf property, a subset of $\mathrm{Hol}\ (\mathbb{D}, X)$ is compact if and only if it is sequentially compact.

We say that $\mathcal{H}$ is *non-degenerate* if $\mathcal{H}$ is normal and $\lim \mathcal{H}$ does not contain any constant map[3].

Let g be a non-exceptional rational map of degree $d \geq 2$. Let μ_g be the maximal entropy measure of g. Let $\mathcal{H} := \{h_i, i \in A\}, \mathcal{H}' := \{h_i', i \in A\}$ be two families of holomorphic maps from $\mathbb{D}$ to $\mathbb{P}^1$. Set $\mathcal{H} \times_A \mathcal{H}' := \{h_i \times h_i' : \mathbb{D} \to \mathbb{P}^1 \times \mathbb{P}^1\} \subseteq \text{Hol}\,(\mathbb{D}, \mathbb{P}^1 \times \mathbb{P}^1)$. It is clear that $\mathcal{H} \times_A \mathcal{H}'$ is a closed subset of $\mathcal{H} \times \mathcal{H}'$. Such a pair $(\mathcal{H}, \mathcal{H}')$ of families is called an *asymptotic symmetry* of g if the following conditions holds:

(i) both $\{h_i, i \in A\}$ and $\{h_i', i \in A\}$ are non-degenerate;
(ii) $h_i(0), h_i'(0) \in \mathcal{J}(g)$;
(iii) for every $\phi = h \times h' \in \lim(\mathcal{H} \times_A \mathcal{H}')$, we have $h^*\mu_g$ and $(h')^*\mu_g$ are proportional.

We note that for an asymptotic symmetry $(\mathcal{H}, \mathcal{H}')$ and every $\phi = h \times h' \in \lim(\mathcal{H} \times_A \mathcal{H}')$, we have

$$0 \in h^{-1}(\mathcal{J}(g)) = \text{supp}\ h^*\mu_g = \text{supp}\ (h')^*\mu_g = (h')^{-1}(\mathcal{J}(g)).$$

Let U be an open subset on $\mathbb{P}^1$. Let $S(U)$ be the set of injective holomorphic maps $\sigma : U \to \mathbb{P}^1$ such that $\sigma^*\mu_g$ and $\mu_g|_U$ are nonzero and proportional. The following theorem was proved in [JX23b, Theorem 1.7].

Theorem 7.4. *For every $\sigma \in S(U)$, there is $\Gamma \in \text{Corr}(\mathbb{P}^1)^g_*$ such that the image of $\text{id} \times \sigma$ is contained in Γ.*

Remark 7.5. To prove our main result Theorem 1.2, we only need the above theorem in the case where g is TCE. This TCE case can be proved by applying Dujardin-Favre-Gauthier's earlier results [DFG22, Theorem A] and [DFG22, Corollary 3.2].

Recall the following theorem of Levin [Lev90, Theorem 1].

Theorem 7.6. *Every non-degenerate family $\mathcal{F} \subseteq S(U)$ is finite.*

Theorem 7.7. *Let $(\mathcal{H}, \mathcal{H}')$ be an asymptotic symmetry of g. Then there is $\Gamma \in \text{Corr}(\mathbb{P}^1)^g_*$ such that for every $\varepsilon > 0$, there is a finite subset E of A such that for every $i \in A \setminus E$ and $t \in 1/2\mathbb{D}$, we have*

$$d((h_i(t), h_i'(t)), \Gamma) < \varepsilon.$$

Proof. For every $\phi \in \text{Hol}\,(\mathbb{D}, \mathbb{P}^1 \times \mathbb{P}^1)$ and $\varepsilon' > 0$, let $N(\phi, \varepsilon')$ be the open neighborhood of ϕ consisting of $\phi' \in \text{Hol}\,(\mathbb{D}, \mathbb{P}^1 \times \mathbb{P}^1)$ such that for every $t \in (9/10)\overline{\mathbb{D}}$, $d(\phi(t), \phi'(t)) < \varepsilon'$.

[3] When $X = \mathbb{P}^1$, such a $\mathcal{H}$ is called "a non-trivial normal family" in [Lev90].

For every $\phi := h \times h' \in \lim(\mathcal{H} \times_A \mathcal{H}')$, there is a point $t_\phi \in (1/2\mathbb{D}) \cap h^{-1}(\mathcal{J}(g))$ and $r_\phi < 1/100$ such that both h and h' are injective on $\mathbb{D}(t_\phi, 10r_\phi)$. There is $s_\phi > 0$ such that $B_\phi := B(h(t_\phi), s_\phi) \subset\subset h(\mathbb{D}(t_\phi, r_\phi))$. There is $\varepsilon_\phi > 0$ such that for every $\psi = l \times l' \in N(\phi, \varepsilon_\phi)$, we have

(i) both l and l' are injective on $\mathbb{D}(t_\phi, 9r_\phi)$;
(ii) $B_\phi \subset\subset l(\mathbb{D}(t_\phi, r_\phi))$;
(iii) $|d(l' \circ (l|_{\mathbb{D}(t_\phi,r_\phi)})^{-1})(t_\phi)| \in \left(9/10\frac{|dh(t_\phi)|}{|dh(t_\phi)|}, 11/10\frac{|dh(t_\phi)|}{|dh(t_\phi)|}\right)$.

Then for every $\psi = l \times l' \in N(\phi, \varepsilon_\phi) \cap \lim(\mathcal{H} \times_A \mathcal{H}')$, $\sigma_\psi := l' \circ (l|_{\mathbb{D}(t_\phi,r_\phi)})^{-1}|_{B_\phi} : B_\phi \to \mathbb{P}^1$ is in $S(B_\phi)$. Moreover, by (iii) and Koebe distortion theorem, the family $\{\sigma_\psi, \psi \in N(\phi, \varepsilon_\phi) \cap \lim(\mathcal{H} \times_A \mathcal{H}')\}$ is non-degenerate. Then it is finite by Theorem 7.6. By Theorem 7.4, there is $\Gamma_\phi \in \mathrm{Corr}(\mathbb{P}^1)^g_*$ such that for every $\psi \in N(\phi, \varepsilon_\phi) \cap \lim(\mathcal{H} \times_A \mathcal{H}')\}$, the image of $\mathrm{id} \times \sigma_\psi$ is contained in Γ_ϕ. Hence the image of ψ is contained in Γ_ϕ.

Since $\lim(\mathcal{H} \times_A \mathcal{H}')$ is compact, there is a finite set $F \subseteq (\mathcal{H} \times_A \mathcal{H}')$ such that $\lim(\mathcal{H} \times_A \mathcal{H}') \subseteq \cup_{\phi \in F} N(\phi, \varepsilon_\phi)$. Then for every $\psi \in \lim(\mathcal{H} \times_A \mathcal{H}')\}$, the image of ψ is contained in $\Gamma := \cup_\phi \Gamma_\phi \in \mathrm{Corr}(\mathbb{P}^1)^g_*$. Set $W := \cup_{\phi \in \lim(\mathcal{H} \times_A \mathcal{H}')} N(\phi, \varepsilon)$. It is an open neighborhood of $\lim(\mathcal{H} \times_A \mathcal{H}')$. Then $E := \{i \in A | \ h_i \times h'_i \notin W\}$ is finite. For every $i \in A$, there is $\phi \in \lim(\mathcal{H} \times_A \mathcal{H}')$ such that $h_i \times h'_i \in N(\phi, \varepsilon)$. Since the image of ψ is contained in Γ and $1/2\mathbb{D} \subset\subset (9/10)\mathbb{D}$, for every $t \in 1/2\mathbb{D}$, we have

$$d((h_i(t), h'_i(t)), \Gamma) \leq d((h_i(t), h'_i(t)), \phi(t)) < \varepsilon.$$

This concludes the proof. □

Proposition 7.8. *Let $f : \mathbb{D} \times \mathbb{P}^1 \to \mathbb{D} \times \mathbb{P}^1$ be a holomorphic family of rational maps as in (2.1). Let a, b be two marked points. Assume that for both a and b, the parameter $0 \in \mathbb{D}$ satisfies*

(i) $\mathrm{PCE}(\lambda_0)$ *for some* $\lambda_0 > 1$;
(ii) $\mathrm{PR}(s)$ *for some* $s > 0$;
(iii) f_0 *is* $\mathrm{TCE}(\lambda)$ *for some* $\lambda > 1$;

Assume further that $\mu_{f,a}$ and $\mu_{f,b}$ are proportional and f_0 is not exceptional. Then there is $\Gamma_0 \in \mathrm{Corr}(\mathbb{P}^1)^{f_0}_*$ such that the $\mathrm{FS}(\Gamma_0)$ condition does not hold.

Proof. By Theorem 4.3, the $\mathrm{PCE}(\lambda_0)$ condition implies the marked Collet-Eckmann condition for $a(0)$ and $b(0)$. In particular, both $a(0)$ and $b(0)$ are contained in $\mathcal{J}(g)$.

By Theorem 5.2, there are $A_a, A_b \subseteq \mathbb{Z}_{\geq 0}$ with $\underline{d}(A_a) > 9/10$ and $\underline{d}(A_b) > 9/10$ and two sequences of positive numbers $\rho_{a,n}, n \in A_a$, and $\rho_{b,n}, n \in A_b$ both tending to 0 as $n \to \infty$ such that $h_{a,n} : \mathbb{D} \to$

$\mathbb{P}^1(\mathbb{C}), n \in A_a$ and $h_{b,n} : \mathbb{D} \to \mathbb{P}^1(\mathbb{C}), n \in A_b$ as in Theorem 5.2 are non-degenerate. Set $A := A_a \cap A_b$. Then $\underline{d}(A) > 0.8$. The families $\mathcal{H}_a := \{h_{a,n}, n \in A\}$ and $\mathcal{H}_b := \{h_{b,n}, n \in A\}$ are non-degenerate.

We claim that $\rho_{a,n}, n \in A$ and $\rho_{b,n}, n \in A$ are equivalent. Otherwise, we may assume that there is a sequence $n_j \in A$ tending to $+\infty$ such that $\rho_{a,n_j}/\rho_{b,n_j} \to +\infty$. After taking subsequence, we may assume that $h_{a,n_j} \to h_a$ and $h_{b,n_j} \to h_b$, where h_a and h_b are non-constant holomorphic maps. Since $a(0), b(0) \in \mathcal{J}(f_0)$, by Proposition 7.1, we have

$$d^{n_j}[\rho_{a,n_j}]^*\mu_{f,a} \to h_a^*\mu_{f_0} \text{ and } d^{n_j}[\rho_{b,n_j}]^*\mu_{f,b} \to h_b^*\mu_{f_0} \tag{7.28}$$

and 0 is contained in both the supports of $h_a^*(\mu_{f_0})$ and $h_b^*(\mu_{f_0})$. Since μ_{f_0} has continuous potential, it does not have atoms. Since $\mu_{f,a} = c\mu_{f,b}$, by Proposition 7.2, $\rho_{a,n_j}, j \geq 0$ and $\rho_{b,n_j}, j \geq 0$ are equivalent, which is a contradiction. Hence our claim holds. After multiplying $\rho_{a,n}$ by a constant in $(0,1]$, we may assume that $\rho_{a,n} \leq \rho_{b,n}$ for every $n \in A$. There is $q' \in (0,1]$ such that $q_n := \rho_{a,n}/\rho_{b,n} \in [q', 1]$ for every $n \in A$. After replacing $\rho_{b,n}$ by $\rho_{a,n} = q_n\rho_{b,n}$ and $h_{b,n}$ by $h_{b,n} \circ [q_n]$, we may assume that $\rho_{a,n} = \rho_{b,n}$ for every $n \in A$. Set $\rho_n := \rho_{a,n} = \rho_{b,n}$. Then for any sequence $n_j \in A$ tending to $+\infty$ satisfying $h_{a,n_j} \to h_a$ and $h_{b,n_j} \to h_b$, (7.28) becomes

$$d^{n_j}[\rho_{n_j}]^*\mu_{f,a} \to h_a^*\mu_{f_0} \text{ and } d^{n_j}[\rho_{n_j}]^*\mu_{f,b} \to h_b^*\mu_{f_0} \tag{7.29}$$

Since $\mu_{f,a}$ and $\mu_{f,b}$ are proportional and 0 is contained in both of $\operatorname{supp} h_a^*(\mu_{f_0})$ and $\operatorname{supp} h_b^*(\mu_{f_0})$, $h_a^*(\mu_{f_0})$ and $h_b^*(\mu_{f_0})$ are non-zero and proportional. This implies that the pair of families $(\mathcal{H}_a, \mathcal{H}_b)$ is an asymptotic symmetry. By Theorem 7.7, there is $\Gamma_0 \in \mathrm{Corr}(\mathbb{P}^1)_*^{f_0}$ such that for every $\varepsilon > 0$, there is a finite subset E_ε of A such that for every $n \in A \setminus E_\varepsilon$, we have

$$d((h_{a,n}(0), h_{b,n}(0)), \Gamma_0) < \varepsilon.$$

Hence, for every $\varepsilon > 0$,

$$\overline{d}(\{n \geq 0 |\ d((h_{a,n}(0), h_{b,n}(0)), \Gamma_0) \geq \varepsilon\}) \leq 1 - \underline{d}(A) < 1/5.$$

So the $\mathrm{FS}(\Gamma_0)$ condition does not hold, which concludes the proof. □

8. Proof of the main theorems

Proof of Theorem 1.3. The direction (ii) implies (i) was proved by DeMarco [DeM16, Section 6.4]. The direction (i) implies (iii) is trivial. We only need to show that (iii) implies (ii).

We may assume that $\phi_f(\Lambda)$ is not contained in the locus of flexible Lattès maps, otherwise Theorem 1.3 trivially holds. As f is defined over $\overline{\mathbb{Q}}$, there is a variety Λ_0 over $\overline{\mathbb{Q}}$ and a morphism $F : \Lambda_0 \times \mathbb{P}^1 \to \mathbb{P}^1$ over $\overline{\mathbb{Q}}$ such that $\Lambda = \Lambda_0 \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ and $f = F \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$. After replacing Λ_0 by a finite ramification cover on it over $\overline{\mathbb{Q}}$, we may assume that F has $2d-2$ marked critical points $a_i, i = 1, \dots, 2d-2$ counted with multiplicity. Let c_i be the base change of a_i. Then we get marked critical points $(c_i)_{1\leq i\leq 2d-2}$.

Since f is not isotrivial and $\phi_f(\Lambda)$ is not contained in the locus of flexible Lattès maps, $\mu_{\mathrm{bif}} \neq 0$. By Corollary 2.4, for every $1 \leq i \leq 2d-2$, μ_{f,c_i} is proportional to μ_{bif}. Moreover, by Theorem 2.2, $\mu_{f,c_i} \neq 0$ if and only if c_i is active. Let c_1, c_2 be two marked critical points, we need to show they are dynamically related. If one of c_1, c_2 is preperiodic, then c_1 and c_2 are dynamically related holds trivially. So we only need to consider the case that c_1 and c_2 are active. Assume for the sake of contradiction that c_1 and c_2 are active but not dynamically related.

By Corollary 3.14 we have that

(a) for μ_{bif}-a.e. point $t \in \Lambda(\mathbb{C})$, for every $\Gamma_t \in \mathrm{Corr}(\mathbb{P}^1)_*^{f_t}$, the pair $c_1(t), c_2(t)$ satisfies the $\mathrm{AS}(\Gamma_t)$ condition for f_t.

Let $1 < \lambda < d^{1/2}$ and $s > 1/2$. By Proposition 4.10 we have

(b) for μ_{bif}-a.e. $t \in \Lambda(\mathbb{C})$, t satisfies $\mathrm{CE}^*(\lambda)$, $\mathrm{PCE}(\lambda)$ and $\mathrm{PR}(s)$ for all c_i that is active.

Definition 8.1. A rational map g of degree at least 2 is called *Collet-Eckmann* $\mathrm{CE}(\lambda)$ for some $\lambda > 1$ if:

(i) There exists $C > 0$ such that for every critical point $c \in \mathcal{J}(g)$, there exists $N > 0$ such that $|dg^n(g^N(c))| \geq C\lambda^n$ for every $n \geq 1$;

(ii) g has no parabolic cycle.

The following proposition is useful. A precise statement can be found in [PRLS03, Main Theorem].

Proposition 8.2 (Przytycki-Rohde [PR98])**.** $\mathrm{CE}(\lambda_0)$ *implies* $\mathrm{TCE}(\lambda)$ *for every* $1 < \lambda < \lambda_0$.

We have the following two results.

Proposition 8.3. *There exists* $\lambda > 1$ *such that*

(c) *for* μ_{bif}*−a.e. point* t*,* f_t *is* $\mathrm{CE}(\lambda)$*.*

Proof. By Theorem 2.2, a marked critical point c is either preperiodic or satisfies $\mu_{f,c} > 0$. Let $I \subseteq \{1, 2, \dots, 2d-2\}$ be the index such that c_i is preperiodic when $i \in I$.

By (b), for every $i \in \{1, 2, \ldots, 2d-2\} \setminus I$, $c_i(t)$ satisfies (i) in Definition 8.1 for μ_{bif}-a.e. point t. This implies that for μ_{bif}-a.e. point t, the critical points are either contained in the Julia set or preperiodic. Note that every parabolic basin contains a critical point, and this critical point can not be preperiodic. This implies that μ_{bif}-a.e. point t satisfies (ii) in Definition 8.1 i.e. f_t has no parabolic cycle.

For each fixed $i \in I$, we let p_i be the cycle which c_i is preperiodic to. We only need to show the following claim: μ_{bif}-a.e. point $t \in \Lambda$ satisfies the following property: if $c_i(t) \in \mathcal{J}(f_t)$, then $c_i(t)$ is preperiodic to a repelling cycle.

Let $\lambda_{p_i} : \Lambda \to \mathbb{C}$ be the holomorphic map defined by the multiplier of the cycle $p_i, i \in I$. There are two cases.

Case 1: λ_{p_i} is a constant map. As f is defined over $\overline{\mathbb{Q}}$, $\lambda_{p_i} \in \overline{\mathbb{Q}}$. If $|\lambda_{p_i}| \neq 1$, then the claim holds. If $|\lambda_{p_i}| = 1$, since for μ_{bif}-a.e. point t, $p_i(t)$ is not a parabolic cycle, λ_{p_i} is not a root of unity. By Siegel's linearization theorem [Mil11, Theorem 11.4] and Baker's theorem (c.f. [Bak22, Theorem 3.1]), $p_i(t)$ is a Siegel cycle. This implies our claim.

Case 2: λ_{p_i} is not a constant map. If the claim is not true for a parameter t, then $c_i(t)$ is preperiodic to a parabolic cycle or a Cremer cycle. Since λ_{p_i} is not a constant map, $\{t \in \Lambda(\mathbb{C}) |\ \lambda_{p_i} \text{ is a root of unity}\}$ is countable. Recall that by Siegel's linearization theorem [Mil11, Theorem 11.4], if $p_i(t)$ is a Cremer cycle, then $\lambda_{p_i}(t) = e^{2\pi i \alpha}$, where α is a Liouville number. Recall that the set of Liouville numbers has Hausdorff dimension 0 [Mil11, Lemma C.7]. Let $\mathcal{CP} \subseteq \Lambda$ be the parameters in Λ such that $p_i(t)$ is Cremer or parabolic. Then the Hausdorff dimension of $\mathcal{CP}$ is 0. On the other hand since μ_{bif} has Hölder continuous potential [DS10, Lemma 1.1], the Hausdorff dimension of μ_{bif} is strictly positive [Sib99, Theorem 1.7.3]. This implies our claim. ☐

Since $\phi_f(\Lambda)$ is not contained in the locus of flexible Lattès maps and not a point, the parameters t such that f_t is exceptional are finite. Since μ_{bif} does not have atoms, we have

(d) for μ_{bif}-a.e. point $t \in \Lambda(\mathbb{C})$,$f_t$ is not exceptional.

Combining (b), (c), (d) and Proposition 7.8, we get that

(e) for μ_{bif}-a.e. point $t \in \Lambda(\mathbb{C})$, there is $\Gamma_t \in \text{Corr}(\mathbb{P}^1)_*^{f_t}$ such that the $\text{FS}(\Gamma_0)$ condition does not hold.

Since (e) contradicts (a), we finish the proof of Theorem 1.2. ☐

We now deduce Theorem 1.2 from Theorem 1.3.

Proof of Theorem 1.2. By Theorem 1.3, we only need to reduce to the case where f is defined over $\overline{\mathbb{Q}}$. Set $\psi := \Psi \circ \phi_f : \Lambda \to \mathcal{M}_d$. Let B be the Zariski closure of $\psi(\Lambda)$ in $\mathcal{M}_d$. Since f is not isotrivial, B is a curve and the map $\psi : \Lambda \to B$ is quasi-finite. We may assume that $\phi_f(\Lambda)$ is not contained in the locus of flexible Lattès maps, otherwise Theorem 1.2 trivially holds. Since there are infinitely many PCF parameters $t \in \Lambda$, by Thurston's rigidity of PCF maps [DH93], $B \cap \mathcal{M}_d(\overline{\mathbb{Q}})$ is infinite. Hence B is defined over $\overline{\mathbb{Q}}$. Since $Z := \Psi^{-1}(B)$ is defined over $\overline{\mathbb{Q}}$, there is a smooth curve Λ_1 and a morphism $\phi_1 : \Lambda_1 \to Z$ defined over $\overline{\mathbb{Q}}$ such that the morphism $\psi_1 := \Psi \circ \phi_1 : \Lambda_1 \to B$ is dominant. Let $g : \Lambda_1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the family of rational maps induced by ϕ_1. It is clear that Theorem 1.2 holds for f if and only if it holds for g, this concludes the proof. $\square$

References

[AGMV19] Matthieu Astorg, Thomas Gauthier, Nicolae Mihalache, and Gabriel Vigny. Collet, Eckmann and the bifurcation measure. *Inventiones mathematicae*, 217(3):749–797, 2019.

[Bak22] Alan Baker. *Transcendental number theory*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2022. With an introduction by David Masser, Reprint of the 1975 original [0422171].

[BDM13] Matthew Baker and Laura De Marco. Special curves and postcritically finite polynomials. *Forum of Mathematics, Pi*, 1, 2013.

[BIJL14] Robert Benedetto, Patrick Ingram, Rafe Jones, and Alon Levy. Attracting cycles in p-adic dynamics and height bounds for postcritically finite maps. *Duke Mathematical Journal*, 163(13):2325–2356, 2014.

[CLN69] S. S. Chern, Harold I. Levine, and Louis Nirenberg. Intrinsic norms on a complex manifold. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 119–139. Univ. Tokyo Press, Tokyo, 1969.

[CLT09] Antoine Chambert-Loir and Amaury Thuillier. Mesures de Mahler et équidistribution logarithmique. *Ann. Inst. Fourier (Grenoble)*, 59(3):977–1014, 2009.

[Dem] Jean-Pierre Demailly. Complex analytic and differential geometry. available at https://www-fourier.ujf-grenoble.fr/ demailly/.

[DeM16] Laura DeMarco. Bifurcations, intersections, and heights. *Algebra and Number Theory*, 10(5):1031–1056, 2016.

[DeM18] Laura DeMarco. Critical orbits and arithmetic equidistribution. In *Proceedings of the International Congress of Mathematicians (ICM 2018) (In 4 Volumes) Proceedings of the International Congress of Mathematicians 2018*, pages 1847–1866. World Scientific, 2018.

[DF08] Romain Dujardin and Charles Favre. Distribution of rational maps with a preperiodic critical point. *Amer. J. Math.*, 130(4):979–1032, 2008.

[DFG22] Romain Dujardin, Charles Favre, and Thomas Gauthier. When do two rational functions have locally biholomorphic Julia sets? *Transactions of the American Mathematical Society*, 2022.

[DH93] Adrien Douady and John H. Hubbard. A proof of Thurston's topological characterization of rational functions. *Acta Math.*, 171(2):263–297, 1993.

[DM18] Laura De Marco. Dynamical moduli spaces and elliptic curves. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 27, pages 389–420, 2018.

[DMWY15] Laura De Marco, Xiaoguang Wang, and Hexi Ye. Bifurcation measures and quadratic rational maps. *Proceedings of the London Mathematical Society*, 111(1):149–180, 2015.

[DPU96] Manfred Denker, Feliks Przytycki, and Mariusz Urbański. On the transfer operator for rational functions on the Riemann sphere. *Ergodic Theory and Dynamical Systems*, 16(2):255–266, 1996.

[DS10] Tien-Cuong Dinh and Nessim Sibony. Dynamics in several complex variables: endomorphisms of projective spaces and polynomial-like mappings. *Holomorphic dynamical systems*, pages 165–294, 2010.

[DTGV21] Henry De Thélin, Thomas Gauthier, and Gabriel Vigny. Parametric Lyapunov exponents. *Bulletin of the London Mathematical Society*, 53(3):660–672, 2021.

[FG18] Charles Favre and Thomas Gauthier. Classification of special curves in the space of cubic polynomials. *International Mathematics Research Notices*, 2018(2):362–411, 2018.

[FG22] Charles Favre and Thomas Gauthier. The arithmetic of polynomial dynamical pairs:(ams-214). *Annals of Mathematics Studies*, 401, 2022.

[Gau21] Thomas Gauthier. Good height functions on quasi-projective varieties: equidistribution and applications in dynamics. *arXiv preprint arXiv:2105.02479*, 2021.

[Gau22] Thomas Gauthier. Dynamical pairs with an absolutely continuous bifurcation measure. *Ann. Fac. Sci. Toulouse Math*, 2022.

[GHT13] Dragos Ghioca, Liang-Chung Hsia, and Thomas Tucker. Preperiodic points for families of polynomials. *Algebra and Number Theory*, 7(3):701–732, 2013.

[GHT15] Dragos Ghioca, L-C Hsia, and Thomas J Tucker. Preperiodic points for families of rational maps. *Proceedings of the London Mathematical Society*, 110(2):395–427, 2015.

[GKN16] Dragos Ghioca, Holly Krieger, and Khoa Nguyen. A case of the dynamical André–Oort conjecture. *International Mathematics Research Notices*, 2016(3):738–758, 2016.

[GKNY17] Dragos Ghioca, Holly Krieger, Khoa Dang Nguyen, and Hexi Ye. The dynamical André–Oort conjecture: unicritical polynomials. *Duke Mathematical Journal*, 166(1):1–25, 2017.

[GV19] Thomas Gauthier and Gabriel Vigny. The Geometric Dynamical Northcott and Bogomolov Properties. arXiv:1912.07907, 2019.

[GY18] Dragos Ghioca and Hexi Ye. A dynamical variant of the André-Oort conjecture. *International Mathematics Research Notices*, 2018(8):2447–2480, 2018.

[Ji23a] Junyi Ji, Zhuchao andXie. The multiplier spectrum morphism is generically injective. *to appear*, 2023.

[Ji23b] Zhuchao Ji. Non-uniform hyperbolicity in polynomial skew products. *International Mathematics Research Notices*, 2023(10):8755–8799, 2023.

[JX23a] Zhuchao Ji and Junyi Xie. Homoclinic orbits, multiplier spectrum and rigidity theorems in complex dynamics. *Forum of Mathematics, Pi*, 11:e11, 2023.

[JX23b] Zhuchao Ji and Junyi Xie. Local rigidity of Julia sets. *arXiv preprint*, 2023.

[Lev90] G. M. Levin. Symmetries on Julia sets. *Mat. Zametki*, 48(5):72–79, 159, 1990.

[Mil11] John Milnor. *Dynamics in One Complex Variable.(AM-160):(AM-160)-*, volume 160. Princeton University Press, 2011.

[PR98] Feliks Przytycki and Steffen Rohde. Porosity of Collet-Eckmann Julia sets. *Fundamenta Mathematicae*, 155(2):189–199, 1998.

[PRLS03] Feliks Przytycki, Juan Rivera-Letelier, and Stanislav Smirnov. Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps. *Inventiones mathematicae*, 151(1):29–63, 2003.

[PST21] Jonathan Pila, Ananth N Shankar, and Jacob and Tsimerman. Canonical heights on Shimura varieties and the André-Oort conjecture, Appendix by Esnault, Hélène and Groechenig, Michael. *arXiv preprint arXiv:2109.08788*, 2021.

[Sib99] Nessim Sibony. Dynamique des applications rationnelles de P^k. *Dynamique et géométrie complexes (Lyon, 1997)*, 8:97–185, 1999.

[Sil07] Joseph H. Silverman. *The arithmetic of dynamical systems*, volume 241 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2007.

[Sil12] Joseph H. Silverman. *Moduli spaces and arithmetic dynamics*, volume 30 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2012.

[Tan90] Lei Tan. Similarity between the Mandelbrot set and Julia sets. *Comm. Math. Phys.*, 134(3):587–617, 1990.

[Ull98] Emmanuel Ullmo. Positivité et discrètion des points algébriques des courbes. *Ann. of Math. (2)*, 147:167–179, 1998.

[Xie15] Junyi Xie. Intersections of valuation rings in $k[x, y]$. *Proc. Lond. Math. Soc. (3)*, 111(1):240–274, 2015.

[XY23] Junyi Xie and Xinyi Yuan. Partial heights, entire curves, and the geometric Bombieri–Lang conjecture. *arXiv preprint arXiv:2305.14789*, 2023.

[YZ17] Xinyi Yuan and Shou-Wu Zhang. The arithmetic Hodge index theorem for adelic line bundles. *Math. Ann.*, 367(3-4):1123–1171, 2017.

[YZ21] Xinyi Yuan and Shou-Wu Zhang. Adelic line bundles over quasi-projective varieties. arXiv preprint arXiv:2105.13587, 2021.

[Zha98] Shou-Wu Zhang. Equidistribution of small points on abelian varieties. *Ann.of Math. (2), 147(1998)*, 147:159–165, 1998.

Institute for Theoretical Sciences, Westlake University, Hangzhou 310030, China

E-mail address: `jizhuchao@westlake.edu.cn`

Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China

E-mail address: `xiejunyi@bicmr.pku.edu.cn`