NUMERICAL SPECTRUMS CONTROL COHOMOLOGICAL SPECTRUMS

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Abstract. Let X be a smooth irreducible projective variety over a field $\mathbf{k}$ with $\dim X = d$. Let $\tau : \mathbb{Q}_l \to \mathbb{C}$ be any field embedding. Let $f : X \to X$ be a surjective endomorphism. We show that for every $i = 0, \dots, 2d$, the spectral radius of f^* on the numerical group $N^i(X) \otimes \mathbb{R}$ and on the l-adic cohomology group $H^{2i}(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \otimes \mathbb{C}$ are the same. As a consequence, if f is q-polarized for some $q > 1$, we show that the norm of every eigenvalue of f^* on the j-th cohomology group is $q^{j/2}$ for all $j = 0, \dots, 2d$. This proves a conjecture of Tate. We also get some apply applications for the counting of fixed points and its "moving target" variant.

Indeed we studied the more general actions of certain cohomological coorespondences and we get the above results as consequences in the endomorphism setting.

1. Introduction

The aim of the paper is to compare the eigenvalues on numerical groups and on l-adic cohomology groups for actions of endomorphisms or more generally for certain cohomological coorespondences on smooth projective varieties.

1.1. **Action of endomorphisms.** Let X be a smooth irreducible projective variety over a field $\mathbf{k}$ with $\dim X = d$. Let $f : X \to X$ be a surjective endomorphism.

Numerical spectrum. For every $i = 0, \dots, d$, the *i-th numerical spectrum* of f is $\mathrm{Sp}_i^{num}(f)$ the multi-set of eigenvalues of

$$f^* : N^i(X) \otimes \mathbb{R} \to N^i(X) \otimes \mathbb{R}.$$

where $N^i(X)$ the group of numerical cycles of codimension i of X. The *i-th numerical spectral radius* is

$$\beta_i(f) := \max\{|a|| \ a \in \mathrm{Sp}_i^{num}(f)\}.$$

By [Tru20, Theorem 1.1(3)] (see also [DS05, Dan20]), the sequence $\beta_i(f), i = 0, \dots, d$ is log-concave i.e.

$$\beta_i(f)^2 \geq \beta_{i-1}(f)\beta_{i+1}(f)$$

for $i = 1, \dots, d-1$.

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Remark 1.1. There is an important notion of *dynamical degrees* $\lambda_i(f), i = 0, \ldots, d$, which are well define even when f is merely a dominant rational self-map. [RS97, DS05, DS04, Tru20, Dan20]. In our setting where f is an endomorphism, these two notion coincide i.e. $\lambda_i(f) = \beta_i(f)$.

By projection formula and the Poincaré duality, we have [c.f. Proposition 2.17]

$$\beta_i^-(f) := \min\{|a||\ a \in \mathrm{Sp}_i^{num}(f)\} = \deg f/\beta_{d-i}(f).$$

Cohomological spectrum. Let l be a prime number with $l \neq \mathrm{char}\,\mathbf{k}$. Let

$$G(X) := \{j = 0, \ldots, 2d|\ H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \neq 0\}.$$

By Poincaré duality, $j \in G(X)$ if and only if $2d - j \in G(X)$. Moreover

$$\{2i|\ i = 0, \ldots, d\} \subseteq G(X).$$

For $j = 0, \ldots, 2d$, the j-th cohomological spectrum $\mathrm{Sp}_j(f)$ is the multi-set of eigenvalues of

$$f^* : H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \to H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l).$$

By Fact 2.1, which is a consequence of Deligne's proof of Weil conjecure [KM74, Theorem 2], all elements in $\mathrm{Sp}_j(f)$ are algebraic integers and moreover $\mathrm{Sp}_j(f)$ is $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-invariant.

Fix any embedding $\tau : \mathbb{Q}_l \hookrightarrow \mathbb{C}$ and view $\mathbb{Q}_l$ as a subfield of $\mathbb{C}$ via τ. The image of $\mathrm{Sp}_j(f)$ in $\mathbb{C}$ does not depend on the choice of τ. We may view $\mathrm{Sp}_j(f)$ as a subset of $\mathbb{C}$. For $j = 0, \ldots, 2d$, the j-th *cohomological spectral radius* is

$$\alpha_j(f) := \max\{|b||\ b \in \mathrm{Sp}_j(f)\}.$$

In particular, if $j \notin G(X)$, then $\alpha_j(f) = 0$. For $j \in G(X)$, we have

$$\alpha_j^-(f) := \min\{|b||\ b \in \mathrm{Sp}_j(f)\} = \frac{\deg(f)}{\alpha_{2d-j}(f)}.$$

Main result and consequences. The following is our main result in the endomorphism case, which is indeed a consequence of the more general result Theorem 1.14 for *bi-finite* correspondences.

Theorem 1.2 (=Corollary 2.18)**.** *For every $i = 0, \ldots, d$, we have*

$$\log \alpha_{2i}^-(f) = \log \beta_i^-(f) \ \textit{and}\ \log \beta_i(f) = \log \alpha_{2i}(f).$$

For every odd $j \in G(X)$, we have

$$\frac{\log \beta_{j-1}^-(f) + \log \beta_{j+1}^-(f)}{2} \leq \log \alpha_j^-(f) \leq \log \alpha_j(f) \leq \frac{\log \beta_{j-1}(f) + \log \beta_{j+1}(f)}{2}.$$

Remark 1.3. Theorem 1.2 can be state in a more geometric way as follows: Consider the following sets of points in $\mathbb{R}^2$:

$$F_{num} := \{(2i, \log|u|)|\ u \in \mathrm{Sp}_i^{num}(f)\} \text{ and } F_{coh} := \{(j, \log|u|)|\ u \in \mathrm{Sp}_j(f))\}.$$

Let C_{num} and C_{coh} be their convex envelops. Then we have

$$C_{num} = C_{coh}.$$

Moreover for every $i = 0, \ldots, d$, the end points of $C_{num} \cap (\{2i\} \times \mathbb{R})$ are contained in F_{num}.

Theorem 1.14 answers a question of Truong [Tru24, Question 2] positively for endomorphisms. By a tensoring product trick originally due to Dinh [Din05, Proposition 5.8], positive answer of [Tru24, Question 2] implies positive answer of [Tru24, Question 4].

Historical notes. When $\mathbf{k} = \mathbb{C}$, by comparison theorem between singular and $\mathbb{Q}_l$-cohomologies, we can consider the singular cohomology instead of the $\mathbb{Q}_l$-cohomology. After such translation, Theorem 1.2 is well known in complex dynamics (see for example [DS17, Section 4]).

In positive characteristic, the first progress was due to Esnault-Srinivas [ES13]. They showed that when f is an automorphism on a smooth projective surface, then $\alpha_2 = \beta_1$. Later Truong [Tru24] generated Esnault-Srinivas's result in any dimension by showing that

$$\max_{j=0,\dots,2d} \alpha_j = \max_{i=0\dots,d} \beta_i.$$

Shuddhodan [Shu19] gave an alternative approach towards this equality using dynamical zeta functions. Recently, Hu proved Theorem 1.2 for endomorphisms of abelian varieties [Hu19, Hu].

Next we apply Theorem 1.2 to endomorphisms satisfying some numerical conditions.

Definition 1.4. For $q \geq 1$, we say that f is *q-straight*, if $\beta_i(f) = q^i$ for all $i = 0, \dots, d$. We say that f is *straight* if it is q-straight for some $q \geq 1$.

Recall that an endomorphism $f : X \to X$ is called *q-polarized* for some $q > 1$, if there is an ample line bundle L such that $f^*L = qL$. For example, if $k = \mathbb{F}_q$, the q-Frobenius is q-polarized. If f is q-polarized, then $\beta_i(f) = q^i$ for $i = 0, \dots, d$. So all polarized endomorphisms are straight. By Theorem 1.2, we get the following consequence.

Corollary 1.5. *Assume that $f : X \to X$ is a q-straight endomorphism for some $q \geq 1$. Then for every $j \in G(X)$, we have*

$$\alpha_j^-(f) = \alpha_j(f) = q^{j/2}.$$

This result generalizes Deligne's theorem [Del74], which proves Weil's Riemann Hypothesis, for any straight endomorphisms. However, Deligne's theorem [Del74] plays a key role in our proof of Theorem 1.2 (hence Corollary 1.5).

Our Corollary 1.5 proved a conjecture proposed by Tate in 1964 [Tat64, §3, Conjecture (d)], see also [Tat65, §3, Conjecture (d)]. In [HT, Conjecture 1.4], Hu and Truong proposed the same conjecture and call it the Generalized Weil's Riemann Hypothesis. When $\mathbf{k} = \mathbb{C}$ and f is polarized, this theorem was proved by Serre [Ser60] using the Hodge structure and Serre viewed his result as an Kählerian analogy of Weil's conjecture.

An endomorphism $f : X \to X$ is called *int-amplified* if there is an ample line bundle L of X such that $f^*L - L$ is ample [Men20]. The following fact was observed by Matsuzawa.

Fact 1.6. [MZ, Proposition 3.7] A surjective endomorphism $f : X \to X$ is int-amplified if and only if $\beta_d(f) > \beta_{d-1}(f)$.

Indeed the log-concavity of $\beta_i(f), i = 0, \dots, d$ implies that the sequence $\beta_i(f), i = 0, \dots, d$ is strictly increasing in the int-amplified case.

By [Fak03, Theorem 5.1], if f is int-amplified, then for every $n \geq 1$, the set of n-periodic points $\mathrm{Fix}(f^n)$ is isolated. Denote by $\#\mathrm{Fix}(f)$ the number of fixed points counting with multiplicity. Combine the Lefschetz fixed point theorem with Corollary 1.5, we get the following estimates.

Corollary 1.7. *Assume that $f : X \to X$ is an int-amplified amplified endmorphism. Then we have*

$$\#\mathrm{Fix}(f^n) = \beta_d^n + O((\beta_d \beta_{d-1})^{n/2}).$$

In particular, if f is q-straight for some $q > 1$, then we have

$$\#\mathrm{Fix}(f^n) = q^{dn} + O(q^{(d-1/2)n}).$$

Remark 1.8. When $\mathbf{k} = \mathbb{C}$, stronger results can be proved. For example, in the recent work [DZ], Dinh-Zhong can counted the periodic points without multiplicity when f is merely a rational map with dominant topological degree.

We also proved a "moving target" version of Corollary 1.7, see Proposition 2.19.

Proposition 1.9 (=Proposition 2.19)**.** *Assume that $f : X \to X$ is an int-amplified amplified endmorphism. Let L be an ample line bundle on X. Let $h_n : X \to X, n \geq 0$ be a sequence of endomorphisms of X with*

$$\limsup_{n \to \infty} (h_n^* L \cdot L^{d-1})^{1/n} < \beta_d(f)/\beta_{d-1}(f).$$

Then for $n \gg 0$, f^n intersects h_n properly in $X \times X$. Moreover, for every $\epsilon > 0$ we have

$$\#\{f^n(x) = h(x)\} = q^{dn} + o((\beta_d \beta_{d-1})^{(1+\epsilon)n})$$

counting with multiplicity.

1.2. **Action of cohomological correspondences.** Let X be a smooth irreducible projective variety over a field $\mathbf{k}$ with pure dimension $\dim X = d$. Here X may not be irreducible.

Denote by $\mathcal{C}(X, X)_{\mathbb{Z}}$ the space of degree 0 cohomological correspondences (for the $\mathbb{Q}_l$-cohomology) from X to itself i.e. the image of the cycle class map

$$cl : \mathrm{CH}^d(X \times X) \to H^d(X_{\overline{\mathbf{k}}} \times X_{\overline{\mathbf{k}}})(d).$$

With the composition $\circ$, it forms a ring. Denote by $\mathcal{C}(X, X) := \mathcal{C}(X, X)_{\mathbb{Z}} \otimes \mathbb{Q}$.

For $c \in \mathcal{C}(X, X)$, we define the numerical and cohomological spectral radius $\beta_i(c), i = 0, \dots, d$, $\alpha_j(c), j = 0, \dots, d$ in the same way as the case for endomorphisms. See Section 2.2 and 2.4 for details.

Every endomorphism $f : X \to X$ can be viewed as a d-cycle in $X \times X$ via its graph, hence induces a cohomological correspondences.

Definition 1.10. A *finite correspondence* (resp. *bi-finite correspondence*) Γ is defined to be an effective cycle in $X \times X$ with $\mathbb{Q}$-coefficients of dimension d such that the projection from Γ to the first (resp. each factor) is finite.

The above notions are generalizations of the notions of endomorphisms and finite endomorphisms.

Definition 1.11. We call a cohomological correspondence $c \in \mathcal{C}(X, X)$ *finite* (resp. *bi-finite*) if it takes form

$$c = cl(\Gamma)$$

where Γ is a finite (resp. bi-finite) correspondence.

Here are some examples of bi-finite cohomological correspondences which I feel especially interesting:

Example 1.12 (Random product)**.** Let $f_1, \dots, f_m$ be finite endomorphisms on X. Define

$$c := \sum_{i=1}^{m} a_i f_i + \sum_{i=1}^{m} b_i {}^{\top} f_i$$

with $a_i, b_i \in \mathbb{Q}_{\geq 0}$. It is clear that c is bi-finite.

If $\sum_{i=1}^{m}(a_i + b_i) = 1$, we may think that c is a random product of $f_1, \dots, f_m$ and their transports ${}^{\top} f_1, \dots, {}^{\top} f_m$ with probability $a_1, \dots, a_m, b_1, \dots, b_m$. One may rely the study of c^n to the random product of matrices.

Example 1.13 (Extensions of endomorhisms)**.** Let $f : X \to X$ be a surjective endomorphism. Let Y be a smooth projecitve variety of pure dimension d. Let $\pi : Y \to X$ be a finite surjective morphism. Let $c_1, \dots, c_m$ be all irreducible components of $(\pi \times \pi)^{-1}(f)$. For $a_i \in \mathbb{Q}_{\geq 0}, i = 0, \dots, m$, the cohomological correspondence c induced by $\sum_{i=1}^{m} a_i c_i$ is bi-fnite. Assume that

$$f \subseteq \operatorname{Supp}(\pi \times \pi)_* c.$$

Then by Lemma 2.9 and the projection formula, we may check that $\beta_i(c) = \beta_i(f)$ for every $i = 0, \dots, d$.

For $c \in \mathcal{C}(X, X)$. Define the *Cohomologcial polygon* for c to be the minimal concave function

$$CP_c : [0, 2d] \to \mathbb{R} \cup \{-\infty\}$$

such that

$$CP_c(j) \geq \log \alpha_j(c)$$

for all $j = 0, \dots, 2d$.

Similarly, define the *numerical polygon* for c is the minimal concave function

$$NP_c : [0, 2d] \to \mathbb{R} \cup \{-\infty\}$$

such that

$$NP_c(2i) \geq \log \beta_i(c)$$

for all $i = 0, \dots, d$.

As the numerical equivalence is weaker than the cohomological one, we have $NP_c \leq CP_c$. The following result shows that in the finite case, the above two polygons are indeed coincide.

Theorem 1.14 (=Theorem 2.12). *If c is finite, then $NP_c = CP_c$.*

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1.3. **Notations.**

- Let V be a finitely dimensional vector space over a field K of characteristic 0 and $f : V \to V$ be an endomorphism. We denote by $\mathrm{Sp}\,(f)$ the spectrum of f, i.e. the multi-set of eigenvalues of f and $P(f)$ the characteristic polynomial. If K is $\mathbb{R}$ or $\mathbb{C}$ and $V \neq \{0\}$, we denote by
$$\rho(f) := \max\{|b||\ b \in \mathrm{Sp}\,(f)\}$$
the spectral radius of f. For the convenience, if $V = \{0\}$, we define $\rho(f) = 0$. If we fix any norm $\|\cdot\|$ and denote by $\|f^n\|$ the operator norm of f^n for $n \geq 0$, we have
$$\rho(f) = \lim_{n\to\infty} \|f^n\|^{1/n}.$$
- A *variety* is a separated reduced scheme of finite type over a field.
- For any projective variety Y of dimension d and $i = 0, \dots, d$, denote by $N_i(X)$ the group of numerical cycles of dimension i. Denote by
$$N_i(X)_{\mathbb{R}} := N_i(X) \otimes \mathbb{R}.$$
If Y is further smooth, denote
$$N^i(Y) := N_{d-i}(Y) \text{ and } N^i(Y)_{\mathbb{R}} := N_{d-i}(Y)_{\mathbb{R}}.$$

2. Cohomological and numerical eigenvalues

Let X be a smooth projective variety over a field $\mathbf{k}$ with pure dimension $\dim X = d$. Let l be a prime number with $l \neq \mathrm{char}\,\mathbf{k}$. We fix an embedding $\tau : \mathbb{Q}_l \hookrightarrow \mathbb{C}$ and view $\mathbb{Q}_l$ as a subfield of $\mathbb{C}$ via τ. Let
$$G(X) := \{i = 0, \dots, 2d|\ H^i(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \neq 0\}.$$
By Poincaré duality, $i \in G(X)$ if and only if $2d - i \in G(X)$.

Denote by $\mathcal{C}(X, X)_{\mathbb{Z}}$ the space of degree 0 cohomological correspondences (for the $\mathbb{Q}_l$-cohomology) from X to itself i.e. the image of the cycle class map
$$cl : \mathrm{CH}^d(X \times X) \to H^d(X_{\overline{\mathbf{k}}} \times X_{\overline{\mathbf{k}}})(d).$$
With the composition $\circ$, it forms a ring. Denote by $\mathcal{C}(X, X) := \mathcal{C}(X, X)_{\mathbb{Z}} \otimes \mathbb{Q}$. For every endomorphism $f : X \to X$, we still denote by f its graph in $X \times X$ and also the cohomological correspondences induced by it.

2.1. **Cohomological and numerical spectrum.** For $c \in \mathcal{C}(X,X)$ and $j = 0,\dots,2d$, define

$$\mathrm{Sp}_j(c) := \mathrm{Sp}\,(c^* : H^j(X_{\overline{\mathbf{k}}},\mathbb{Q}_l) \to H^j(X_{\overline{\mathbf{k}}},\mathbb{Q}_l))$$

and

$$P_j(c) := P(c^* : H^j(X_{\overline{\mathbf{k}}},\mathbb{Q}_l) \to H^j(X_{\overline{\mathbf{k}}},\mathbb{Q}_l)).$$

For $j \notin G(X)$, we have $\mathrm{Sp}_j(c) = \emptyset$ and $P_j = 1$.

Fact 2.1. For every $j = 0,\dots,2d$, $P_j(c)$ has coefficients in $\mathbb{Q}$. In particular, all elements of $\mathrm{Sp}_j(c)$ are algebraic. Moreover, if $c \in \mathcal{C}(X,X)_{\mathbb{Z}}$, then $P_j(c)$ has coefficients in $\mathbb{Z}$. In particular, all elements of $\mathrm{Sp}_j(c)$ are algebraic integers.

Proof. By the proper base change of étale cohomology and the spreading out argument, we may assume that $\mathbf{k} = \mathbb{F}_q$ is a finite field. We conclude the proof by [KM74, Theorem 2]. □

For $c \in \mathcal{C}(X,X)$, and $i = 0,\dots,d$, define

$$\mathrm{Sp}_i^{num}(c) := \mathrm{Sp}\,(c^* : N^i(X)\otimes\mathbb{R} \to N^i(X)\otimes\mathbb{R}).$$

and

$$P_j^{num}(c) := P(c^* : N^i(X)\otimes\mathbb{R} \to N^i(X)\otimes\mathbb{R}).$$

It is clear that $\mathrm{Sp}_i^{num}(c) \subseteq \mathrm{Sp}_{2i}(c)$,[1] and $P_j^{num}(c)$ has coefficient in $\mathbb{Q}$. Moreover if $c \in \mathcal{C}(X,X)_{\mathbb{Z}}$, then $P_i^{num}(c)$ has coefficients in $\mathbb{Z}$.

Easy to see the following fact.

Fact 2.2. Let X_1, X_2 be smooth projective varieties of pure dimension d and $X := X_1 \sqcup X_2$. Let $c_1 \in \mathcal{C}(X_1,X_1)$ and $c_2 \in \mathcal{C}(X_2,X_2)$ be bi-finite cohomological correspondences. Then $c := c_1 \sqcup c_2 \in \mathcal{C}(X,X)$ is bi-finite. Moreover, we have

$$\mathrm{Sp}_j(c) = \mathrm{Sp}_j(c_1) \sqcup \mathrm{Sp}_j(c_2)$$

for every $j = 0,\dots,2d$ and

$$\mathrm{Sp}_i^{num}(c) = \mathrm{Sp}_i^{num}(c_1) \sqcup \mathrm{Sp}_i^{num}(c_2)$$

for $i = 0,\dots,d$.[2]

If $f : X \to X$ is a surjective endomorphism of X, then there is $m \geq 1$ such that f^m maps every irreducible component of X to itself. This is not true for general correspondence. Using Fact 2.2, many problems on f can be reduced to the case where X is irreducible.

[1] Here the inclusion is the inclusion for multi-sets i.e. for every element $a \in \mathrm{Sp}_i(c)$, it is in $\mathrm{Sp}_{2i}(c)$ with larger or equal multiplicity.

[2] Here the disjoint union is the disjoint union for multi-sets e.g. $\{a,a,b\} \sqcup \{a,c\} = \{a,a,a,b,c\}$.

2.2. **Cohomological spectral radius.** Fix any embedding $\tau : \mathbb{Q}_l \hookrightarrow \mathbb{C}$ and view $\mathbb{Q}_l$ as a subfield of $\mathbb{C}$ via τ. For $c \in \mathcal{C}(X,X)$ and $j = 0, \dots, 2d$, define

$$\alpha_j(c) := \rho(c^* : H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C} \to H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C}).$$

By Fact 2.1, $\alpha_j(c)$ does not depend on the choice of τ. We call all $\alpha_j(c)$ the i-th *cohomological spectral radius* of c. Note that, if $j \notin G(X)$, then $\alpha_j(c) = 0$.

We say the c has Condition (A) if there is a unique $i \in \{0, \dots, 2d\}$ such that

$$\alpha_i(c) = \max_{j=0,\dots,2d} \alpha_j(c).$$

Denote by $\Delta \in \mathcal{C}(X,X)$ the diagonal of $X \times X$.

Lemma 2.3. *If Condition (A) holds for $c \in \mathcal{C}(X,X)$, then*

$$\limsup_{n\to\infty} \langle c^n, \Delta\rangle^{1/n} = \max_{j=0,\dots,2d} \alpha_j(c).$$

Proof. We have

$$\langle c^n, \Delta\rangle = \sum_{j=0}^{2d} (-1)^j \mathrm{Tr}((c^n)^* : H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C}). \tag{2.1}$$

It is clear that for $j = 0, \dots, d$.

$$\limsup_{n\to\infty} |\mathrm{Tr}((c^n)^* : H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C})|^{1/n} = \alpha_j(c). \tag{2.2}$$

There is a unique $i \in \{0, \dots, 2d\}$ such that

$$\alpha_i(c) = \max_{j=0,\dots,2d} \alpha_j(c). \tag{2.3}$$

Combine (2.2) with (2.3), we get

$$\limsup_{n\to\infty} |\mathrm{Tr}((c^n)^* : H^i(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C})^{1/n}| = \alpha_i(c)$$

and there is $\alpha < \alpha_i(c)$ such that for $n \gg 0$, we have

$$|\sum_{j\neq i} \mathrm{Tr}((c^n)^* : H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C})| \leq \alpha^n.$$

So we get

$$\limsup_{n\to\infty} (\sum_{j=0}^{2d} \mathrm{Tr}((c^n)^* : H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \mathbb{C}))^{1/n} = \alpha_i(c),$$

which concludes he proof. □

2.3. **Finite cohomological correspondence.**

Definition 2.4. We call a cohomological correspondence $c \in \mathcal{C}(X,X)$ *effective* if it takes form
$$c = cl(\Gamma)$$
where Γ is an effective cycle in $X \times X$ of dimension d.

If $c \in \mathcal{C}(X,X)$ is effective, then its transport ${}^{\top}c$ is also effective. If $c_1, c_2 \in \mathcal{C}(X,X)$ are effective and $r_1, r_2 \in \mathbb{Q}_{\geq 0}$, $r_1c_1 + r_2c_2$ is effective.

Definition 2.5. We call a cohomological correspondence $c \in \mathcal{C}(X,X)$ finite (resp.*bi-finite*) if it takes form
$$c = cl(\Gamma)$$
where Γ is an effective cycle in $X \times X$ of dimension d such that the projection from Γ to the first (resp. each) factor is finite.

It is clear that bi-finite correspondences are effective. We have the following basic properties:

(i) If $c \in \mathcal{C}(X,X)$ is bi-finite, then its transport ${}^{\top}c$ is also bi-finite.
(ii) If $c_1, c_2 \in \mathcal{C}(X,X)$ are finite (resp. bi-finite), then $c_1 \circ c_2$ is finite (resp. bi-finite).
(iii) If $c_1, c_2 \in \mathcal{C}(X,X)$ are finite (resp. bi-finite) and $r_1, r_2 \in \mathbb{Q}_{\geq 0}$ then $r_1c_1 + r_2c_2$ is finite (resp. bi-finite).

In particular, for every if $c \in \mathcal{C}(X,X)$ is finite (resp. bi-finite), then c^n is finite (resp. bi-finite) for every $n \geq 0$.

For a bi-finite $c \in \mathcal{C}(X,X)$, we say the c has a *bi-finite inverse* if it has an inverse $c^{-1} \in \mathcal{C}(X,X)$ which is also bi-finite. If a bi-finite $c \in \mathcal{C}(X,X)$ has a bi-finite inverse, then for every $n \in \mathbb{Z}$, c^n is bi-finite.

Example 2.6. Let $f : X \to X$ be a surjective endomorphism. By [Fak03, Lemma 5.6], f is finite. If we view f as a cohomological correspondence in $C(X,X)$, then f is bi-finite. As
$$f^{-1} = (\deg f)^{-1\top} f \in C(X,X),$$
f^{-1} is bi-finite.

Lemma 2.7. *Let $c_1, c_2, e \in \mathcal{C}(X,X)$. Assume that c_1, c_2 are bi-fnite and e is effective. Then $c_1 \circ e \circ c_2$ is effective.*

Proof. It is clear that for every bi-finite $c \in \mathcal{C}(X,X)$ and effective $e \in \mathcal{C}(X,X)$, $e \circ c$ is effective. Taking transport, we get that $c \circ e$ is also effective. Then we conclude the proof. ☐

2.4. **Numerical spectral radius.** For $c\in \mathcal{C}(X,X)$, define

$$\beta_i(c):=\rho(c^*:N^i(X)_{\mathbb{R}}\to N^i(X)_{\mathbb{R}}),$$

and call it the i-th *numerical spectral* of c. As the cohomological equivalence is finner than the numerical equivalence, we have

$$\beta_i(c)\leq \alpha_{2i}(c). \tag{2.4}$$

Let $\mathrm{Psef}^i(X)$ be the pseudo-effective cone in $N^i(X)$. It is a closed convex cone with non-empty interior. Moreover it is salient i.e. $\mathrm{Psef}^i(X)\setminus\{0\}$ is convex. Let L be an ample line bundle on X. It induces a norm $\|\cdot\|_L$ on $N^i(X)_{\mathbb{R}}$ as follows: for $u\in N^i(X)_{\mathbb{R}}$,

$$\|u\|_L:=\inf\{(u^+\cdot L^{d-i})+(u^-\cdot L^{d-i})|\ u^+,u^-\in \mathrm{Psef}^i(X), u=u^+-u^-\}.$$

Easy to check that $\|\cdot\|_L$ is a norm and for every $u\in \mathrm{Psef}^i(X)$, $\|u\|_L=(u\cdot L^{d-i})$. If $c\in\mathcal{C}(X,X)$ is bi-finite, then both c^* and c_* preserve $\mathrm{Psef}^i(X)$. The following lemma give another description of $\beta_i(c)$.

Lemma 2.8. *If $c\in \mathcal{C}(X,X)$ is finite, then for every $i=0,\dots,d$, we have*

$$\beta_i(c)=\lim_{n\to\infty}((c^n)^*(L^i)\cdot L^{d-i})^{1/n}.$$

In particular, $\beta_i(c)\geq 1$.

Proof. For every $u\in \mathrm{Psef}^i(X)$, there is $C>0$ such that $CL^i-u\in \mathrm{Psef}^i(X)$. Then for every $n\geq 0$, we have $(c^n)^*(CL^i-u)\in \mathrm{Psef}^i(X)$. So we have

$$(C\|(c^n)^*L^i\|_L)^{1/n}\geq \|(c^n)^*u\|_L^{1/n}$$

for every $n\geq 0$. As $\mathrm{Psef}^i(X)$ has non-empty interior, the above inequality shows that

$$\lim_{n\to\infty}\|(c^n)^*L^i\|_L^{1/n}=\rho(c^*:N^i(X)_{\mathbb{R}}\to N^i(X)_{\mathbb{R}})=\beta_i(c).$$

□

Let $\pi_i: X\times X\to X, i=1,2$ be the projection to the first and the second coordinate. Set $L_i:=\pi_i^*L$. Then L_1+L_2 is ample on $X\times X$.

Lemma 2.9. *Assume that $c_1,c_2\in\mathcal{C}(X,X)$ are bi-finite and c_1-c_2 is effective. Then for every $i=0,\dots,d$, we have $\beta_i(c_1)\geq\beta_i(c_2)$.*

Proof. Set $e:=c_1-c_2$. Then for $n\geq 0$,

$$c_1^n-c_2^n=\sum_{l=0}^{n-1}c_2^l\circ e\circ c_1^{n-1-l}$$

is effective. So we have $((c_1^n-c_2^n)\cdot L_1^i\cdot L_2^{d-i})\geq 0$. We conclude the proof by Lemma 2.8. □

Denote by $\Delta\in\mathcal{C}(X,X)$ the diagonal of $X\times X$.

Lemma 2.10. *Assume that $c\in\mathcal{C}(X,X)$ is finite. Then we have*

$$\limsup_{n\to\infty}\langle c^n,\Delta\rangle^{1/n}\leq \max_{i=0}^{d}\beta_i(c).$$

Proof. Consider the linear function $I : N^d(X\times X)_{\mathbb{R}} \to \mathbb{R}$ defined by $u \mapsto (u\cdot\Delta)$. There is $C>0$ such that

$$I(u) \le C\|u\|_{L_1+L_2}$$

for every $u \in N^d(X\times X)_{\mathbb{R}}$.

We still denote by c^n the numerical class in $N^d(X\times X)$ induced by c^n. Then we have

$$\langle c^n, \Delta\rangle = (c^n\cdot\Delta) = I(c^n) \le C\|c^n\|_{L_1+L_2}. \tag{2.5}$$

As

$$\|c^n\|_{L_1+L_2} = \sum_{i=0}^{d}\binom{d}{i}((c^n)^*L^i\cdot L^{d-i}),$$

we get

$$\lim_{n\to\infty}\|c^n\|_{L_1+L_2}^{1/n} = \max_{i=1}^{d}\lim_{n\to\infty}((c^n)^*L^i\cdot L^{d-i})^{1/n} = \max_{i=0}^{d}\beta_i(c).$$

We conclude the proof by (2.5). □

By Lemma 2.10, Lemma 2.3 and (2.4), we get the following consequence.

Corollary 2.11. *Assume that $c\in\mathcal{C}(X,X)$ is finite and has Condition (A). Then we have*

$$\max_{j=0}^{2d}\alpha_i(c) = \max_{i=0}^{d}\beta_i(c).$$

2.5. **Cohomologcial and numerical polygons.** Let $c\in\mathcal{C}(X,X)$. Define the *Cohomologcial polygon* for c to be the minimal concave function

$$CP_c : [0,2d] \to \mathbb{R}\cup\{-\infty\}$$

such that

$$CP_c(j) \ge \log\alpha_j(c)$$

for all $j=0,\dots,2d$. Observe that c has condition (A) if and only if there is a unique $j\in\{0,\dots,2d\}$ such that $CP_c(j)$ takes the maximal value.

Similarly, define the *numerical polygon* for c is the minimal concave function

$$NP_c : [0,2d] \to \mathbb{R}\cup\{-\infty\}$$

such that

$$NP_c(2i) \ge \log\beta_i(c)$$

for all $i=0,\dots,d$.

We note that

$$\max_{i=0,\dots d} NP_c(2i) = \max_{i=0,\dots,d}\beta_i(c)$$

and

$$\max_{j=0,\dots 2d} CP_c(j) = \max_{j=0,\dots,2d}\alpha_j(c).$$

By (2.4), we have

$$NP_c \le CP_c. \tag{2.6}$$

By Lemma 2.8, if c is finite, then we have $CP_c \ge NP_c$ on $[0,2d]$. Moreover, if c is bi-finite, then $NP_c > -\infty$.

Theorem 2.12. *If c is finite, then $NP_c = CP_c$.*

Proof. By the proper base change of étale cohomology and the spreading our argument, we may assume that $\mathbf{k} = \mathbb{F}_q$ is a finite field. The following lemma is the key ingredient of our proof, which treats the even vertexes.

Lemma 2.13. *We have $NP_c(2i) = CP_c(2i)$ for all $i = 0, \ldots, d$.*

For every $l \geq 2$, consider the finite cohomological correspondence $c(l) := \prod_{u=1}^{l} c \in \mathcal{C}(X^l, X^l)$. Denote by L an ample line bundle on X. Let $\pi_j : X^l \to X$ the j-th projection. Set $M(j) := \pi_j^* L$. For every $n \geq 0$, set

$$M_n(j) := (c(l)^n)^* M(j) = \pi_j^*((c^n)^* L).$$

For every $i = 0, \ldots, d$ and $n \geq 0$, denote by

$$\deg_i(c^n) := ((c^n)^* L^i \cdot L^{d-i}).$$

By Lemma 2.8, we have

$$\lim_{n\to\infty} \deg_i(c^n)^{1/n} = \beta_i(c). \tag{2.7}$$

By projection formula, we get the following fact.

Fact 2.14. For $s_j, t_j \geq 0, j = 1, \ldots, l$ with $\sum_{j=1}^{l}(s_j + t_j) = ld$ and $n \geq 0$, we have

$$(M_n(1)^{s_1} \cdots M_n(l)^{s_l} \cdot M(1)^{t_1} \cdots M(l)^{t_l}) \neq 0$$

only when $s_j + t_j = d$ for every $j = 1 \ldots, l$. Moreover, for $s_j \in \{0, \ldots, d\}$, $j = 1, \ldots, l$, we have

$$(M_n(1)^{s_1} \cdots M_n(l)^{s_l} \cdot M(1)^{d-s_1} \cdots M(l)^{d-s_l}) = \prod_{j=1}^{l} \deg_{s_j}(c^n).$$

By Fact 2.14, we have

$$((\sum_{j=1}^{l} M_n(j))^i \cdot (\sum_{j=1}^{l} M(j))^{2d-i}) = \sum_{s_j \geq 0, j=1,\ldots,l \text{ and } \sum_{j=1}^{l} s_j = i} \binom{i}{s_1, \ldots, s_l} \prod_{j=1}^{l} \deg_{s_j}(c^n).$$

By Lemma 2.8, for every $i = 0, \ldots, ld$, we have

$$\lim_{n\to\infty} ((\sum_{j=1}^{l} M_n(j))^i \cdot (\sum_{j=1}^{l} M(j))^{2d-i})^{1/n} = \beta_i(c(l)). \tag{2.8}$$

So we get

$$\log \beta_i(c(l)) = \max_{s_j \geq 0, j=1,\ldots,l \text{ and } \sum_{j=1}^{l} s_j = i} \sum_{j=1}^{l} \log(\beta_{s_j}(c)). \tag{2.9}$$

As $\log(\beta_{s_j}(c)) \leq NP_c(2s_j)$, we get

$$\log \beta_i(c(l)) \leq \max_{s_j \geq 0, j=1,\ldots,l \text{ and } \sum_{j=1}^{l} s_j = i} \sum_{j=1}^{l} NP_c(2s_j). \tag{2.10}$$

Pick $l := (2d)!$. By (2.9), (2.10) and the concavity of $NP_c(j)$, for every $i = 0, \dots, 2d$, we get

$$\begin{aligned}
lNP_{c(l)}(li) =& l \max\{\log \beta_{li/2}(c(l)), \max_{0\le u<i<v\le ld}\{\frac{v-li/2}{v-u}\log\beta_u(c(l)) + \frac{li/2-u}{v-u}\log\beta_v(c(l))\}\} \\
\le& \max_{0\le t_w, w=1,\dots,l \text{ and } \sum_{w=1}^{l} t_w = l^2 i/2} \sum_{j=1}^{l} \beta_{t_w}(c(l)) \\
\le& \max_{0\le t_w, w=1,\dots,l \text{ and } \sum_{w=1}^{l} t_w = l^2 i} (\sum_{j=1}^{l} \max_{s_j\ge 0, j=1,\dots,l \text{ and } \sum_{j=1}^{l} s_j = t_w} \sum_{j=1}^{l} NP_c(s_j)) \\
\le& \max_{0\le s_y, y=1,\dots,l^2 \text{ and } \sum_{y=1}^{l^2} s_y = l^2 i} NP_c(s_y) \\
\le& l^2 \times NP_c(i).
\end{aligned}$$

Hence

$$NP_{c(l)}(li) \le l \times NP_c(i). \tag{2.11}$$

The definition of NP_c implies that

$$NP_c(i) = \frac{1}{l}\log(\max_{s_j\ge 0, j=1,\dots,l \text{ and } \sum_{j=1}^{l} s_j = li/2} \prod_{j=1}^{l} \beta_{s_j}(c)).$$

By (2.9), we get

$$l \times NP_c(i) = \beta_{li/2}(c(l)) \le NP_{c(l)}(li). \tag{2.12}$$

Combine (2.12) with (2.11), we get

$$NP_{c(l)}(li) = l \times NP_c(i) = \beta_{li/2}(c(l)). \tag{2.13}$$

Apply Lemma 2.13 for $c(l)$, we get

$$l \times NP_c(i) = NP_{c(l)}(li) = CP_{c(l)}(li). \tag{2.14}$$

By Künneth formula, we have

$$\log \alpha_{li}(c(l)) \ge l \log \alpha_i(c).$$

Hence for every $i = 0, \dots, 2d$ we get

$$NP_c(i) \ge \alpha_i(c). \tag{2.15}$$

As NP_c is concave, we get

$$NP_c(i) \ge CP_c(i)$$

for all $i = 0, \dots, 2d$. This concludes the proof by equation (2.6). □

Proof of Lemma 2.13. We may approximate c by $c_m := c + m^{-1}\Delta$ with $m \to \infty$. For each $m \ge 1$, as $c = c_m - m^{-1}\Delta$ is effective. As $m^{-1}\Delta$ is bi-finite, and $m^{-1}\Delta$ commutes with c_m, the proof of Lemma 2.9 indeed shows that

$$\beta_i(c_m) \ge \beta_i(m^{-1}\Delta) > 0$$

for every $i = 0, \dots, d$. After replacing c by $c_m, m \ge 1$, we may assume that $\beta_i(c) > 0$ for every $i = 0, \dots, d$.

For $i=1,\dots,d$, set

$$\mu_i := NP_c(i) - NP_c(i-1).$$

As NP_c is concave, μ_i is decreasing on i. Set $\alpha_j := CP_c(j), j=0,\dots,2d$ and $\beta_i := NP_c(2i), i=0,\dots,d$.

By (2.6), we only need to show that $\beta_i \geq \alpha_{2i}$ for all $i=0,\dots,d$. If $i=0$, it is clear. So only need to prove it for $i\geq 1$.

Denote by Φ_q the cohomological correspondence in $\mathcal{C}(X,X)$ induced by the q-Frobenius on X. So Φ_q is bi-finite and $\Phi_q^{-1} = q^{-d\top}\Phi_q$. Deligne's theorem [Del74] implies that all eigenvalues of $\Phi_q^*: H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l)\otimes\mathbb{C} \to H^j(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l)\otimes\mathbb{C}$ has norm $q^{j/2}$. For $s\in\mathbb{Z}, t\in\mathbb{Z}_{\geq 0}$, define $c_{s,t} := \Phi_q^s\circ c^t$. As Φ_q commutes with c, we have the following properties:

(i) for $i=1,\dots,d$, $NP_{c_{s,t}}(2i) = (s\log q)i + t\beta_i$;
(ii) for $j=1,\dots,2d$, $CP_{c_{s,t}}(j) = (\frac{1}{2}s\log q)j + t\alpha_j$.

Set

$$E := \{\frac{2(\log\alpha_{j+1} - \log\alpha_j)}{\log q} |\ j=0,\dots,2d-1\},$$

which is finite. Set $\mathbb{Q}_* := \mathbb{Q}\setminus E$ which is dense in $\mathbb{R}$. Then for every $(s,t)\in\mathbb{Z}\times\mathbb{Z}_{\geq 0}$, $c_{s,t}$ has condition (A). Now fix $i\in\{1,\dots,d\}$. Set $r_i := \log\mu_i/\log q$. For every $m\geq 1$, pick $(s_m,t_m)\in\mathbb{Z}\times\mathbb{Z}_{\geq 0}$ such that $s_m/t_m\in\mathbb{Q}_*$ and

$$r_i < s_m/t_m < r_i + \frac{1}{m}.$$

Then c_{s_m,t_m} has Condition (A). For $l=1,\dots,d,$, set

$$\theta_l := NP_{c_{s_m,t_m}}(2l) - NP_{c_{s_m,t_m}}(2l-2) = t_m(\frac{s_m}{t_m}\log q + \log\mu_l),$$

which is decreasing. Then we have

$$\theta_i = t_m(r_i\log q + \log\mu_i) + t_m(\frac{s_m}{t_m} - r_i)\log q = t_m(\frac{s_m}{t_m} - r_i)\log q \in (0, \frac{1}{m}t_m\log q).$$

Then when $l<i$, we get

$$NP_{c_{s_m,t_m}}(2l) < NP_{c_{s_m,t_m}}(2i).$$

When $l>i$, we get

$$NP_{c_{s_m,t_m}}(2l) = NP_{c_{s_m,t_m}}(2i) + (\theta_{i+1} + \cdots + \theta_j)$$

$$\leq NP_{c_{s_m,t_m}}(2i) + \theta_i(j-i) < NP_{c_{s_m,t_m}}(2i) + \frac{1}{m}(j-i)t_m\log q.$$

Combine the above two inequalities, we get

$$NP_{c_{s_m,t_m}}(2i) \geq \max_{l=0,\dots,d} NP_{c_{s_m,t_m}}(2l) - \frac{1}{m}dt_m\log q \tag{2.16}$$

By Corollary 2.11 and (2.16), we get

$$CP_{c_{s_m,t_m}}(2i) \leq \max_{j=0,\dots 2d} CP_c(j) \tag{2.17}$$

$$= \max_{l=0,\dots,d} NP_{c_{s_m,t_m}}(2l) \tag{2.18}$$

$$\leq NP_{c_{s_m,t_m}}(2i) + \frac{1}{m} dt_m \log q. \tag{2.19}$$

Then we get

$$is_m \log q + t_m \alpha_{2i} \leq is_m \log q + t_m \beta_i + \frac{1}{m} dt_m \log q,$$

hence

$$\alpha_{2i} \leq \beta_i + \frac{1}{m} d \log q.$$

Let $m \to \infty$, we get conclude the proof. □

2.6. **Case of endomorphisms.** In this section, we assume that X is **irreducible**. Let $f : X \to X$ be a surjective endomorphism. We still denote by f the cohomological correspondence in $\mathcal{C}(X, X)$ induced by f.

For any bi-finite $c \in \mathcal{C}(X, X)$, we say that c is *numerically log-concave*, if

$$NP_c(2i) = \log \beta_i(c). \tag{2.20}$$

By [Tru20, Theorem 1.1(3)] (see also [DS05, Dan20]), if X is irreducible, the sequence $\beta_i(f), i = 0, \dots, d$ is log-concave. So f is numerically log-concave. Indeed, as the transport of the graph of f^n is irreducible for all $n \geq 0$, [Tru20, Theorem 1.1(3)] also implies the $^\top f$ is numerically log-concave.

The following gives an example of bi-finite cohomological correspondence $c \in \mathcal{C}(X, X)$ which is not numerically log-concave.

Example 2.15. Let $f_i, i = 1, 2$ be two surjective endomorphisms on $\mathbb{P}^d$ of algebraic degree $q_i \geq 1$ i.e. $f_i^* O(1) = O(q_i)$. Define $c := \frac{1}{2}(f_1 + f_2) \in \mathcal{C}(X, X)$. We may think c as a random product of f_1, f_2 independently of probability $1/2$ for each i. Set $L := O(1)$. For $i = 0, \dots, d$, we have

$$\beta_i(c) = \lim_{n\to\infty} ((c^n)^* L^i \cdot L^{d-i})^{1/n} = \lim_{n\to\infty} (2^{-n}(q_1^i + q_2^i)^n)^{1/n} = (q_1^i + q_2^i)/2. \tag{2.21}$$

Cauchy inequality implies that for every $x > 0$, $i \geq 1$, we have

$$(1 + x^{i-1})(1 + x^{i+1}) \geq (1 + x^i)^2,$$

and the equality holds if and only if $x = 1$. Applying the above inequality, we get that for every $i = 1, \dots, d - 1$, we get

$$\beta_{i-1}(c)\beta_{i+1}(c) \geq \beta_i(c)^2,$$

and the equality holds if and only if $q_1 = q_2$. In particular, if $q_1 \neq q_2$, then

$$NP_c(2i) > \log \beta_i(c)$$

for all $i = 1, \dots, d - 1$.

In general the sequence $\alpha_i(f), i = 0, \ldots, 2d$ may not be log-concave even when X is irreducible. For example, it is possible that $H^i(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l)$ vanishes for some odd i. Then this case $\alpha_i(f) = 0$. As shown in the following example, every when $H^i(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \neq 0$ for every $i = 0, \ldots, 2d$, $\alpha_i(f), i = 0, \ldots, 2d$ could not be log-concave.

Example 2.16. Let $X = \mathbb{P}^1 \times E$ where E is an elliptic curve. Let $g : \mathbb{P}^1 \to \mathbb{P}^1$ be the square map $[x : y] \mapsto [x^2 : y^2]$. Set $f := g \times \mathrm{id} : X \to X$. Easy to see that $H^i(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l) \neq 0$ for all $i = 0, \ldots, 4$. Easy to compute that $\alpha_0(f) = 1, \alpha_1(f) = 1, \alpha_2(f) = 2$. As

$$1 = \alpha_1(f)^2 < \alpha_0(f)\alpha_1(f) = 2,$$

$\alpha_i(f), i = 0, \ldots, 4$ is not log-concave.

For a finite multi-set $A \subseteq \mathbb{C}^*$ and $b \in \mathbb{C}^*$, write

$$A^{-1} = \{a^{-1} |\ a \in A\}, bA := \{ba|\ a \in A\} \text{ and } |A| = \{|a||\ a \in A\}.$$

Now we fix an embedding $\tau : \mathbb{Q}_l \hookrightarrow \mathbb{C}$. By Poincaré duality, we have the following facts: For every $j \in G(X), i = 0, \ldots, d$, we have

(i) $0 \notin \mathrm{Sp}_j(f)$ and $0 \notin \mathrm{Sp}\,(f^* : N^i(X)_{\mathbb{R}} \to N^i(X)_{\mathbb{R}})$,

(ii) $\mathrm{Sp}_j({}^\top f) = (\deg f)\mathrm{Sp}_j(f)^{-1}$ and

$$\mathrm{Sp}\,(({}^\top f)^* : N^i(X)_{\mathbb{R}} \to N^i(X)_{\mathbb{R}}) = (\deg f)\mathrm{Sp}\,(f^* : N^i(X)_{\mathbb{R}} \to N^i(X)_{\mathbb{R}})^{-1};$$

(iii) $\mathrm{Sp}_j(f) = \mathrm{Sp}_{2d-j}({}^\top f)$ and

$$\mathrm{Sp}\,(f^* : N^i(X)_{\mathbb{R}} \to N^i(X)_{\mathbb{R}}) = \mathrm{Sp}\,(({}^\top f)^* : N^{d-i}(X)_{\mathbb{R}} \to N^{d-i}(X)_{\mathbb{R}}).$$

For $j \in G(X)$, set

$$\alpha_j^-(f) := \min |\mathrm{Sp}_j(f)|.$$

For $i = 0, \ldots, d$, set

$$\beta_i^-(f) := \min |\mathrm{Sp}\,(f^* : N^i(X)_{\mathbb{R}} \to N^i(X)_{\mathbb{R}})|.$$

By Fact 2.1, $\rho_j^-(f)$ does not depend on the choice of the embedding τ. By the above facts (ii) and (iii), we get the following result.

Proposition 2.17. *For every $j \in G(X)$, we have*

$$\alpha_j^-(f) = \frac{\deg(f)}{\alpha_{2d-j}(f)} \text{ and } \beta_i^-(f) = \frac{\deg(f)}{\beta_{d-i}(f)}.$$

By Theorem 2.12, we get the following consequence.

Corollary 2.18. *For every $j \in G(X)$, we have*

$$NP_f(2d) - NP_f(2d - j) \leq \log \alpha_j^-(f) \leq \log \alpha_j(f) \leq NP_f(j)$$

Moreover, for every $i = 0, \ldots, d$, we have

$$NP_f(2d) - NP_f(2d - 2i) = \log \alpha_{2i}^-(f) = \log \beta_i^-(f)$$

and

$$\log \beta_i(f) = \log \alpha_{2i}(f) = NP_f(2i).$$

Note that $\log \deg f = NP_f(2d)$.

In the end, we prove the "moving target" version of Corollary 1.7.

Proposition 2.19. *Assume that $f : X \to X$ is an int-amplified amplified endmorphism. Let L be an ample line bundle on X. Let $h_n : X \to X, n \geq 0$ be a sequence of endomorphisms of X with*

$$\limsup_{n\to\infty}(h_n^*L \cdot L^{d-1})^{1/n} < \beta_d(f)/\beta_{d-1}(f).$$

Then for $n \gg 0$, f^n intersects h_n properly in $X \times X$. Moreover, for every $\epsilon > 0$ we have

$$\#\{f^n(x) = h_n(x)\} = q^{dn} + o((\beta_d\beta_{d-1})^{(1+\epsilon)n})$$

counting with multiplicity.

Proof. Let $\pi_i : X\times X \to X, i = 1, 2$ be the i-th projection. Set $\mu_d := \beta_d(f)/\beta_{d-1}(f)$. As f is int-amplified amplified, $\mu_d > 1$. By Corollary 2.18, every eigenvalue of $f^* : N^1(X) \otimes \mathbb{R} \to N^1(X) \otimes \mathbb{R}$ has norm at least μ_d. Hence every eigenvalue of $f_* : N^{d-i}(X) \otimes \mathbb{R} \to N^{d-i}(X) \otimes \mathbb{R}$ has norm at least μ_d. So there is $\delta > 0$, such that for every $Z \in \mathrm{Psef}^{d-i}(X)$ and $n \geq 0$ we have

$$(f_*^n(Z) \cdot L) = \|f_*^n(Z)\|_L \geq \delta\mu_d^n\|Z\|_L. \tag{2.22}$$

There is a constant $A > 0$, such that for for every $M \in N^1(X), Z \in N^{d-1}(X)$, we have

$$(Z \cdot M) \leq A\|M\|_L\|Z\|_L. \tag{2.23}$$

Pick $\eta \in (\limsup_{n\to\infty}(h_n^*L \cdot L^{d-1}), \mu_d)$. There is $B > 0$ such that

$$(h_n^*L \cdot L^{d-1}) < B\eta^n. \tag{2.24}$$

We first prove that f^n intersects h_n properly for $n \gg 0$. Assume that there is an irreducible curve $C' \subseteq f^n \cap h_n \subseteq X \times X$ for some $n \geq 0$. Set $C := \pi_1(C')$. We have $(f^n)^*L|_C = (h_n)^*L|_C$. So we get

$$(L \cdot f_*^n(C)) = (h_n^*L \cdot C).$$

By (2.22), (2.23) and (2.24), we get

$$\delta\mu_d^n\|C\|_L \leq \|f_*^n(C)\|_L = (L\cdot f_*^n(C)) = (h_n^*L\cdot C) \leq A(h_n^*L\cdot L^{d-1})\|C\|_L \leq AB\eta^n\|C\|_L.$$

Hence

$$n \leq \frac{\log A + \log B - \log\delta}{\log\mu_d - \log\eta}.$$

Fix a field embedding $\tau : \mathbb{Q}_l \hookrightarrow \mathbb{C}$. For every $i = 0, \dots, 2d$, $V_i := H^i(X_{\overline{\mathbf{k}}}, \mathbb{Q}_l)\otimes_{\mathbb{Q}_l} \mathbb{C}$. Fix a norm $\|\cdot\|$ on each V_i. For every endomorphism $g : V_i \to V_i$, denote by $\|g\|$ the operator norm of g. There is a constant $D > 0$ such that for every $i = 0, \dots, 2d$ and $g : V_i \to V_i$, we have

$$|\mathrm{Tr}(g)| \leq D\|g\|.$$

For $n \gg 0$, we have

$$\#\{f^n(x) = h(x)\} = (f^n \cdot h_n) = \sum_{i=0}^{2d}(-1)^i\mathrm{Tr}((^\top h)^* \cdot (f^n)^* : V_i \to V_i)$$

$$= \mu_d^n + \sum_{i=0}^{2d-1}(-1)^i \mathrm{Tr}((^\top h)^* \cdot (f^n)^* : V_i \to V_i).$$

We only need to bound $\sum_{i=0}^{2d-1}(-1)^i \mathrm{Tr}((^\top h)^* \cdot (f^n)^* : V_i \to V_i)$. For every $i = 0, \ldots, 2d-1$, we have

$$|\mathrm{Tr}((^\top h)^* \cdot (f^n)^*|_{V_i})| \leq D\|(^\top h)^*|_{V_i}\|\|(f^n)^*|_{V_i}\| \leq D\|(^\top h)^*|_{V_i}\|\alpha_i(f)^{(1+\epsilon/2)n}.$$

As $\beta_i, i = 1, \ldots, d$ is increasing, by Corollary 2.18, for every $i = 0, \ldots, 2d-1$, we get $\alpha_i(f) \leq (\beta_{d-1}\beta_d)^{1/2}$. This concludes the proof. □

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