NUMERICAL ACTION FOR ENDOMORPHISMS

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Abstract. Let $f : X \to X$ be a surjective endomorphism of a projective variety of dimension d. The aim of this paper is to study the action of f on the numerical group of divisors.

For this, we introduce a notion of spectrum for an open and salient invariant cone. Let V be a finitely dimensional vector space over $\mathbb{R}$ and $g : V \to V$ be a linear endomorphism. Let $\mathcal{C}$ be an open and salient g-invariant cone. We define $\mathrm{Sp}\,(g, \mathcal{C})$ to be the set of $\alpha > 0$ such that no $v \in V$ satisfying $gv - \alpha v \in \mathcal{C}$. We call $\mathrm{Sp}\,(g, \mathcal{C})$ the spectrum of g for $\mathcal{C}$. We prove some basic properties for this spectrum. In particular, we get the following result: For any subset S of $\mathbb{C}$, let $E_S(\mathbb{C}) \subseteq V \otimes_{\mathbb{R}} \mathbb{C}$ be the sum of generalized eigenspaces for g with eigenvalues in S. Then $E_S(\mathbb{C}) \cap \mathcal{C} \neq \emptyset$ if and only if S contains $\mathrm{Sp}\,(g, \mathcal{C})$. Here we view $\mathcal{C}$ as a subset of $V \otimes_{\mathbb{R}} \mathbb{C}$ via the natural embedding $V \subseteq V \otimes_{\mathbb{R}} \mathbb{C}$.

Let $\lambda_i(f), i = 0, \ldots, d$ be the dynamical degrees and $\mu_i(f) := \frac{\lambda_i(f)}{\lambda_{i-1}(f)}, i = 1, \ldots, d$ be the cohomological Lyapunov exponents. Let $\mathrm{Big}(X)$ and $\mathrm{Amp}(X)$ be the big and ample cones in $N^1(X)_{\mathbb{R}}$. We show that $\mathrm{Sp}\,(f^*, \mathrm{Big}(X)) = \{\mu_1(f), \ldots, \mu_d(f)\}$. We also compute the spectrum for the ample cone using the cohomological Lyapunov exponents of all periodic irreducible subvarieties. In particular, f is quasi-amplified if and only if it is cohomologically hyperbolic; and f is amplified if and only if every subsystem of (X, f) is cohomologically hyperbolic. As a consequence, we show that every factor of an amplified (resp. quasi-amplified) endomorphism is amplified (resp. quasi-amplified). This implies a positive answer of a question of Krieger-Reschke.

We introduce a notion of generalized (positive) cycles, which can be viewed as an algebraic analogy of (positively) closed currents in complex geometry. This notion plays a key role in our proof of the ample cone case.

1. Introduction

Let $\mathbf{k}$ be a field. Let X be a projective variety over $\mathbf{k}$ of dimension d and $f : X \to X$ be a surjecive endomorphism of X. Denote by $N_{d-i}(X)$ the group of numerical classes of X of codimension i and $N^i(X) := \mathrm{Hom}(N_{d-i}(X), \mathbb{Z})$. Set $N_{d-i}(X)_{\mathbb{R}} := N_{d-i}(X) \otimes \mathbb{R}$ and $N^i(X)_{\mathbb{R}} := N^i(X) \otimes \mathbb{R}$. Denote by $\mathrm{Amp}(X)$ and $\mathrm{Big}(X)$ the ample cone and the big cone in $N^1(X)_{\mathbb{R}}$. Both of them are open, f^*-invariant and salient i.e. do not contain any line. The ample and big cones gives two natural notions of positivity on $N^1(X)_{\mathbb{R}}$ which contain many geometric information of X. The aim of this paper is to study the action of f^* on $N^1(X)_{\mathbb{R}}$. In the follow-up works, we will apply the results and the method developed in this paper to the Kawaguchi-Silverman conjecture and the Dynamical Mordell-Lang conjecture.

Date: March 14, 2025.

The author is supported by the NSFC Grant No.12271007.

In general, $\dim N^1(X)_{\mathbb{R}}$ can be very large comparing with $\dim X$. A priori, the action of f^* on $N^1(X)_{\mathbb{R}}$ may have many different eigenvalues. It is natural to ask, whether there is a natural f^*-invariant subspace of $N^1(X)_{\mathbb{R}}$ which has smaller dimension, but still captures a large part of dynamical information. For this purpose, we want such space cuts the ample (resp. big) cone. For this, we introduce a notion of spectrum for a good cone.

1.1. **Spectrum for a good invariant cone.** Let W be a finite dimensional $\mathbb{R}$-vector space and $g : W \to W$ be an endomorphism. Set $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$ and $g_{\mathbb{C}} : W_{\mathbb{C}} \to W_{\mathbb{C}}$ the endomorphism induced by g. View W as a $\mathbb{R}$-subspace of $W_{\mathbb{C}}$. We have $g = g_{\mathbb{C}}|_W$. Denote by $\mathrm{Sp}\,(g)$ the set of eigenvalues of g.

Generalized eigenspaces. For every $c \in \mathbb{C}$, denote by $E_c(\mathbb{C})$ the generalized eigenspace of $g_{\mathbb{C}} : W_{\mathbb{C}} \to W_{\mathbb{C}}$ i.e.

$$E_c(\mathbb{C}) := \cup_{n\geq 0} \ker((g - c\mathrm{id})^n : W_{\mathbb{C}} \to W_{\mathbb{C}}).$$

It is clear that $E_c(\mathbb{C}) \neq 0$ if and only if $c \in \mathrm{Sp}\,(g)$. When $c \in \mathbb{R}$, write $E_c := E_c(\mathbb{C}) \cap W$. We have $E_c(\mathbb{C}) = E_c \otimes_{\mathbb{R}} \mathbb{C}$. For every $S \subseteq \mathbb{C}$, set

$$E_S(\mathbb{C}) := \oplus_{c\in S} E_c(\mathbb{C}) \text{ and } E_S := E_S(\mathbb{C}) \cap W.$$

We note that, if $S = \overline{S}$, then $E_S(\mathbb{C}) = E_S \otimes_{\mathbb{R}} \mathbb{C}$. We mainly interest in the case where $S \subseteq \mathbb{R}$. In this case $E_S = \oplus_{c\in S} E_c$.

Good invariant cone. Let $\mathcal{C}$ be a non-empty open convex cone in W satisfying the following properties:

(i) $g(\mathcal{C}) \subseteq \mathcal{C}$;
(ii) $\overline{\mathcal{C}}$ is salient i.e. $\overline{\mathcal{C}} \setminus \{0\}$ is convex.

We call such $\mathcal{C}$ a *good invariant cone* for g.

Example 1.1. If $W = N^1(X)_{\mathbb{R}}$ for a projective variety X, $g = f^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$ for some surjective endomorphism $f : X \to X$, then the ample cone $\mathrm{Amp}(X)$ and the big cone $\mathrm{Big}(X)$ are good invariant cone for g.

Let $\mathcal{C}$ be a good invariant cone for g.

Definition 1.2. For $\alpha \in \mathbb{R}$, we say that g is α-*amplified* for $\mathcal{C}$, if there is $N \in W$ such that $gN - \alpha N \in \mathcal{C}$. Define the $\mathcal{C}$-spectrum $\mathrm{Sp}\,(g, \mathcal{C})$ for g to be the set of $\alpha \in \mathbb{R}$ such that g is **not** α-amplified.

If $\alpha \notin \mathrm{Sp}\,(g)$, then $g - \alpha$ is invertible on W, hence g is α-amplified for $\mathcal{C}$. In particular, we have $\mathrm{Sp}\,(g, \mathcal{C}) \subseteq \mathrm{Sp}\,(g)$. It is clear that $\mathrm{Sp}\,(g, \mathcal{C})$ is decreasing on $\mathcal{C}$. The following result gives a description of the spectrum $\mathrm{Sp}\,(g, \mathcal{C})$ using generalized eigenspaces.

Theorem 1.3. *For every subset $S \subseteq \mathbb{C}$, $\mathcal{C} \cap E_S \neq \emptyset$ if and only if $\mathrm{Sp}\,(g, \mathcal{C}) \subset S$.*

1.2. **Cohomological Lyapunov exponents.** For every $i = 0, \ldots, d$, the *i-th dynamical degree* $\lambda_i(f)$ of f is is defined to be the spectral radius of

$$f^* : N^i(X)_{\mathbb{R}} \to N^i(X)_{\mathbb{R}}.$$

It can be also defined as follows: Let L be any ample line bundle on X, then

$$\lambda_i(f) = \lim_{n\to\infty} ((f^n)^* L^i \cdot L^{d-i})^{1/n}.$$

The later definition can be generated to rational self-maps [RS97, DS05, DS04, Tru20, Dan20].

By [Tru20, Theorem 1.1(3)] (see also [DS05, Dan20]), the sequence $\lambda_i(f), i = 0, \ldots, d$ is log-concave i.e.

$$\lambda_i(f)^2 \geq \lambda_{i-1}(f)\lambda_{i+1}(f)$$

for $i = 1, \ldots, d-1$. As in [Xie24], we define $\mu_i(f) := \lambda_i(f)/\lambda_{i-1}(f)$ for $i = 1, \ldots, d$ and $\mu_{d+1}(f) := 0$. We call them the cohomological Lyapunov exponents. The log-concavity of $\lambda_i(f), i = 0, \ldots, d$ implies that the sequence $\mu_i(f), i = 1, \ldots, d+1$ is decreasing.

1.3. **Main results.** For $\alpha \in \mathbb{R}_{>0}$, we say that f is *α-quasi-amplified* (resp. *α-amplified*) if

$$\alpha \notin \mathrm{Sp}\,(f^*, \mathrm{Big}(X)) \text{ (resp. } \alpha \notin \mathrm{Sp}\,(f^*, \mathrm{Amp}(X)))$$

i.e. there is $N \in N^1(X)_{\mathbb{R}}$ such that $f^*N - \alpha N$ is big (resp. ample).

As defined in [Men23, Definition 2.2] and in [Fak03], f is called *quasi-amplified* (resp. amplified), if there is $N \in N^1(X)_{\mathbb{R}}$ such that $f^*N - N$ is big (resp. ample). So f is quasi-amplified (resp. amplified) if and if f is 1-quasi-amplified (resp. 1-amplified).

The following result computes the spectrum for the big cone.

Theorem 1.4. *We have*

$$\mathrm{Sp}\,(f^*, \mathrm{Big}(X)) = \{\mu_i(f) |\ i = 1, \ldots, d\}.$$

In other words, for $\alpha \in \mathbb{R}_{>0}$, f is α-quasi-amplified if and only if

$$\alpha \notin \{\mu_i(f) |\ i = 1, \ldots, d\}.$$

In particular, f is quasi-amplified if and only if f is cohomologically hyperbolic.

As a consequence of Theorem 1.4, every $\mu_i(f), i = 1, \ldots, d$ is an eigenvalue of $f^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}}$, hence an algebraic integer.

The proof of Theorem 1.4 relies on the recursive inequalities constructed in [Xie24, Theorem 3.7] and the computation of mixed degrees in [Xie24, Corollary 3.4].

Next, we compute the spectrum for the ample cone. Let V be an irreducible and periodic subvariety of X of dimension $d_V \geq 0$. Define

$$\mu_i(V, f) := \mu_i(f|_V^{r_V} : V \to V)^{1/r_V}$$

where $r_V \geq 1$ is a period of V. It does not depend on the choice of r_V.

Theorem 1.5. *We have*

$$\mathrm{Sp}\,(f^*, \mathrm{Amp}(X)) = \cup_V \{\mu_i(V, f)|\ i = 1, \dots, d_V\}$$

where the union taken over all irreducible periodic subvarieties.

In other words, for $\alpha \in \mathbb{R}_{>0}$, f is α-amplified if and only if for every periodic irreducible subvariety V, $f^{r_V}|_V$ is α-quasi-amplified, where $r_V \geq 1$ is a period of V.

To prove Theorem 1.5, we introduce a notion of *generalized (positive) cycles* (c.f. Section 5), which can be viewed as the algebraic analogy of (positively) closed currents in complex geometry. We prove some basic properties of it. In particular, we prove a decomposition theorem of generalized positive cycles (c.f. Theorem 5.9), which can be viewed as an algebraic analogy of Siu's decommposition theorem [Siu74].

As a consequence of Theorem 1.4 and Theorem 1.5, we get the following result.

Corollary 1.6 (c.f. Corollary 6.6 and 3.6)**.** *Let Y be a projective variety over* $\mathbf{k}$ *and $g : Y \to Y$ be an endomorphism. Let $\pi : X \to Y$ be a surjective morphism such that $\pi \circ f = g \circ \pi$. If f is α-amplified (resp. α-quasi-amplified) for some $\alpha \in \mathbb{R}_{>0}$, then g is α-amplified (resp. α-quasi-amplified).*

In [KR17, Question 1.10], Krieger and Reschke asked the following question:

Question 1.7. Let X be a normal[1] projective variety over an algebraically closed field. Does an amplified (resp. polarized) endomorphism of X induce an amplified (resp. polarized) endomorphism of $\mathrm{Alb}(X)$?

The following result is a consequence of Corollary 1.6, which affirmatively answers the part of amplified endomorphisms in Question 1.7. Indeed, we proved a more general version for quasi-amplified endomorphisms.

Corollary 1.8 (=Corollary 6.7)**.** *Let X be a normal projective variety over an algebraically closed field. Let $f : X \to X$ be a quasi-amplified endomorphism. Then the induced endomorphism on* $\mathrm{Alb}(X)$ *is amplified.*

The part of Question 1.7 for polarized endomorphisms was already proved to be true by Matsuzawa-Meng-Shibata-Zhang-Zhong [MMS+22, Theorem 1.9] (see [CMZ20, Theorem 1.2] for the separable case). Hence, combining Corollary 1.8 with [MMS+22, Theorem 1.9], Question 1.7 is fully answered affirmatively.

Remark 1.9. In the end of [KR17], Krieger and Reschke conjectured that all unity-free[2] endomorphisms of a complex abelian variety are amplified. By [KR17, Theorem 1.9], the above conjecture implies a positive answer to Question 1.7 over $\mathbb{C}$. Moreover, they proved this conjecture in dimension two c.f. [KR17, Proposition 4.3]. However, this conjecture is not true in general. Here is a

[1] In [KR17, Question 1.10], they didn't ask X to be normal. However, we usually need the normality to define the Albanese variety $\mathrm{Alb}(X)$.

[2] A surjective endomorphism g on a complex abelian variety A is unity-free if no eigenvalue of $g^*|_{H^{1,0}(A)}$ is a root of unity c.f. [KR17, Definition 1.7].

counter-example: Consider the polynomial $P(x) = x^4 - 2x^3 - 2x + 1$, which is irreducible over $\mathbb{Q}$. It has two roots α_1, α_2 on the unit circle and two real roots β_1, β_2 with $0 < \beta_1 < 1 < \beta_2$. Let E be an elliptic curve over $\mathbb{C}$. Let $g : A := E^4 \to E^4$ be the endomorphism defined by

$$(x_1, x_2, x_3, x_4) \mapsto (x_2, x_3, x_4, 2x_4 + 2x_2 - x_1).$$

Easy to check that the eigenvalues of $g^*|_{H^{1,0}(A)}$ are $\alpha_1, \alpha_2, \beta_1, \beta_2$. So g is unity-free. Easy to check that the four cohomological Lyapunov exponents of g are $\beta_2^2 > 1 = 1 > \beta_1^2$. So g is not cohomologically hyperbolic. By Theorem 1.4, g is not quasi-amplified, hence not amplified.

Hyperbolicity. The case $\alpha = 1$ in Theorem 1.4 and 1.5 are especially interesting as they clarify two different algebraic analogies of the notion of hyperbolicity.

The hyperbolicity is one of the most important notion in smooth dynamical system. Basically, it means that the tangent bundle can be decomposed to the direct sum of the attracting and the repelling parts. Further, if we only have the repelling part, such map is called *expanding*. These notion are analytic, which do not make sense for algebraic dynamical system over an abstract field. Indeed, even when $\mathbf{k} = \mathbb{C}$, very few algebraic dynamical system could be Anosov (which is a strong version of hyperbolicity) c.f. [Ghy95, Can04, XZ24].

In our setting (or more generally for rational self-maps), f is called *cohomologically hyperbolic* if the cohomological Lyapunov exponents $\mu_i(f) \neq 1$ for every $i = 1, \ldots, d$. From the point of view of ergodic theory, this notion gives an algebraic analogy of the hyperbolicity.

In algebraic geometry, we may view an ample line bundle as an analogy of a Riemannian metric. From this point of view, the algebraic analogy of an expanding map should be an *int-amplified endomorphism*[3]. It was observed by Matsuzawa, that f is int-amplified if and only if $\mu_d(f) > 1$ (c.f. [MZ, Proposition 3.7]). In other words, the metric and ergodic theory styles analogies of expanding maps are the same.

Further, we view an arbitrary line bundle as an analogy of a pseudo-Riemannian metric. From this point of view, the algebraic analogy of a hyperbolic map should be an amplified endomorphism. Then the $\alpha = 1$ case of Theorem 1.4 and 1.5 connects the metric and ergodic theory styles analogies of the hyperbolicity.

Acknowledgement. The author would like to thank Sheng Meng, De-Qi Zhang and She Yang for their helpful comments for the first version of the paper. I especially thank Sheng Meng for telling me that our Corollary 1.6 implies a positive answer of Krieger-Reschke's Question 1.7.

[3]An endomorphism $f : X \to X$ is called int-amplified if there is an ample line bundle L of X such that $f^*L - L$ is ample [Men20].

2. Linear algebra for good invariant cones

In this section W is a finite dimensional $\mathbb{R}$-vector space and $g : W \to W$ be an endomorphism. Set $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$ and $g_{\mathbb{C}} : W_{\mathbb{C}} \to W_{\mathbb{C}}$ the endomorphism induced by g. View W as a $\mathbb{R}$-subspace of $W_{\mathbb{C}}$. We have $g = g_{\mathbb{C}}|_W$. Denote by $\mathrm{Sp}\,(g)$ the set of eigenvalues of g.

2.1. **Speed of growth.** Let $\|\cdot\|$ by any norm on W. Then for every $v \in W$, we have

$$\lim_{n\to\infty} \|f^n(v)\|^{1/n} = \min\{r \in \mathbb{R}_{\geq 0}|\ v \in E_{\overline{\mathbb{D}}_r}\} \tag{2.1}$$

$$= \min\{|c||\ c \in \mathrm{Sp}\,(g), v \in E_{\overline{\mathbb{D}}_{|c|}}\}. \tag{2.2}$$

Here $\overline{\mathbb{D}}_r$ is the closed disc in $\mathbb{C}$ with center 0 and radius r.

The following lemma is useful. It's proof is a simple application of the Jordan normal form. We leave it to the readers.

Lemma-Definition 2.1. Assume that $\mathrm{Sp}\,(g) \subseteq \mathbb{R}_{>0}$. Then for every $Z \in W^{\vee}$ and $v \in W$, there is a unique $(\beta(v), a(v)) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0}$ such that $(g^n(v) \cdot Z) = C(v)\beta(v)^n n^{a(v)} + O(\beta(v)^n n^{a(v)-1})$ with $C(v) \neq 0$. Moreover, there is a unique $(\beta, a) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0}$ such that the following holds:

(i) for every $v \in W$, $(\beta, a) \geq (\beta(v), a(v))$ for the lexicographical order $\geq$;
(ii) $\{v \in W|\ (\beta, a) > (\beta(v), a(v))\}$ is a proper closed subspace of W.

We say that $v \in W$ has *maximal growth* for Z if $(\beta, a) = (\beta(v), a(v))$.

Let $\mathcal{C}$ be a non-empty open convex cone in W satisfying the following properties:

(i) $g(\mathcal{C}) \subseteq \mathcal{C}$;
(ii) $\overline{\mathcal{C}}$ is salient i.e. $\overline{\mathcal{C}} \setminus \{0\}$ is convex.

We call such $\mathcal{C}$ a *good invariant cone* for g.

Lemma 2.2. *Let $Z \in W^{\vee}$. Assume that for every $v \in \overline{\mathcal{C}} \setminus \{0\}$, $(v, Z) > 0$. Then $\log \frac{(g^n(v),Z)}{\|g^n(v)\|}$ is bounded. In particular, if $\mathrm{Sp}\,(g) \subseteq \mathbb{R}_{>0}$, then every $v \in \mathcal{C}$ has maximal growth for Z.*

Proof. The cone $\mathcal{C}$ with Z induces a norm $\|\cdot\|'$ on W as follows: for every $v \in W$, $\|v\|' := \inf(Z(v_1)+Z(v_2))$ where (v_1, v_2) are taken over all pairs in $\overline{\mathcal{C}}$ with $v_1 - v_2 = v$. Easy to check that $\|\cdot\|$ is a norm and for every $v \in \overline{\mathcal{C}}$, $\|v\|' = Z(v)$. As all norms on W are equivalent, we conclude the proof. □

2.2. **Spectrum for good invariant cones.** We prove the following lemma in a setting more general than what we need. In our paper, we only need the case where $\mathcal{C}$ is a good invariant cone for g.

Lemma 2.3. *Let $\mathcal{C}$ be a convex subset of W with $g(\mathcal{C}) \subseteq \mathcal{C}$. Let V be a g-invariant subspace of W. Then $V \cap \mathcal{C} \neq \emptyset$ if and only if $V \cap E_{\mathbb{R}_{>0}} \cap \mathcal{C} \neq \emptyset$.*

Proof. The "if" parts are trivial. We now prove the "only if" parts. Assume that $V \cap \mathcal{C} \neq \emptyset$. There is $m \geq 0$, such that $g^m(W) = E_{\mathbb{C}^*}$. We have

$$\emptyset \neq g^m(V \cap \mathcal{C}) \subseteq V \cap \mathcal{C} \cap E_{\mathbb{C}^*}.$$

After replacing W by $E_{\mathbb{C}^*}$, we may assume that g is invertible i.e. $E_0 = 0$. After replacing W by the subspace spans by $\mathcal{C}$, we may assume that $\mathcal{C}$ spans W. After replacing $\mathcal{C}$ by $\mathrm{cone}(\mathcal{C}) := \mathbb{R}_{>0}\mathcal{C}$, we may assume that $\mathcal{C}^\circ \neq \emptyset$.

Pick $L \in V \cap \mathcal{C}$. By contradiction, assume that $V \cap E_{\mathbb{R}>0} \cap \mathcal{C} = \emptyset$. Then

$$V \cap E_{\mathbb{R}>0} \cap \mathrm{cone}(\mathcal{C}) = \emptyset.$$

By Hahn-Banach theorem, there is $Z \in W^\vee$ such that $V \cap E_{\mathbb{R}_{>0}} \subseteq Z^\perp$ and for every $N \in \mathcal{C}$, $(N \cdot Z) > 0$.

Set $S := \mathrm{Sp}\,(f^*|_V)$. We have $S = \overline{S}$. Consider the canonical decomposition in $V \otimes_{\mathbb{R}} \mathbb{C}$ that

$$L = \sum_{c \in S} L_c$$

where $L_c \in E_c(\mathbb{C})$. It is clear that $L_{\overline{c}} = \overline{L_c}$. For $n \geq 0$, write $L_n := g^n(L)$ and $(L_c)_n := g^n(L_c)$. As $V \cap E_{\mathbb{R}>0} \subseteq Z^\perp$, we have

$$(L_n \cdot Z) = \sum_{c \in S \setminus \mathbb{R}_{>0}} ((L_c)_n \cdot Z).$$

As $L \in V \cap \mathcal{C}$ and for every $N \in \mathcal{C}$, $(N \cdot Z) > 0$, we have

$$h(n) := (L_n \cdot Z) > 0 \tag{2.3}$$

for every $n \geq 0$. Easy to see that $h(n)$ is an exponential-polynomial function i.e. $h(n)$ is a finite sum of terms having form $u^n n^s$ where $u \in \mathbb{C}^*$ and $s \in \mathbb{Z}_{\geq 0}$. Moreover, we may ask the u above are contained in $\mathrm{Sp}\,(g) \setminus \mathbb{R}_{>0}$. We note that $0 \notin \mathrm{Sp}\,(g) \setminus \mathbb{R}_{>0}$. Consider $I := (\mathrm{Sp}\,(g) \setminus \mathbb{R}_{>0}) \times \mathbb{Z}_{\geq 0}$. For every $p = (u, s) \in I$, set $H_p(n) := u^n n^s$. Then there is a finite set $S \subseteq I$ such that

$$h(n) = \sum_{p \in S} c_p H_p(n)$$

where $c_p \in \mathbb{C}^*$. By (2.3), h is not constantly equal to 0, hence $S \neq \emptyset$. As $h(n) \in \mathbb{R}$ for every $n \geq 0$, S is invariant under the complex conjugacy i.e. for every $p = (u, s) \in S$, we have $\overline{p} := (\overline{u}, s) \in S$. Moreover, we have $c_{\overline{p}} = \overline{c_p}$.

Set $J := \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0}$ with the lexicographical order $\geq$. For $p = (u, s) \in I$, write $|p| := (|u|, s) \in J$. There is a unique maximal element $(\beta, a) \in \{|p|\,|\, p \in S\}$. Set $S^+ := \{p \in S|\ |p| = (\beta, a)\}$ and $S^- = S \setminus S^+$. Then S^+, S^- are invariant under complex conjugacy. Set $h^+ := \sum_{p \in S^+} c_p H_p$ and $h^- := \sum_{p \in S^-} c_p H_p$. We have $h = h^+ + h^-$. There is $C > 0$ such that

$$|h^-(n)| \leq C' \beta^n n^{a-1}. \tag{2.4}$$

for every $n \geq 1$.

For every $p \in S^+$, write

$$H_p(n) = e^{i\theta_p n} \beta^n n^a$$

where $\theta_p \in \mathbb{R}/2\pi\mathbb{Z}$. Note that, we have $\theta_p \neq 0$ for every $p \in S^+$. The map

$$p \in S^+ \mapsto \theta_p \in (\mathbb{R}/2\pi\mathbb{Z}) \setminus \{0\}$$

is injective. Set

$$C(n) := \sum_{p\in S^+} c_p e^{i\theta_p n}.$$

For every $n \geq 0$, we have $C(n) \in \mathbb{R}$. We have $h^+(n) = C(n)\beta^n n^a$. Using Vandermonde determinant, easy to show that $C(n)$ is not constantly equal to 0.

By Poincaré recurrence theorem, there is a strict increasing sequence $n_i \in \mathbb{Z}_{\geq 0}, i \geq 0$ such that

$$\lim_{i\to\infty} \theta_p n_i = 0 \in \mathbb{R}/2\pi\mathbb{Z}$$

for every $p \in S^+$.

Lemma 2.4. *There is $m \geq 0$ such that $C(m) < 0$.*

Set $m_i := m + n_i$, then m_i is strictly increasing and $C(m_i) \to C(m)$ as $i \to \infty$. By (2.4), we get

$$\lim_{i\to\infty} \frac{h(m_i)}{\beta^{m_i} m_i^a} = \lim_{i\to\infty} \frac{h^+(m_i)}{\beta^{m_i} m_i^a} = \lim_{i\to\infty} C(m_i) = C(m) < 0.$$

This contradicts with (2.3). □

Proof of Lemma 2.4. By contradition, we assume that $C(n) \geq 0$ for every $n \geq 0$. There is $m' \geq 0$ such that $C(m') \neq 0$. So we have $C(m') > 0$.

For every $l \geq 0$, we have

$$0 \leq \sum_{n=0}^{l} C(n) = \sum_{p\in S^+} c_p \sum_{n=0}^{l} e^{i\theta_p n} = \sum_{p\in S^+} c_p \frac{1 - e^{i\theta_p(l+1)}}{1 - e^{i\theta_p}} \leq \sum_{p\in S^+} \frac{2|c_p|}{1 - e^{i\theta_p}}. \tag{2.5}$$

As

$$\lim_{i\to\infty} \theta_p n_i = 0 \in \mathbb{R}/2\pi\mathbb{Z},$$

for every $p \in S^+$,

$$C(m' + n_i) \to C(m)$$

as $i \to \infty$. Then we have

$$\liminf_{j\to\infty} \sum_{n=0}^{m'+n_j} C(n) \geq \lim \sum_{i=0}^{j} C(m' + n_i) = +\infty.$$

This contradicts with (2.5). We concludes the proof. □

Let $\mathcal{C}$ be a good invariant cone for g.

Definition 2.5. For $\alpha \in \mathbb{R}$, we say that g is *α-amplified* for $\mathcal{C}$, if there is $N \in W$ such that $gN - \alpha N \in \mathcal{C}$. Define the $\mathcal{C}$-spectrum $\mathrm{Sp}\,(g, \mathcal{C})$ for g to be the set of $\alpha \in \mathbb{R}$ such that g is **not** α-amplified.

If $\alpha \notin \mathrm{Sp}\,(g)$, then $g - \alpha$ is invertible on W, hence g is α-amplified for $\mathcal{C}$. In particular, we have $\mathrm{Sp}\,(g, \mathcal{C}) \subseteq \mathrm{Sp}\,(g)$. It is clear that $\mathrm{Sp}\,(g, \mathcal{C})$ is decreasing on $\mathcal{C}$.

Lemma 2.6. *For $\alpha \in \mathbb{R}$, the following statement are equivalent:*

(i) *g is α-amplified for $\mathcal{C}$;*

(ii) *there is* $n \geq 1$*, such that* g^n *is* α^n*-amplified for* $\mathcal{C}$*;*
(iii) *for every* $n \geq 1$*, such that* g^n *is* α^n*-amplified for* $\mathcal{C}$*.*

Proof. It is clear that (iii) implies (i) and (i) implies (ii).

Now we show that (i) implies (iii). By (i), there is $N \in W$ such that

$$L := g(N) - \alpha N \in \mathcal{C}.$$

For every $n \geq 1$, we have

$$g^n(N) - \alpha^n N = \sum_{j=1}^{n} (\alpha^{n-j} g^j(N) - \alpha^{n-j+1} g^{j-1}(N)) = \sum_{j=0}^{n-1} g^j(L) \in \mathcal{C}.$$

This implies (iii).

We only need to show that (ii) implies (i). By (ii), there is $n \geq 1$ and $N \in W$ such that

$$M := g^n(N) - \alpha^n N \in \mathcal{C}.$$

Set $N' := \sum_{j=0}^{n-1} \alpha^{n-j} g^j(N)$. Then

$$f^* N' - \alpha N' = \alpha g^n(N) - \alpha^{n+1} N = \alpha M \in \mathcal{C}.$$

This implies (i). □

Lemma 2.6 has the following direct consequence.

Corollary 2.7. *For every* $n \geq 1$, $\mathrm{Sp}\,(g^n, \mathcal{C}) = \{\alpha^n |\ \alpha \in \mathrm{Sp}\,(g, \mathcal{C})\}$.

The following lemma shows that $\mathrm{Sp}\,(g, \mathcal{C}) \subseteq \mathbb{R}_{>0} \cap \mathrm{Sp}\,(g)$ and gives constrains of g-invariant subspace V meeting $\mathcal{C}$. We will reinforce this lemma in Corollary 2.11 latter.

Lemma 2.8. *Let* V *be a* g*-invariant subspace such that* $V \cap \mathcal{C} \neq \emptyset$*, then we have* $\mathrm{Sp}\,(g, \mathcal{C}) \subseteq \mathrm{Sp}\,(g|_V, \mathcal{C} \cap V)$*. As a consequence, we have* $\mathrm{Sp}\,(g, \mathcal{C}) \subseteq \mathbb{R}_{>0}$ *and* $\dim V \geq \#\mathrm{Sp}\,(g, \mathcal{C})$.

Proof. By contradiction, assume that $\mathcal{C} \cap V \neq \emptyset$ and $\mathrm{Sp}\,(g, \mathcal{C}) \not\subseteq \mathrm{Sp}\,(g|_V)$. Pick $c \in S \setminus \mathrm{Sp}\,(g, \mathcal{C})$. As $g - c\mathrm{id}$ is invertible on V, $V \subseteq \mathrm{Im}(g - c\mathrm{id})$. So $\mathrm{Im}(g - c\mathrm{id}) \cap \mathcal{C}$ contains $\mathcal{C} \cap V \neq \emptyset$. By Lemma 2.3, we have $\mathrm{Sp}\,(g, \mathcal{C}) \subseteq \mathbb{R}_{>0}$. □

We give an example that $\mathrm{Sp}\,(g, \mathcal{C}) = \mathbb{R}_{>0} \cap \mathrm{Sp}\,(g)$.

Example 2.9. Let $W = \mathbb{R}^d$ with standard base $e_1, \ldots, e_d$. Let $a_1, \ldots, a_d \in \mathbb{R}_{>0}$. Let $g : W \to W$ be the morphisms sending e_i to $a_i e_i$, $i = 1, \ldots, d$. Let

$$\mathcal{C} := \{x_1 e_1 + \cdots + x_d e_d |\ x_1, \ldots, x_d \in \mathbb{R}_{>0}\}.$$

Then $\mathcal{C}$ is a good invariant cone for g. Easy to check that $\mathrm{Sp}\,(g, \mathcal{C}) = \{a_1, \ldots, a_d\} = \mathbb{R}_{>0} \cap \mathrm{Sp}\,(g)$.

The following result gives a description of the spectrum $\mathrm{Sp}\,(g, \mathcal{C})$ using generalized eigenspaces.

Theorem 2.10. *(=Theorem 1.3) For every subset* $S \subseteq \mathbb{C}$, $\mathcal{C} \cap E_S \neq \emptyset$ *if and only if* $\mathrm{Sp}\,(g, \mathcal{C}) \subset S$.

Proof. The "only if" part follows from Lemma 2.8. We now prove the "if" part. We only need to show that when $S = \mathrm{Sp}\,(g, \mathcal{C})$, $\mathcal{C} \cap E_S = \emptyset$.

By contradiction, assume that $\mathcal{C} \cap E_S = \emptyset$. By Hahn-Banach theorem, there is $Z \in W^\vee$ such that $E_S \subseteq Z^\perp$ and for every $N \in \mathcal{C}$, $(N \cdot Z) > 0$. By Lemma 2.3, $\mathcal{C} \cap E_{\mathbb{R}_{>0}} \neq \emptyset$. As $\mathrm{Sp}\,(g|_{E_{\mathbb{R}_{>0}}}) \subseteq \mathbb{R}_{>0}$, by Lemma 2.2, every $L' \in \mathcal{C} \cap E_{\mathbb{R}_{>0}}$ has maximal growth for Z. More precisely, there is $(\beta, a) \in \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0}$ and $C' \in \mathbb{R}^*$ such that

$$(L'_n \cdot Z) = C'\beta^n n^a + O(\beta^n n^{a-1}),$$

where $L'_n := g^n(L')$. Moreover, for every $N \in W$, $(g^n(N) \cdot Z) = O((L'_n \cdot Z))$.

After replacing g by $\beta^{-1}g$, we may assume that $\beta = 1$. As $E_S \subseteq Z^\perp$, $1 \notin S = \mathrm{Sp}\,(g, \mathcal{C})$. So there is $N \in W$ such that

$$g(N) - N = L$$

for some $L \in \mathcal{C}$. Set $N_n := g^n(N), L_n := g^n(L)$ for $n \geq 0$. Then

$$(L_n, Z) = C\beta^n n^a + O(\beta^n n^{a-1}),$$

for some $C \in \mathbb{R}^*$. We have

$$N_{n+1} - N_n = L_n. \tag{2.6}$$

As L has maximal growth for Z, there is $m \geq 0$ such that for every $n \geq m$,

$$C/2 < \frac{h(n)}{n^a} < 2C. \tag{2.7}$$

As $L_n \in \mathcal{C}$, $h(n) > 0$. Hence $C > 0$. By (2.6), for every $n \geq m$, we get

$$\sum_{i=m}^{n} L_i = N_{n+1} - N_m \tag{2.8}$$

There is $B > 0$ such that $BL \pm N \in \mathcal{C}$. Then we get

$$\sum_{i=m}^{n} h(i) = \sum_{i=m}^{n} (L_i \cdot Z) \leq B((L_{n+1} \cdot Z) + (L_m \cdot Z)) \leq B(h(n+1) + h(m)).$$

By (2.7), we get

$$(C/2)\sum_{i=m}^{n} i^a \leq 2BC((n+1)^a + m^a).$$

Let $n \to \infty$, we get a contradiction. This concludes the proof. □

Corollary 2.11. *Let V be a g-invariant subspace such that $V \cap \mathcal{C} \neq \emptyset$, then we have* $\mathrm{Sp}\,(g, \mathcal{C}) = \mathrm{Sp}\,(g|_V, \mathcal{C} \cap V)$.

Proof. By Lemma 2.8, we have $\mathrm{Sp}\,(g, \mathcal{C}) \subseteq \mathrm{Sp}\,(g|_V, \mathcal{C} \cap V)$. We only need to show the inverse direction. Apply Theorem 2.10 to $g|_V$, we get

$$E_{\mathrm{Sp}\,(g|_V, \mathcal{C}\cap V)} \cap V \cap \mathcal{C} = (E_{\mathrm{Sp}\,(g|_V, \mathcal{C}\cap V)} \cap V) \cap (V \cap \mathcal{C}) \neq \emptyset.$$

Hence $E_{\mathrm{Sp}\,(g|_V, \mathcal{C}\cap V)} \cap \mathcal{C} \neq \emptyset$. We conclude the proof by Theorem 2.10. □

3. The spectrum for the big cone

Let X be a projective variety over $\mathbf{k}$ of dimension d and $f : X \to X$ is a surjective endomorphism. Denote by $\mu_i := \mu_i(f), i = 1, \ldots, d+1$, the cohomological Lyapunov exponents of f.

Theorem 3.1. *(=Theorem 1.4) We have*

$$\mathrm{Sp}\,(f^*, \mathrm{Big}(X)) = \{\mu_i |\ i = 1, \ldots, d\}.$$

In other words, for $\alpha \in \mathbb{R}_{>0}$, f is α-quasi-amplified if and only if

$$\alpha \notin \{\mu_i |\ i = 1, \ldots, d\}.$$

In particular, f is quasi-amplified if and only if f is cohomologically hyperbolic.

Proof of Theorem 3.1. Let L be an ample line bundle on X. Set $L_n := (f^n)^*L$.

We first assume that $\alpha \notin \{\mu_i |\ i = 1, \ldots, d\}$. There is a minimal $i = 0, \ldots, d$ such that $\mu_{i+1} < \alpha$. If $i = 0$, then $\alpha > \mu_1$. As μ_1 is the spectral radius of

$$f^* : N^1(X)_{\mathbb{R}} \to N^1(X)_{\mathbb{R}},$$

$f^* - \alpha\mathrm{id}$ is invertible on $N^1(X)_{\mathbb{R}}$. So for every big class $L \in N^1(X)_{\mathbb{R}}$, there is $N \in N^1(X)_{\mathbb{R}}$ such that

$$f^*N - \alpha N = L.$$

Now assume that $i \geq 1$. Then we have $\alpha \in (\mu_i, \mu_{i+1})$. Pick $\epsilon \in (\max\{\alpha/\mu_i, \mu_{i+1}/\alpha\}, 1)$. There is $m_0 \geq 1$, such that for every $m \geq m_0$,

$$\epsilon^m \mu_i^m - \alpha^m - \mu_i^m \mu_{i+1}^m \alpha^{-m} > 0. \tag{3.1}$$

By [Xie24, Theorem 3.7], there is $m_1 > m_0$, such that for every $m \geq m_1$,

$$M := L_{2m} + \mu_i^m \mu_{i+1}^m L - \epsilon^m \mu_i^m L_m$$

is big. Set $N := L_m - \mu_i^m \mu_{i+1}^m \alpha^{-m} L$. Then we have

$$(f^m)^*N - \alpha^m N = (L_{2m} - \alpha^m L_m) - \mu_i^m \mu_{i+1}^m \alpha^{-m}(L_m - \alpha^m L) \tag{3.2}$$

$$= M + (\epsilon^m \mu_i^m - \alpha^m - \mu_i^m \mu_{i+1}^m \alpha^{-m}) L_m. \tag{3.3}$$

By (3.1), $(f^m)^*N - \alpha^m N$ is big. So f^m is α^m-quasi-amplified. By Lemma 2.6, f is α-quasi-amplifed.

Now assume that $\alpha = \mu_i$ for some $i = 1, \ldots, d$ and want to show that f is not α-quasi-amplifed. Otherwise, we assume that f is α-quasi-amplifed, then there is $N \in N^1(X)_{\mathbb{R}}$ such that

$$M := f^*N - \alpha N = f^*N - \mu_i N$$

is big. As M is big, after replacing N by a suitable multiple, we may assume that $M \geq L$ i.e. $M - L$ is peudo-effective. There is $B > 0$ such that

$$-BL \leq N \leq BL.$$

For $n \geq 0$, set $N_n := (f^n)^*N$ and $M_n := (f^n)^*M$. Set

$$h(n, m) := \mu_i^{-m}(L_n^{i-1} \cdot L_m \cdot L^{d-i}).$$

For $m,n\in\mathbb{Z}_{\geq 0}$ and $m_0\geq 0$, we have

$$\sum_{j=m_0}^{m}\mu_i^{-j}(L_n^{i-1}\cdot M_j\cdot L^{d-i})=\mu_i^{-m-1}(L_n^{i-1}\cdot N_{m+1}\cdot L^{d-i})-\mu_i^{-m_0}(L_n^{i-1}\cdot N_{m_0}\cdot L^{d-i}). \tag{3.4}$$

As $M\geq L$, we have

$$\sum_{j=m_0}^{m}\mu_i^{-j}(L_n^{i-1}\cdot M_j\cdot L^{d-i})\geq\sum_{j=m_0}^{m}h(n,j). \tag{3.5}$$

As $-BL\leq N\leq BL$, we have

(3.6)

$$\mu_i^{-m-1}(L_n^{i-1}\cdot N_{m+1}\cdot L^{d-i})-\mu_i^{-m_0}(L_n^{i-1}\cdot N_{m_0}\cdot L^{d-i})\leq Bh(n,m+1)+Bh(n,m_0)$$

Combining (3.4), (3.5) and (3.6), we have

$$\sum_{j=m_0}^{m}h(n,j)\leq Bh(n,m+1)+Bh(n,m_0). \tag{3.7}$$

One may check that $h:\mathbb{Z}\geq 0\times\mathbb{Z}_{\geq 0}\to\mathbb{R}\subseteq\mathbb{C}$ is an exponential-polynomial function.

Lemma 3.2. *Let $h:\mathbb{Z}\geq 0\times\mathbb{Z}_{\geq 0}\to\mathbb{C}$ be an exponential-polynomial function i.e. $h(n,m)$ is a finite sum of terms having form*

$$u^n v^m n^s m^t$$

where $u,v\in\mathbb{C}^*$ and $s,t\in\mathbb{Z}_{\geq 0}$. Assume that h has the following conditions:

(i) *h is real and positive i.e. $h(n,m)\in\mathbb{R}_{>0}$ for every $n,m\in\mathbb{Z}_{\geq 0}$;*

(ii) *there is $D>0$ such that*

$$\max\{h(n+1,m),h(n,m+1)\}\leq Dh(n,m)$$

for every $n,m\in\mathbb{Z}_{\geq 0}$.

Then there is $C>1$, $\epsilon_0\in(0,1)$, $\beta,\gamma>0$, $a,b\in\mathbb{Z}_{\geq 0}$ such that

$$C^{-1}\leq\frac{h(n,m)}{\beta^n\gamma^m n^a m^b}\leq C$$

for

$$(n,m)\in\mathcal{N}:=\{(n,m)\in\mathbb{Z}_{\geq 0}|\ (\log n)^2\leq m\leq n^{\epsilon_0}\}.$$

Moreover, if the following holds,

(iii) *there is $\lambda>0$ such that for every $\delta\in(0,1)$, there is $D_\delta>1$ such that for every $n\geq m\geq 0$, we have*

$$D_\delta^{-1}\delta^n\leq\frac{h(n,m)}{\lambda^n}\leq D_\delta\delta^{-n},$$

then we may take $\beta=\lambda$ and $\gamma=1$.

It is clear that assumptions (i) in Lemma 3.2 are satisfied for our h. As L is ample, there is $D' > 0$ such that $f^*L \leq D'L$. So we have

$$h(n+1,m) = \mu_i^{-m}(L_{n+1}^{i-1} \cdot L_m \cdot L^{d-i}) \tag{3.8}$$
$$= \mu_i^{-m}((f^n)^*(f^*L)^{i-1} \cdot L_m \cdot L^{d-i}) \tag{3.9}$$
$$\leq D'^{i-1}\mu_i^{-m}((f^n)^*L^{i-1} \cdot L_m \cdot L^{d-i}) \tag{3.10}$$
$$= D'^{i-1}h(n,m). \tag{3.11}$$

and

$$h(n,m+1) = \mu_i^{-m-1}(L_n^{i-1} \cdot L_{m+1} \cdot L^{d-i}) \tag{3.12}$$
$$= \mu_i^{-m-1}(L_n^{i-1} \cdot (f^m)^*(f^*L) \cdot L^{d-i}) \tag{3.13}$$
$$\leq (D'/\mu_i)\mu_i^{-m}((f^n)^*L^{i-1} \cdot L_m \cdot L^{d-i}) \tag{3.14}$$
$$= (D'/\mu_i)h(n,m). \tag{3.15}$$

Taking $D := \max\{D'^{i-1}, D'/\mu_i\}$ we get condition (ii) in Lemma 3.2. By the computation of mixed degrees in [Xie24, Corollary 3.4], h satisfies (iii) for $\lambda = \lambda_{i-1}$.

There is $(n, m_0) \in \mathcal{N}$ such that $m_0 \geq 2 \times 3^b BC^2$ and $(n, 2m_0 - 1) \subseteq \mathcal{N}$. By (3.7) and Lemma 3.2, we have

$$\sum_{j=m_0}^{2m_0-1} j^b \leq BC^2(2m_0)^b + m_0^b) \leq BC^2 2^{b+1} m_0^b \tag{3.16}$$

On the other hand, we have

$$\sum_{j=m_0}^{2m_0-1} j^b \geq m_0^{b+1} \geq 2 \times 3^b BC^2 m_0^b.$$

This contradicts (3.16). We conclude the proof. □

Proof of Lemma 3.2. For $p := (u, v, s, t) \in I := \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, write

$$H_p := u^n v^m n^s m^t.$$

For $p := (u, v, s, t) \in I$, let $\overline{p} := (\overline{u}, \overline{v}, s, t) \in I$ be its complex conjugation.

There is a finite set S of I such that

$$h(n,m) = \sum_{p \in S} c_p H_p(n,m)$$

for every $n, m \in \mathbb{Z}_{\geq 0}$. We may assume further that $c_p \neq 0$ for every $p \in S$. By assumption (i), after replacing h by $(h + \overline{h})/2$, we may assume that S is invariant under complex conjugacy and we have $c_{\overline{p}} = \overline{c_p}$. We may assume that h is not identically 0, then $S \neq \emptyset$.

Consider $J := \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with the lexicographical order $\geq$. This is a total order. For $p = (u, v, s, t) \in I$, define $|p| := (|u|, |v|, s, t) \in J$. Pick $\epsilon_0 \in (0, 1)$ sufficiently close to 0 and set

$$\mathcal{N} := \{(n,m) \in \mathbb{Z}_{\geq 0} |\ (\log n)^2 \leq m \leq n^{\epsilon_0}\}$$

and

$$\mathcal{N}' := \{(n,m) \in \mathbb{Z}_{\geq 0} |\ 2^{-1}(\log n)^2 \leq m \leq 2n^{\epsilon_0}\}.$$

It is clear that for $p, p' \in S$, $|p| > |p'|$ if and only if

$$\lim_{m\to\infty} \sup\{\frac{|H_{p'}(n,m)|}{|H_p(n,m)|} |\ (n,m) \in \mathcal{N}'\} \to 0. \tag{3.17}$$

There is a unique maximal element $(\beta, \gamma, a, b) \in \{|p||\ p \in S\}$. Set $S^+ := \{p \in S|\ |p| = (\beta, \gamma, a, b)\}$ and $S^- = S \setminus S^+$. Then S^+, S^- are invariant under complex conjugacy. Set $h^+ := \sum_{p\in S^+} c_p H_p$ and $h^- := \sum_{p\in S^-} c_p H_p$. We have $h = h^+ + h^-$. By (3.17), there is a function $e : \mathbb{Z}_{\geq 0} \to \mathbb{R}_{>0}$ with $\lim_{m\to\infty} e(m) = 0$, such that

$$|h^-(n,m)| \leq e(m)\beta^n\gamma^m n^a m^b \tag{3.18}$$

for every $(n,m) \in \mathcal{N}$.

For every $p \in S^+$, write

$$H_p(n,m) = e^{i\theta_p n} e^{i\phi_p m} \beta^n \gamma^m n^a m^b$$

where $\theta_p, \phi_p \in \mathbb{R}/2\pi\mathbb{Z}$. The map

$$p \in S^+ \mapsto (\theta_p, \phi_p) \in \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$$

is injective. Set

$$C(n,m) := \sum_{p\in S^+} c_p e^{i\theta_p n} e^{i\phi_p m}.$$

We have $h^+(n,m) = C(n,m)\beta^n\gamma^m n^a m^b$. Using Vandermonde determinant, easy to show that $C(n,m)$ is not constantly equal to 0.

Define $\Theta : \mathbb{Z}_{\geq 0} \to (\mathbb{R}/2\pi\mathbb{Z})^{S^+}$ to be the map

$$\Theta : n \mapsto (n\theta_p)_{p\in S^+}.$$

Define $\Phi : \mathbb{Z}_{\geq 0} \to (\mathbb{R}/2\pi\mathbb{Z})^{S^+}$ to be the map

$$\Phi : m \mapsto (m\phi_p)_{p\in S^+}.$$

Define $R := \Theta \times \Phi : \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to (\mathbb{R}/2\pi\mathbb{Z})^{S^+} \times (\mathbb{R}/2\pi\mathbb{Z})^{S^+}$ to be the map

$$R : (n,m) \mapsto ((n\theta_p)_{p\in S^+}, (m\phi_p)_{p\in S^+}).$$

Let $q : (\mathbb{R}/2\pi\mathbb{Z})^{S^+} \times (\mathbb{R}/2\pi\mathbb{Z})^{S^+} \to \mathbb{R}$ be the function

$$(x,y) \mapsto \sum_{p\in S^+} c_p e^{ix} e^{iy}.$$

It is continuous. We have

$$C(n,m) = q \circ R(n,m) = q(\Theta(n), \Phi(m)).$$

Let Z_1, Z_2 be the closures of $\Theta(\mathbb{Z}_{\geq 0})$ and $\Phi(\mathbb{Z}_{\geq 0})$ respectively. It is clear that $Z_1 \times Z_2$ is the closure of $R(\mathbb{Z}^2_{\geq 0})$. By Poincaré recurrence theorem, there are strictly increasing sequences $n_i, m_i \in \mathbb{Z}_{\geq 0}, i \geq 0$ such that

$$\Theta(n_i) \to 0 \text{ and } \Phi(m_i) \to 0.$$

Lemma 3.3. *The subsets* Z_1 *and* Z_2 *are subgroups of* $(\mathbb{R}/2\pi\mathbb{Z})^{S^+}$.

Lemma 3.4. *For every non-empty open subset* $U \subseteq Z_1 \times Z_2$*, there is* $B_U \geq 0$ *such that for every* $z \in Z_1 \times Z_2$*, there is* $(n, m) \in \{0, \ldots, B_U\}^2$ *such that*

$$z + R(n, m) \in U.$$

We claim that $Z_1 \times Z_2$ is the limit set of $R(\mathcal{N})$. Let V be any non-empty open subset of $Z_1 \times Z_2$. Let B_V as in Lemma 3.4. For every $T \geq 0$, there is $(n_0, m_0) \in \mathcal{N}$ with $(n_0, m_0) + \{0, \ldots, B_V\}^2 \subseteq \mathcal{N}$ and $m \geq T$. By Lemma 3.4, there is $(n', m') \in \{0, \ldots, B_V\}^2$ such that $R(n_0 + n', m_0 + m') \in V$. This implies that claim.

We claim that $q \geq 0$ on $Z_1 \times Z_2$. As $Z_1 \times Z_2$ is the limit set of $R(\mathcal{N})$, for every $z \in Z_1 \times Z_2$, there is a sequence $(n'_i, m'_i) \in \mathcal{N}$ such that $m'_i \to \infty$ and $R(n'_i, m'_i) \to z$. Then we have

$$0 \leq \lim_{i\to\infty} \frac{h(n'_i, m'_i)}{\beta^{n'_i}\gamma^{m'_i}{n'_i}^a{m'_i}^b} = \lim_{i\to\infty} \frac{h_+(n'_i, m'_i)}{\beta^{n'_i}\gamma^{m'_i}{n'_i}^a{m'_i}^b} = \lim_{i\to\infty} q(R(n'_i, m'_i)) = q(z). \tag{3.19}$$

This implies the claim.

As q is not constantly zero on $Z_1 \times Z_2$, there is $A_1 > 0$ and a non-empty open subset U of $Z_1 \times Z_2$ such that $q|_U > A_1$. Let B_U as in Lemma 3.4.

Lemma 3.5. *For every* $z \in Z_1 \times Z_2$,

$$q(z) > \min\{1, \beta\}^{B_U} \min\{\gamma, \beta\}^{B_U} D^{-2B_U} A_1.$$

Set

$$A_2 := \min\{1, \beta\}^{B_U} \min\{\gamma, \beta\}^{B_U} D^{-2B_U} A_1$$

and

$$A_3 := \max q(Z_1 \times Z_2).$$

Pick $T_0 \geq 0$ such that $|e(m)| < 0.1A_2$ for every $m \geq T_0$. Then for every $(n, m) \in \mathcal{N}'$ with $m \geq T_0$, we have

$$h(n, m) = h^+(n, m) + h^-(n, m) \leq (A_3 + 0.1A_2)\beta^n\gamma^m n^a m^b$$

and

$$h(n, m) = h^+(n, m) + h^-(n, m) \geq 0.9A_2\beta^n\gamma^m n^a m^b.$$

As $h(n, m) > 0$ for every $n, m \geq 0$, there is $C > 0$ such that

$$C^{-1} \leq \frac{h(n, m)}{\beta^n\gamma^m n^a m^b} \leq C \tag{3.20}$$

for $(n, m) \in \mathcal{N}$.

We now assume that (iii) holds and prove the last statement.

Now assume that (iii) holds: there is $\lambda > 0$ such that for every $\delta \in (0, 1)$, there is $D_\delta > 1$ such that for every $n \geq m \geq 0$, we have

$$D_\delta^{-1}\delta^n \leq \frac{h(n, m)}{\lambda^n} \leq D_\delta\delta^{-n}. \tag{3.21}$$

For $n \gg 0$, $(n, \lfloor (\log n)^2 \rfloor + 1) \in \mathcal{N}$. By (3.21), we have

$$\lim_{n\to\infty} h(n, \lfloor (\log n)^2 \rfloor + 1\rfloor)^{1/n} = \lambda.$$

On the other hand, by (3.20),

$$\lim_{n\to\infty} h(n, \lfloor (\log n)^2 \rfloor + 1\rfloor)^{1/n} = \beta.$$

Then we get $\beta = \lambda$.

We only need to show that $\gamma = 1$. Let

$$S_0 := \{(u, v, s, t) \in S |\ |u| = \beta, |v| = \gamma\}.$$

Then S_0 is invariant under complex conjugacy and $S^+ \subseteq S_0$. Set $C_0 := \max\{s + t | (u, v, s, t) \in S\}$. Set

$$S' := \{(u, v, s, t) \in S_0 |\ s + t = C_0\}.$$

Then S' is non-empty and invariant under complex conjugacy. Set $S'' := S \setminus S'$. For $\epsilon_1 \in (0, 1)$, set

$$\mathcal{M}(\epsilon_1) := \{(n, m) |\ 2^{-1}\epsilon_1 n \le m \le \epsilon_1 n\}.$$

It is clear that there is $\epsilon' \in (0, 1)$ such that for every $\epsilon_0 \in (0, \epsilon')$ and $p \in S'$, we have

$$\lim_{m\to\infty} \sup\{\frac{|H_p(n, m)|}{\beta^n \gamma^m n^{C_0}} |\ (n, m) \in \mathcal{M}(\epsilon_1)\} \to 0. \tag{3.22}$$

Set $h' := \sum_{p\in S'} c_p H_p$ and $h'' := \sum_{p\in S''} c_p H_p$. We have $h = h' + h''$. By (3.22), there is a function $e' : \mathbb{Z}_{\ge 0} \to \mathbb{R}_{>0}$ with $\lim_{m\to\infty} e'(m) = 0$, such that

$$|h'(n, m)| \le e'(m)\beta^n \gamma^m n^{C_0} \tag{3.23}$$

for every $(n, m) \in \mathcal{N}$. For every $p \in S'$, write

$$H_p(n, m) = e^{i\theta_p n} e^{i\phi_p m} \beta^n \gamma^m n^{C_0} (\frac{m}{n})^{b_p}$$

where $\theta_p, \phi_p \in \mathbb{R}/2\pi\mathbb{Z}$. Set

$$D(n, m) := \sum_{p\in S'} c_p e^{i\theta_p n} e^{i\phi_p m} (\frac{m}{n})^{b_p}.$$

We have $h'(n, m) = D(n, m)\beta^n \gamma^m n^{C_0}$. Set $b_- := \min\{t |\ (u, v, s, t) \in S'\}$. Set $S'_+ := \{(u, v, s, t) \in S' |\ t = b_-\}$. So S'_+ is non-empty and invariant under complex conjugacy. Set $S'_- := S' \setminus S'_+$. Set

$$C_+(n, m) := \sum_{p\in S'_+} c_p e^{i\theta_p n} e^{i\phi_p m}$$

and

$$D_+(n, m) := C_+(n, m)(\frac{m}{n})^{b_-}.$$

Set

$$D_-(n, m) := \sum_{p\in S'_-} c_p e^{i\theta_p n} e^{i\phi_p m} (\frac{m}{n})^{b_p}$$

Set

$$A_- := \sum_{p\in S'_-} |c_p|.$$

Then for $(n,m)\in\mathcal{M}(\epsilon_1)$,

$$|D_-(n,m)| \le A_-\epsilon_1^{b_-+1}. \tag{3.24}$$

Define $\Theta' : \mathbb{Z}_{\ge 0} \to (\mathbb{R}/2\pi\mathbb{Z})^{S'_+}$ to be the map

$$\Theta' : n \mapsto (n\theta_p)_{p\in S'_+}.$$

Define $\Phi' : \mathbb{Z}_{\ge 0} \to (\mathbb{R}/2\pi\mathbb{Z})^{S'_+}$ to be the map

$$\Phi : m \mapsto (m\phi_p)_{p\in S'_+}.$$

Define $R' := \Theta\times\Phi : \mathbb{Z}_{\ge 0}\times\mathbb{Z}_{\ge 0} \to (\mathbb{R}/2\pi\mathbb{Z})^{S'_+}\times(\mathbb{R}/2\pi\mathbb{Z})^{S'_+}$ to be the map

$$R' : (n,m) \mapsto ((n\theta_p)_{p\in S'_+}, (m\phi_p)_{p\in S'_+}).$$

Let $q' : (\mathbb{R}/2\pi\mathbb{Z})^{S'_+}\times(\mathbb{R}/2\pi\mathbb{Z})^{S'_+} \to \mathbb{R}$ be the function

$$(x,y) \mapsto \sum_{p\in S'_+} c_p e^{ix} e^{iy}.$$

It is continuous. We have

$$C_+(n,m) = q'\circ R'(n,m) = q'(\Theta'(n),\Phi'(m)).$$

Note that the map

$$p\in S'_+ \mapsto (\theta_p,\phi_p) \in \mathbb{R}/2\pi\mathbb{Z}\times\mathbb{R}/2\pi\mathbb{Z}$$

is injective. Using Vandermonde determinant, easy to show that $C'(n,m)$ is not constantly equal to 0. In particular q' is not constantly equal to 0 on $Z'_1\times Z'_2$.

Let Z'_1, Z'_2 be the closures of $\Theta'(\mathbb{Z}_{\ge 0})$ and $\Phi(\mathbb{Z}_{\ge 0})$ respectively. Same proof as Lemma 3.3 shows that Z'_1, Z'_2 are closed subgroups in $(\mathbb{R}/2\pi\mathbb{Z})^{S'_+}$. There is a non-empty open subset $U'\subseteq Z'_1\times Z'_2$ and a constant $A_3>0$ such that for every $z\in U'$, $|q'(z)|>A_3$. For every $(n,m)\in\mathcal{M}(\epsilon_1)$, if $R(n,m)\in U'$, then

$$|D_+(n,m)| \ge 2^{-b_-}A_3\epsilon_1^{b_-} \tag{3.25}$$

Let $A_4 := \max\{|q(z)|| \ z\in Z'_1\times Z'_2\}$. Then

$$|D_+(n,m)| \le A_4\epsilon_1^{b_-} \tag{3.26}$$

Fix $\epsilon_1\in(0,\epsilon')$ with $\epsilon_1 < \frac{0.1A_3}{A_1 2^{b_-}}$. Then for every $(n,m)\in\mathcal{M}(\epsilon_1)$, if $R(n,m)\in U'$, we have

$$|D(n,m)| \ge 0.9\times 2^{-b_-}A_3\epsilon^{b_-} \text{ and } |D(n,m)| \le 2A_3\epsilon^{b_-}. \tag{3.27}$$

Same proof as Lemma 3.4 shows that there is $B'\ge 0$ such that for every $z\in Z'_1\times Z'_2$, there is $(n,m)\in\{0,\dots,B'\}^2$ such that

$$z + R'(n,m) \in U'.$$

For every $n\ge 0$, there is $(N(n),M(n)) \in (\lfloor 0.75\epsilon n\rfloor, n) + \{0,\dots,B'\}^2$ such that

$$R'(N(n),M(n)) \in U'.$$

By (3.27) and (3.23), there is $C_1>1$ such that for every $(n,m)\in\mathcal{M}(\epsilon_1)$, we have

$$C_1^{-1}<\frac{h(N(n),M(n))}{\beta^{N(n)}\gamma^{M(n)}N(n)^{C_0}}<C_1. \tag{3.28}$$

Then we have

$$\lim_{n\to\infty}h(N(n),M(n))^{1/n}=\lim_{n\to\infty}(\beta^{N(n)}\gamma^{M(n)}N(n)^{C_0})^{1/n}=\beta\times\gamma^{\epsilon_0}.$$

By (iii) and the fact $\beta=\lambda$, we get

$$\lim_{n\to\infty}h(N(n),M(n))^{1/n}=\lambda.$$

Then we get $\gamma=1$, which concludes the proof. □

Proof of Lemma 3.3. As Θ and Φ are group homeomorphisms, both Z_1 and Z_2 contain 0 and are closed under addition. For every $z\in Z_1$, there is an increasing sequence $l_i\in\mathbb{Z}_{\geq 0}$ such that

$$\Theta(l_i)\to z.$$

Note that l_i may not be strictly increasing. After taking subsequence, we may assume that $n_i\geq l_i$. Then we have

$$\Theta(n_i-l_i)\to -z.$$

This implies that Z_1 is a subgroup. Similarly, Z_2 is a subgroup. □

Proof of Lemma 3.4. Pick $w\in U$. There is an open neighborhood W of 0 such that $W=-W$ and $W+W\subseteq U-w$. Set $V:=w+W\subseteq U$. We have $W+V\subseteq U$. As $Z_1\times Z_2$ is a group, for every $z\in Z_1\times Z_2$, $-z\in Z_1\times Z_2$. So there is $(n_z,m_z)\in\mathbb{Z}^2_{\geq 0}$ such that $R(n_z,m_z)\in V-z$. As $Z_1\times Z_2$ is compact, there is a finite subset F of $Z_1\times Z_2$ such that $F+V=Z_1\times Z_2$. For every $z'\in Z_1\times Z_2$, there is $z''\in F$ such that $z'\in z''+W$. Then we have

$$z'+R(n_{z''},m_{z''})=z''+R(n_{z''},m_{z''})+(z'-z'')\subseteq V+W\subseteq U.$$

Setting $B_U:=\max\{\max\{n_z,m_z\}|z\in F\}$, we conclude the proof. □

Proof of Lemma 3.5. Pick a sequence $(n_i',m_i')\in\mathcal{N}$ such that $m_i'\to\infty$, and $R(n_i',m_i')\to z$. There is $(b_1,b_2)\in\{0,\dots,B_U\}^2$ such that

$$R(b_1,b_2)+z\in U.$$

Note that for m sufficiently large, we have $\mathcal{N}+\{0,\dots,B_U\}^2\subseteq\mathcal{N}'$.

By (3.19), we have

$$\lim_{i\to\infty}\frac{h(n_i',m_i')}{\beta^{n_i'}\gamma^{m_i'}{n_i'}^a{m_i'}^b}=q(z).$$

Similarly, we have

$$\beta^{-b_1}\gamma^{-b_2}\lim_{i\to\infty}\frac{h(n_i'+b_1,m_i'+b_2)}{\beta^{n_i'}\gamma^{m_i'}{n_i'}^a{m_i'}^b}=\lim_{i\to\infty}\frac{h(n_i'+b_1,m_i'+b_2)}{\beta^{n_i'}\gamma^{m_i'}(n_i'+b_1)^a(m_i'+b_2)^b}$$
$$=q(R(b_1,b_2)+z)>A_1.$$

Our condition (ii) implies that

$$h(n_i'+b_1,m_i'+b_2)\leq D^{b_1+b_2}h(n_i',m_i')\leq D^{2B_U}h(n_i',m_i').$$

Then we get

$$A_1 < \beta^{-b_1}\gamma^{-b_2}D^{2B_U}\lim_{i\to\infty}\frac{h(n'_i,m'_i)}{\beta^{n'_i}\gamma^{m'_i}{n'_i}^a{m'_i}^b} = \beta^{-b_1}\gamma^{-b_2}D^{2B_U}q(z).$$

So we have

$$q(z) > \beta^{b_1}\gamma^{b_2}D^{-2B_U}A_1 \geq \min\{1,\beta\}^{B_U}\min\{\gamma,\beta\}^{B_U}D^{-2B_U}A_1,$$

which concludes the proof. □

Corollary 3.6. *Let Y be a projective variety over $\mathbf{k}$ and $g : Y \to Y$ be an endomorphism. Let $\pi : X \to Y$ be a surjective morphism such that $\pi \circ f = g \circ \pi$. If f is α-quasi-amplified for some $\alpha \in \mathbb{R}_{>0}$, then g is α-quasi-amplified.*

Proof. The product formula for relative dynamical degrees (c.f. [DN11], [Dan20] and [Tru20, Theorem 1.3]) shows that

$$\{\mu_i(V,g)|\ i = 1,\dots,d_V\} \subseteq \{\mu_i(W,f)|\ i = 1,\dots,d_W\}.$$

We conclude the proof by Theorem 3.1. □

4. Measures for the constructible topology

Let X be a reduced projective scheme over $\mathbf{k}$ of dimension d. For every $x \in X$, denote by $Z_x := \overline{\{x\}}$.

4.1. **Constructible topology.** Denote by $|X|$ the underling set of X with the constructible topology; i.e. the topology on a X generated by the constructible subsets (see [Gro64, Section (1.9) and in particular (1.9.13)]). In particular every constructible subset is open and closed. This topology is finer than the Zariski topology on X. Moreover $|X|$ is (Hausdorff) compact.

Let $\mathcal{C}(|X|)$ be the space of continuous functions on $|X|$ endowed with the norm $\|h\| := \max\{f(x)|\ x \in |X|\}$. Let $\mathcal{A}(|X|)$ be the set of constructible subsets of X. For every $U \in \mathcal{A}(|X|)$, denote by 1_U the characteristic function of U i.e. $1_U(x) = 1$ if $x \in U$ and $1_U(x) = 0$ if $x \notin U$. Such functions are continuous, as constructible subsets are both open and closed in $|X|$. For two distinct points $x, y \in |X|$, there is a constructible subset U of X such that $x \in U$ and $y \notin U$. So we have $1_U(x) \neq 1_U(y)$. By Stone-Weierstrass theorem, we get the following result.

Lemma 4.1. *The $\mathbb{R}$-algebra generated by $\{1_U|\ U \in \mathcal{A}(|X|)\}$ is dense in $\mathcal{C}(|X|)$.*

4.2. **Measures.** Let $\mathcal{A}(|X|)$ be the set of constructible subsets of X. It is a field of sets in $|X|$ i.e. it has the following properties:

(i) $\emptyset \in \mathcal{A}(|X|), |X| \in \mathcal{A}(|X|)$;
(ii) for $U, V \in \mathcal{A}(|X|)$, $U \cap V \in \mathcal{A}$ and $U \setminus V \in \mathcal{A}(|X|)$.

Moreover, the Borel σ-algebra $\mathcal{B}(|X|)$ on $|X|$ is generated by $\mathcal{A}(|X|)$.

Proposition 4.2. *Every finite measure Borel measure on $|X|$ is a Radon measure i.e. $|X|$ is a Radon space.*

Proof. Let μ be a finite Borel measure on $|X|$. By Riesz-Markov-Kakutani representation theorem (c.f. [Rud87, Theorem 2.14]), there is a Radon measure μ' such that for every continuous function h on $|X|$, $\int h\mu = \int h\mu'$. So for every constructible subset V of X, we have

$$\mu'(V) = \int 1_V \mu' = \int 1_V \mu = \mu(V).$$

As the restriction of μ and μ' on $\mathcal{A}$ are the same and both of them are finite, the uniqueness part of the Carathéodory's extension theorem, implies that $\mu = \mu'$. So μ is a Radon measure, which concludes the proof. $\square$

Denote by $\mathcal{M}(|X|)$ the space of finite Borel measures on X endowed with the weak-$*$ topology. Combining [Xie23, Theorem 1.12] with Proposition 4.2, we get the following result.

Theorem 4.3. *Every $\mu \in \mathcal{M}(|X|)$ takes form*

$$\mu = \sum_{i \geq 0} a_i \delta_{x_i}$$

where δ_{x_i} is the Dirac measure at $x_i \in X$, $a_i \geq 0$ with $\sum_{i\geq 0} a_i < +\infty$.

In [Xie23, Theorem 1.12], X is assumed to be irreducible. However the proof still work without the irreducibility. Let $\mathcal{M}^1(|X|)$ be the space of probability measures on $|X|$. Since $|X|$ is compact, $\mathcal{M}^1(|X|)$ is compact. Combining [Xie23, Theorem 1.12] with Proposition 4.2, we get the following result.

Corollary 4.4. *The space $\mathcal{M}^1(|X|)$ is sequentially compact.*

In [Xie23, Theorem 1.12], X is assumed to be irreducible. However the proof still works without the irreducibility.

4.3. **Vector valued measures.** Let V be a finitely dimensional $\mathbb{R}$-space. Let $\mathcal{P}$ be a convex closed salient cone in V with $\mathcal{P}^\circ \neq \emptyset$. Let $\mathcal{P}^\vee$ be its dual cone in $V^\vee$. Then $\mathcal{P}^\vee$ is a convex closed salient cone in $V^\vee$ with $(\mathcal{P}^\vee)^\circ \neq \emptyset$. Let $\|\cdot\|$ be any norm on V.

Remark 4.5. For every $\omega \in (\mathcal{P}^\vee)^\circ$, we may define a norm $\|\cdot\|_\omega$ on V as follows: for every $v \in V$, $\|v\|_\omega := \inf\{((v_1+v_2)\cdot\omega)|\ v_1, v_2 \in \mathcal{P}, v_1 - v_2 = v\}$. Easy to check that $\|\cdot\|_\omega$ is a norm. Moreover, for every $v \in \mathcal{P}$, we have $\|v\|_\omega = (v\cdot\omega)$.

We define a $\mathcal{P}$-valued Borel measure on $|X|$ to be a map $\gamma : \mathcal{B}(|X|) \to \mathcal{P}$ such that

(i) $\gamma(\emptyset) = 0$;
(ii) for a countable collection of disjoint Borel subsetes $E_n, n \geq 0$, we have

$$\gamma(\sqcup_{n\geq 0} E_n) = \sum_{n\geq 0} \gamma(E_n).$$

Denote by $\mathcal{M}(|X|,\mathcal{P})$ the set of $\mathcal{P}$-valued Borel measure on $|X|$. There is a map

$$(\beta,\gamma)\in\mathcal{P}^\vee\times\mathcal{M}(|X|,\mathcal{P})\to(\beta\cdot\gamma)\in\mathcal{M}(|X|)$$

such that for every $E\in\mathcal{B}(|X|)$, $(\beta\cdot\gamma)(E)=(\beta\cdot\gamma(E))$. Easy to check that $(\beta\cdot\gamma)$ is a Borel measure on $|X|$. Define the weak-* topology on $\mathcal{M}(|X|,\mathcal{P})$ by the topology generated by the sets of form

$$\{\gamma\in\mathcal{M}(|X|,\mathcal{P})|\ \int h(\beta\cdot\gamma)<c\}$$

where $\beta\in\mathcal{P}^\vee, h\in C(|X|)$ and $c\in\mathbb{R}$.

On the other hand, once we have a continuous map

$$\phi:\mathcal{P}^\vee\to\mathcal{M}(|X|)$$

such that for $r_1,r_2\geq 0, v_1,v_2\in\mathcal{P}^\vee$, we have

$$\phi(r_1v_1+r_2v_2)=r_1\phi(v_1)+r_2\phi(v_2),$$

then ϕ defined a $\mathcal{P}$-valued Borel measure $\nu_\phi\in\mathcal{M}(|X|,\mathcal{P})$ such that for every $E\in\mathcal{B}(|X|)$, $\nu(E)$ is the unique element in $\mathcal{P}$ such that for every $v\in\mathcal{P}^\vee$,

$$(v,\nu(B))=\phi(v)(E).$$

Now we give a concrete description of $\mathcal{M}(|X|,\mathcal{P})$. Let $\mathcal{M}^\pm(|X|)=\mathcal{M}(|X|)\otimes_{\mathbb{R}_{>0}}\mathbb{R}$ be the space of signed measures. By Theorem4.3, every $\mu\in\mathcal{M}^\pm(|X|)$ takes form

$$\mu=\sum_{i\geq 0}a_i\delta_{x_i}$$

where δ_{x_i} are the Dirac measures at distinct points $x_i\in X$ and $a_i\in\mathbb{R}$ with $\sum_{i\geq 0}|a_i|<+\infty$. Consider the $\mathbb{R}$-space $\mathcal{M}(|X|)\otimes_{\mathbb{R}_{>0}}V:=\mathcal{M}^\pm\otimes_{\mathbb{R}}V$. Assume that $\dim V=s$ and fix $\beta_1^\vee,\ldots,\beta_s^\vee\in(\mathcal{P}^\vee)^\circ$ which forms a base of $V^\vee$. Let $\beta_1,\ldots,\beta_s$ be its dual basis in V. Then $\mathcal{M}(|X|)\otimes_{\mathbb{R}_{>0}}V=\oplus_{i=1}^s\mathcal{M}^\pm(|X|)\beta_i$. So every element in $\mathcal{M}(|X|)\otimes_{\mathbb{R}_{>0}}V$ takes form

$$\mu=\sum_{i\geq 0}\alpha_i\delta_{x_i}$$

where δ_{x_i} are the Dirac measures at distinct points $x_i\in X$ and $\alpha_i\in V$ with $\sum_{i\geq 0}\|\alpha_i\|<+\infty$.

Consider the map

$$\Theta:\gamma\in\mathcal{M}(|X|,\mathcal{P})\mapsto(\beta_i^\vee\cdot\gamma)\beta_i\in\mathcal{M}(|X|)\otimes_{\mathbb{R}_{>0}}V.$$

This map induces a homeomorphism from $\mathcal{M}(|X|,\mathcal{P})$ to its image. The pairing $\mathcal{P}^\vee\times\mathcal{M}(|X|,\mathcal{P})\to\mathcal{M}(|X|)$ extends to the pairing

$$\mathcal{M}(|X|)\otimes_{\mathbb{R}_{>0}}V\times V^\vee\to\mathcal{M}^\pm(|X|)$$

defined in the obvious way:

$$((\oplus_{i=1}^s\mu_i\beta_i)\cdot(\sum_{i=1}^s a_i\beta_i^\vee)):=\sum_{i=1}^s a_i\mu_i.$$

Then the image of Θ is the intersection

$$\Theta(\mathcal{M}(|X|,\mathcal{P})) = \cap_{\beta\in\mathcal{P}^\vee}\{\alpha \in \mathcal{M}(|X|)\otimes_{\mathbb{R}_{>0}} V|\ (\alpha\cdot\beta)\in\mathcal{M}(|X|)\}.$$

For every Borel set $E \in \mathcal{B}(|X|)$, the characteristic function 1_E is measurable. In particular, for every $E \in \mathcal{B}(|X|)$, and $\gamma \in \mathcal{M}(|X|,\mathcal{P})$, we get $1_E\gamma \in \mathcal{M}(|X|,\mathcal{P})$. If further $E \in \mathcal{A}(|X|)$, 1_E is continuous. So the endomorphism

$$\gamma \in \mathcal{M}(|X|,\mathcal{P}) \to 1_E\gamma \in \mathcal{M}(|X|,\mathcal{P})$$

is continuous.

For every $x \in |X|$ and $\alpha \in \mathcal{P}$, define $\alpha\delta_x \in \mathcal{M}(|X|,\mathcal{P})$ as follows: for every $E \in \mathcal{B}(|X|)$, $\alpha\delta_x(E) = \alpha$ if $x \in V$ and $\alpha\delta_x(E) = 0$ if $x \notin E$. More generally, for an at most countable subset S of $|X|$, with a sequence of vectors $\alpha_x \in \mathcal{P}, x \in S$ with $\sum_{x\in S}\|\alpha_x\| < +\infty$, we may define $\sum_{x\in S}\alpha_x\delta_x \in \mathcal{M}(|X|,\mathcal{P})$ in the obvious way: for every $V \in \mathcal{B}(|X|)$,

$$(\sum_{x\in S}\alpha_x\delta_x)(E) := \sum_{x\in S}\alpha_x\delta_x(E).$$

Easy to check that

$$\Theta(\sum_{x\in S}\alpha_x\delta_x) = \sum_{x\in S}\alpha_x\delta_x \in \mathcal{M}(|X|)\otimes_{\mathbb{R}_{>0}} E.$$

On the other hand, for every $\gamma \in \mathcal{M}(|X|,\mathcal{P})$, we may write $\Phi(\gamma) = \sum_{x\in S}\alpha_x\delta_x$ for an at most countable subset S of $|X|$, $\alpha_x \in V$ with $\sum_{x\in S}\|\alpha_x\| < +\infty$. Then for every $\beta \in \mathcal{P}^\vee$,

$$(\beta\cdot\gamma) = \sum_{x\in S}(\beta\cdot\alpha_x)\delta_x \in \mathcal{M}(|X|).$$

So $(\beta\cdot\alpha_x) \geq 0$ for every $x \in S$. So we have $\alpha_x \in \mathcal{P}$. We then get the following generalization of Theorem 4.3.

Proposition 4.6. *Every $\gamma \in \mathcal{M}(|X|,\mathcal{P})$ takes form*

$$\gamma = \sum_{x\in S}\alpha_x\delta_x$$

for an at most countable subset S of $|X|$, $\alpha_x \in \mathcal{P}$ with $\sum_{x\in S}\|\alpha_x\| < +\infty$.

For $\gamma \in \mathcal{M}(|X|,\mathcal{P})$, the support $\operatorname{Supp}\gamma$ of γ is defined to be the support of the measure $(\beta\cdot\gamma)$ where $\beta \in (\mathcal{P}^\vee)^\circ$. It does not depend on the choice of β. Indeed $|X|\setminus\operatorname{Supp}\gamma$ is the maximal open subset U of $|X|$ satisfying $\gamma(U) = 0$.

Let $\mathcal{M}^1(|X|,\mathcal{P})$ be the subset elements γ of $\mathcal{M}(|X|,\mathcal{P})$ with $\|\gamma(X)\| \leq 1$. By Proposition 4.6 and Corollary 4.4, $\mathcal{M}^1(|X|,\mathcal{P})$ is compact and sequentially compact.

5. GENERALIZED CYCLES

Let X be a reduced projective scheme over $\mathbf{k}$ of dimension d. For every $i = 0, 1, \dots, d$, denote by X_i the set of (scheme-theoretic) points $x \in X_i$ with $\dim Z_x = i$ where $Z_x := \overline{\{x\}}$. Denote by $Z_i(X)_{\mathbb{R}}$ the space of i-cycles with $\mathbb{R}$-coefficients. Every $Z \in Z_i(X)_{\mathbb{R}}$ can be uniquely written as

$$Z = \sum_{x \in X_i} m(Z, x) Z_x,$$

where $m(Z, x) = 0$ for all but finitely many $x \in X_i$. Let $c(Z)$ be the set of x with $m(Z, x) \neq 0$. Denote by $\mathrm{Eff}_i(X)$ the subset of effective i-cycles in $Z_i(X)_{\mathbb{R}}$. For every subset U of X. Denote by $Z_i(U, X)_{\mathbb{R}}$ the set of i-cycles Z with $c(Z) \subseteq U$; and $\mathrm{Eff}_i(U, X)$ the subset of effective i-cycles in $Z_i(U, X)_{\mathbb{R}}$. Denote by $\mathrm{Psef}_i(X)$ the cone of pseudo-effective classes in $N_i(X)_{\mathbb{R}}$.

Let U be a locally closed subset of X i.e. U is open in its Zariski closure $\overline{U}^{\mathrm{zar}}$. Let $\mathrm{Psef}^i(U, X)$ be the closure of the convex cone in $\overline{U}^{\mathrm{zar}}$ generated by effective i-cycles of form

$$\sum_{j=1}^{m} a_j Z_{x_j}$$

where $a_j \geq 0$, $x_j \in U$. For $U_1 \subseteq U_2$ with the same Zariski closure, we have $\mathrm{Psef}^i(U_1, X) \subseteq \mathrm{Psef}^i(U_2, X)$.

5.1. **Generalized cycles.** Let $\mathcal{Z}(X)$ be the set of all non-empty Zairski closed subsets of X. Moreover, for Zariski closed subsets $V_1 \subseteq V_2$, we have natural map $\iota_{V_1 \subseteq V_2} : N_i(V_1)_{\mathbb{R}} \to N_i(V_2)_{\mathbb{R}}$ induced by the inclusion $V_1 \hookrightarrow V_2$. **For simplifying the notations, we often omit the morphism $\iota_{V_1 \subseteq V_2}$.** We note that if $\dim V < i$, then $N_i(V)_{\mathbb{R}} = 0$. For every $V \in \mathcal{Z}(X)_{\mathbb{R}}$, set $R_V : Z_i(X)_{\mathbb{R}} \to N_i(V)_{\mathbb{R}}$ sending Z to

$$R_V(Z) := \sum_{x \in X_i \cap V} m(Z, x)[Z_x].$$

Define

$$\Phi := \prod_{V \in \mathcal{Z}(X)} R_V : Z_i(X)_{\mathbb{R}} \to \prod_{V \in \mathcal{Z}(X)} N_i(V)_{\mathbb{R}}.$$

It is clear the Φ is injective. We endow $\prod_{V \in \mathcal{Z}(X)} N_i(V)_{\mathbb{R}}$ the product topology. Define $\mathcal{G}_i(X)$ to be the closure of $\Phi(Z_i(X)_{\mathbb{R}})$ with the induced topology. We call elements in $\mathcal{G}_i(X)$ the *generalized i-cycles*. We now identify $Z_i(X)_{\mathbb{R}}$ with its image in $\mathcal{G}_i(X)$. We would like to think generalized i-cycles as an analogy of the notion of closed current of bidimension (i, i) in complex geometry. For every $V \in \mathcal{Z}(X)$, the morphism R_V is just the restriction of the projection $\prod_{W \in \mathcal{Z}(X)} N_i(W)_{\mathbb{R}} \to N_i(V)_{\mathbb{R}}$ to $Z_i(X)_R$. So R_V extends to a continuous morphism

$$R_V : \mathcal{G}_i(X) \to N_i(V)_{\mathbb{R}}.$$

The continuity of $R_V, V \in \mathcal{Z}(X)$ implies the following cut-and-paste relations: for every $V_1, V_2 \in \mathcal{Z}$, we have

$$R_{V_1} + R_{V_2} = R_{V_1\cup V_2} + R_{V_1\cap V_2} \tag{5.1}$$

in $N_i(V_1 \cup V_2)_{\mathbb{R}}$.

Lemma 5.1. *The following holds:*

(i) $\mathcal{G}_i(X) = \{\alpha \in \prod_{V\in\mathcal{Z}(X)} N_i(V)_{\mathbb{R}}|$ (5.1) *holds at* $\alpha\}$.

(ii) *the restriction of the projection* $\Psi : \prod_{V\in\mathcal{Z}(X)} N_i(V)_{\mathbb{R}} \to \prod_{x\in X} N_i(Z_x)_{\mathbb{R}}$ *induces an isomorphism*

$$\Psi|_{\mathcal{G}(X)} : \mathcal{G}(X) \simeq \prod_{x\in X} N_i(Z_x)_{\mathbb{R}}.$$

Proof. Set $H := \{\alpha \in \prod_{V\in\mathcal{Z}(X)} N_i(V)_{\mathbb{R}}|$ (5.1) holds at $\alpha\}$. By (5.1), we have $\mathcal{G}_i(X) \subseteq H$.

Next we show that $\Psi|_H : H \to \prod_{x\in X} N_i(Z_x)_{\mathbb{R}}$ is an isomorphism. For this, only need to construct an inverse h of $\Psi|_H$. Consider $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{>0}$ with lexicographical order $\geq$ which is a well ordered set. For every $V \in \mathcal{Z}(X)$, let s_V be the number of irreducible components of V. Define $d(V) := (\dim V, s_V)$. For $\alpha := (\alpha_x)_{x\in|X|} \in \prod_{x\in X} N_i(Z_x)_{\mathbb{R}}$, we define $h(\alpha)_V$ by induction on $d(V)$. If $s_V = 1$, let η be the generic point of V. Define $h(\alpha)_V := \alpha_\eta$. In particular, $h(\alpha)_V$ defined when $d(V) = (0, 1)$. Now assume that $s_V > 1$. Let $V_1, \dots, V_s$ by the generic points of the irreducible components of V. Note that $s \geq 2$. Set $W := V_2 \cup \dots \cup V_s$. We have

$$\max\{d(V_1 \cap W), d(V_1), d(W)\} < d(V)$$

Define

$$h(\alpha_V) := h(\alpha)_{V_1} + h(\alpha)_W - h(\alpha)_{V_1\cap W}$$

in $N_i(V)_{\mathbb{R}}$. Easy to check that $h(\prod_{x\in X} N_i(Z_x)_{\mathbb{R}}) \subseteq H$, $\Psi\circ h = \mathrm{id}$, $h\circ\Psi|_H = \mathrm{id}$ and h is continuous. This implies that $\Psi|_H : H \to \prod_{x\in X} N_i(Z_x)_{\mathbb{R}}$ is an isomorphism.

We now only need to show that $\Psi(Z_i(X)_{\mathbb{R}})$ is dense in $\prod_{x\in X} N_i(Z_x)_{\mathbb{R}}$. We only need to show that for every $x_1, \dots, x_s \in X$ and $\alpha_l \in N_i(Z_{x_l})_{\mathbb{R}}$, there is $\alpha \in Z_i(X)_{\mathbb{R}}$ such that $\Psi(\alpha)_{x_l} = \alpha_l$. For every $l = 1, \dots, s$, let $D(l)$ be the set of index j such that $x_j \in Z_{x_l}$. Set $W_l := Z_{x_l} \cap (\cup_{j\neq l} Z_{x_j})$.

Lemma 5.2. *For every* $l = 1, \dots, s$, *there is* $\beta_l \in Z_i(Z_{x_l})_{\mathbb{R}}$ *such that the image of* β_l *in* $N(Z_{x_l})_{\mathbb{R}}$ *is* $\alpha_l - \sum_{j\in D(l)} \alpha_j$ *and we have* $R_{W_l}(\beta_l) = 0$.

Easy to check that $\Psi(\sum_{l=1}^s \beta_l)_j = \alpha_j$ for every $j = 1, \dots, s$. This concludes the proof. □

Proof of Lemma 5.2. By De Jong's alteration theorem [dJ96], there is a smooth projective variety Y with a generically finite and surjective morphism $q : Y \to V_i$. Pick any $\gamma \in Z_i(Z_{x_l})_{\mathbb{R}}$ such that the image of γ in $N(Z_{x_l})_{\mathbb{R}}$ is $\alpha_l - \sum_{j\in D(l)} \alpha_j$. Pick $\gamma' \in Z_i(Y)_{\mathbb{R}}$ such that $q_*\gamma' = \gamma$. By Chow's moving lemma [Sta19, Lemma 43.24.1], there is $\gamma'' \in Z_i(Y)_{\mathbb{R}}$ which is linearly equivalent to γ' and no irreducible component of $\mathrm{Supp}\,\gamma'$ contained in $q^{-1}(W_l)$. Then $\beta_l := \gamma''$ satisfies the conditions we need. □

Let $g : Y \to X$ be a morphisms between reduced projective schemes over $\mathbf{k}$. Easy to check that the pushforward $g_* : Z_i(Y) \to Z_i(X)$ extends to a continuous morphism $g_* : \mathcal{G}_i(Y) \to \mathcal{G}_i(X)$.

Restriction map. For every $V \in \mathcal{Z}(X)$, consider the restriction map

$$\pi_V : \prod_{W \in \mathcal{Z}(X)} N_i(W)_{\mathbb{R}} \to \prod_{W' \in \mathcal{Z}(V)} N_i(W')_{\mathbb{R}}$$

sending $(\alpha_W)_{W \in \mathcal{Z}(X)}$ to $(\alpha_{W' \cap V})_{W' \in \mathcal{Z}(X)}$. It is clear that π_V is continuous. For every $Z \in Z_i(X)$, we have

$$\pi_V \circ \Phi(Z) = \Phi(\sum_{x \in X_i \cap V} m(Z, x) Z_x).$$

So $\pi_V(\mathcal{G}_i(X)) \subseteq \mathcal{G}_i(V)$. We still denote by π_V its restriction on $\mathcal{G}_i(X)$. Let $i_V : V \hookrightarrow X$ be the inclusion morphism, then we have

$$\pi_V \circ (i_V)_* = (i_V)_*.$$

In particular, $(i_V)_*$ gives a natural embedding from $\mathcal{G}_i(V)$ to $\mathcal{G}_i(X)$. We identify $\mathcal{G}_i(V)$ with its image under $(i_V)_*$ in $\mathcal{G}_i(X)$.

We now extend the above definition to any constructible subset by induction on its dimension. Let W be a constructible subset. If $\dim W = 0$, then W is closed. So π_W is defined. Now assume that $\dim W = r \geq 1$ and the restriction map is defined for every constructible set of dimension $< r$. We define

$$\pi_W := \pi_{\overline{W}} - \pi_{\overline{W} \setminus W}.$$

The two terms in the right hand side are defined as $\overline{W}$ is closed and $\dim(\overline{W} \setminus W) < r$. By induction, easy to check that π_W is continuous and for every $Z \in Z_i(X)$, we have

$$\pi_W \circ \Phi(Z) = \Phi(\sum_{x \in X_i \cap W} m(Z, x) Z_x). \tag{5.2}$$

Then for two disjoint constructible sets W_1, W_2, we get

$$\pi_{W_1 \sqcup W_2} = \pi_{W_1} + \pi_{W_2} \text{ and } \pi_{W_1} \circ \pi_{W_2} = 0. \tag{5.3}$$

Then for constructible sets W, W' with $W \subseteq W'$, we have

$$\pi_W = \pi_W \circ \pi'_W.$$

Define $\mathcal{G}_i(W, X) := \pi_W(\mathcal{G}_i(X))$. By (5.3), $\mathcal{G}_i(W, X)$ is the closure of $Z_i(W, X)_{\mathbb{R}}$ in $\mathcal{G}_i(X)$.

Support of generalized cycles. Let $Z \in \mathcal{Z}_i(X)$. For every $x \in |X|$, we say that x is **not** *in the support of* Z if there is a constructible subset W containing x such that $\pi_W(Z) = 0$. Otherwise, we say that x is in the support of Z. Define $\operatorname{Supp} Z$ to be the set of all $x \in |X|$ which is in the support of Z. As constructible subsets are open in $|X|$, $\operatorname{Supp} Z$ is closed in $|X|$. By compactness of constructible subsets, for every constructible subset $W \subseteq |X| \setminus \operatorname{Supp} Z$, $\pi_W(Z) = 0$.

For every closed subset Y of $|X|$, we say that Z is supported on Y if $\mathrm{Supp}\, Z \subseteq Y$. If Y is a constructible subset, then $\mathcal{R}_i(Y, X) = \pi_Y(\mathcal{R}_i(X))$ is exactly the subspace of $Z \in \mathcal{R}_i(X)$ supported on Y.

Let $X = \sqcup_{i=1}^s Y_i$ be a finite partition of X by constructible subsets Y_i. Then we get a direct composition

$$\mathcal{G}_i(X) = \oplus_{i=1}^s \mathcal{G}_i(Y, X).$$

Intersection numbers. For every $\alpha \in \mathcal{G}_i(X)$, $Z \in \mathcal{Z}(X)$ and $\beta \in N^i(Z)_{\mathbb{R}}$, define

$$(\alpha \cdot \beta) := (R_Z(\alpha) \cdot \beta).$$

It is clear that for every $Z \in \mathcal{Z}(X)$, the map

$$(\alpha, \beta) \in \mathcal{G}_i(X) \times N^i(Z)_{\mathbb{R}} \mapsto (\alpha \cdot \beta) \in \mathbb{R}$$

is a continuous bilinear form.

5.2. **Positive generalized cycles.** Define $\mathcal{G}_i^+(X)$ to be the closure of $\mathrm{Eff}_i(X)$ in $\mathcal{G}_i(X)$. It is a closed convex cone of $\mathcal{G}_i(X)$. We call the induced topology on $\mathcal{G}^+(X)$ the *weak topology*. We view $\mathcal{G}_i^+(X)$ as an analogy of the notion of positive closed currents of bidimension (i, i) in complex geometry. It is clear that the projection $R_X : \mathcal{G}_i(X) \to N_i(X)_{\mathbb{R}}$ maps $\mathcal{G}_i^+(X)$ onto $\mathrm{Psef}_i(X)$. Set

$$R^+ := R_X|_{\mathcal{G}_i^+(X)} : \mathcal{G}_i^+(X) \twoheadrightarrow \mathrm{Psef}_i(X).$$

Lemma 5.3. *The map $R^+ : \mathcal{G}_i^+(X) \twoheadrightarrow \mathrm{Psef}_i(X)$ is proper. In other words, for every ample line bundle L of X and $B \geq 0$,*

$$\{\alpha \in \mathcal{G}_i^+(X) |\ (\alpha \cdot L^i) \leq B\}$$

is compact. In particular, $(R^+)^{-1}(0) = 0$.

Proof. For every $V \in \mathcal{Z}(X)$, set $K_V := \{\beta \in \mathrm{Psef}_i(V) |\ (\beta \cdot L^i) \leq B\}$, which is compact. Then we have

$$\{\alpha \in \mathcal{G}_i^+(X) |\ (\alpha \cdot L^i) \leq B\} \subseteq \prod_{V \in \mathcal{Z}(X)} K_V.$$

By Tychonoff theorem, the right hand side is compact. As $\{\alpha \in \mathcal{G}_i^+(X) |\ (\alpha \cdot L^i) \leq B\}$ is close, it is compact. As

$$\mathbb{R}_{>0} \times (R^+)^{-1}(0) = (R^+)^{-1}(0)$$

and $(R^+)^{-1}(0)$ is compact, we get $(R^+)^{-1}(0) = 0$. This concludes our proof. □

For every constructible subset V of X, define $\mathcal{G}_i^+(V, X) := \mathcal{G}_i(V, X) \cap \mathcal{G}_i^+(X)$. We may check that $\pi_V(\mathrm{Eff}_i(X)) \subseteq \mathrm{Eff}_i(V, X)$. By the continuity, we get

$$\pi_V(\mathcal{G}_i^+(X)) = \mathcal{G}_i^+(V, X). \tag{5.4}$$

Indeed, $\mathcal{G}_i^+(V, X)$ is the closure of $\mathrm{Eff}_i(V, X)$ in $\mathcal{G}^+(X)$ and is exactly the space of $\alpha \in \mathcal{G}_i^+(X)$ supported in V. If V is locally closed, we may check that

$$R_{\overline{V}^{zar}}(\mathcal{G}_i^+(V, X)) = \mathrm{Psef}_i(V, X). \tag{5.5}$$

We then have the following application on the positivity of some intersection numbers.

Proposition 5.4. *Let D be an effective Cartier divisor of X. Let $\alpha \in \mathcal{G}_1^+(X)$. Assume that* $\operatorname{Supp}\alpha \cap D = \emptyset$, *then* $(\alpha \cdot D) \geq 0$.

Proof. As $\alpha \in \mathcal{G}_i^+(X \setminus D, X)$, we have $R_X(\alpha) \subseteq \mathrm{Psef}_i(X \setminus D, X)$. Hence we have $(\alpha \cdot D) \geq 0$. □

Strong topology. Define $\mathcal{DG}_i^+ := \mathcal{G}_i^+ - \mathcal{G}_i^+ = \{\alpha - \beta |\ \alpha, \beta \in \mathcal{G}_i^+\}$ which is a subspace of $\mathcal{G}_i(X)$. Let L be any ample line bundle on X. For every $\alpha \in \mathcal{DG}_i^+$, define

$$\|\alpha\|_L := \inf\{((\alpha_1 + \alpha_2) \cdot L^i)|\ \alpha_1, \alpha_2 \in \mathcal{G}_i^+, \alpha_1 - \alpha_2 = \alpha\}.$$

Easy to check that $\|\cdot\|$ is a norm. Moreover, for any different ample line bundles L_1, L_2, the norms $\|\cdot\|_{L_1}, \|\cdot\|_{L_2}$ are equivalent i.e. there is $C > 1$ such that

$$C^{-1}\|\cdot\|_{L_2} \leq \|\cdot\|_{L_1} \leq C\|\cdot\|_{L_2}.$$

For the simplicity, we now fix L and write $\|\cdot\|$ for $\|\cdot\|_{L_1}$. Easy to check that for every $\alpha \in \mathcal{G}_i^+$, we have

$$\|\alpha\| = (\alpha \cdot L^i). \tag{5.6}$$

We call the topology on $\mathcal{G}_i^+(X)$ induced by $\|\cdot\|$ the strong topology. As the map $\alpha \mapsto (\alpha \cdot L^i) = \|\alpha\|$ is continuous on $\mathcal{G}_i^+(X)$, the map

$$\mathrm{id} : (\mathcal{DG}_i^+, \|\cdot\|) \to \mathcal{G}_i^+(X)$$

is continuous. So the strong topology on $\mathcal{G}^+(X)$ is stronger than the weak topology.

The following result shows that $(\mathcal{DG}_i^+, \|\cdot\|)$ is a Banach space.

Proposition 5.5. *Let $\alpha_n \in \mathcal{G}_i^+$. The followings are equivalent:*

(i) $\sum_{n\geq 0} \|\alpha_n\| < +\infty$;
(ii) $\sum_{n\geq 0} \alpha_n$ *converges for weak topology;*
(iii) $\sum_{n\geq 0} \alpha_n$ *converges for* $\|\cdot\|$.

In particular, the norm $\|\cdot\|$ is complete on $\mathcal{DG}_i^+$.

Proof. It is clear that (ii) implies (i) and (iii) implies (i).

Assume (i), set $B := \sum_{n\geq 0} \|\alpha_n\|$. By Lemma 5.3, $K := \{\alpha \in \mathcal{G}_i^+(X)|\ (\alpha \cdot L^i) \leq B\}$ is compact. For $m \geq 0$, set $S_m := \sum_{n=0}^m \alpha_n$. The $S_m \in K$ for every $m \geq 0$. Pick β in the limit set. of $\{S_m|\ m \geq 0\}$. We first show that $\sum_{n\geq 0} \alpha_n \to \beta$ in weak topology. For every $m \geq 0$, $\beta - S_m$ is contained in the limit set of

$$\{S_{m+n} - S_m|\ n \geq 0\} \subseteq K \subseteq \mathcal{R}_i^+(X).$$

Then we have

$$\beta - S_m \in K$$

for every $m \geq 0$. For every $V \in \mathcal{Z}(X)$, we have $R_V(\beta - S_m) \subseteq \mathrm{Psef}_i(V)$. As

$$0 \leq ((\beta - S_m) \cdot L^i) \leq \sum_{n\geq m} \|\alpha_n\| \to 0,$$

we get $R_V(\beta - S_m) \to 0$ as $m \to \infty$. So we have $\sum_{n\geq 0} \alpha_n \to \beta$ as $n \to \infty$. Then we get that (i) implies (iii). As $\beta - S_m \in K$, $\|\beta - S_m\| = ((\beta - S_m) \cdot L^i) \to 0$. So $\sum_{n\geq 0} \alpha_n \to \beta$ in $(\mathcal{DG}_i^+, \|\cdot\|)$. So (i) implies (iii). This concludes the proof. □

The following example shows that the stronger topology is strictly stronger than the weak topology in general.

Example 5.6. Let $X=\mathbb{P}^2_{\mathbb{C}}$. Let $Z_n, n\geq 0$ be the line defined by $y-x-nz=0$. Let $L:=\mathcal{O}_{\mathbb{P}^2_{\mathbb{C}}}(1)$. It is clear that Z_n converges weakly. But for every distinct $n,m\geq 0$, $\|Z_n-Z_m\|=1$. So there is no convergence subsequence for the strong topology.

However, as the morphism $\alpha\mapsto(\alpha\cdot L^i)=\|\alpha\|$ is continuous on $\mathcal{G}_i^+(X)$, a sequence $\alpha_n\in\mathcal{G}_i^+(X)$ tends to 0 weakly if and only if it tends to 0 strongly.

Induced measure. Denote by $\mathrm{P}^i(X)$ the dual cone of $\mathrm{Psef}_i(X)$ in $N^i(X)_{\mathbb{R}}$. As $\mathrm{Psef}_i(X)$ has non-empty interior and is salient, $\mathcal{P}^i(X)$ has non-empty interior and is salient. We have the following morphism

$$\nu:\mathrm{P}^i(X)\times\mathcal{G}_i^+(X)\to\mathcal{M}(|X|)$$

as follows: For every $\beta\in\mathrm{P}^i(X)$, $\alpha\in\mathcal{G}_i^+(X)$ and every constructible set $V\in\mathcal{A}(|X|)$, define

$$\nu(\beta,\alpha)(V):=(\pi_V(\alpha)\cdot\beta)\geq 0.$$

We define $\nu(\beta,\alpha)$ to be the unique measure in $\mathcal{M}(|X|)$ make the above equality holds for every $V\in\mathcal{A}(|X|)$.

Such measure exists and is unique. Indeed by Carathéodory's extension theorem, we only need to show the following: Let $V_n, n\geq 0$ be a sequence of disjoint constructible sets. Let $V\in\mathcal{A}(|X|)$ satisfying $V=\sqcup_{n\geq 0}V_n$. Then

$$\nu(\beta,\alpha)(V)=\sum_{n\geq 0}\nu(\beta,\alpha)(V_n).$$

We note that constructible sets are open and closed. The compactness of V implies that $V_n=\emptyset$ for $n\gg 0$. By (5.3), we get $\nu(\beta,\alpha)(V)=\sum_{n\geq 0}\nu(\beta,\alpha)(V_n)$.

Lemma 5.7. *The map*

$$\nu:\mathrm{P}^i(X)\times\mathcal{G}_i^+(X)\to\mathcal{M}(|X|)$$

is continuous. Here $\mathcal{M}(|X|)$ is endowed with the weak- topology and $\mathcal{G}_i^+(X)$ is endowed with the weak topology.*

Proof. For the continuity, we only need to show that for every continuous function h on $|X|$, ν^*h is continuous on $\mathrm{P}^i(X)\times\mathcal{G}_i^+(X)$. By Lemma 4.1, we may assume that h takes form 1_U where $U\in\mathcal{A}(|X|)$. Then the function μ^*h sends (β,α) to $(\beta\cdot\pi_V(\alpha))$ which is continuous. □

Then every $\alpha\in\mathcal{G}_i^+(X)$ defines a unique vector-valued measure ν_α^X such that for every $\beta\in\mathrm{P}^i(X)$,

$$\nu(\beta\cdot\alpha)=(\beta\cdot\nu_\alpha^X).$$

Easy to check that $\mathrm{Supp}\,\alpha=\mathrm{Supp}\,\nu_\alpha^X$.

Atomic decomposition. For $x \in |X|$, define

$$\mathrm{Psef}_i(x, X) := \cap_U \mathrm{Psef}^i(U, X)$$

in $N_i(Z_x)_{\mathbb{R}}$, where U is taken over all non-empty Zariski open subsets of Z_x. If $\dim Z_x \leq i-1$, it is clear that $\mathrm{Psef}_i(x, X) = \{0\}$.

Lemma 5.8. *For $x \in |X|$ with $\dim Z_x = l \geq i$, then $\mathrm{Psef}_i(x, X)$ is a closed, convex, salient cone with non-empty interior.*

Proof. Pick L an ample line bundle on X. For every non-empty Zariski open subset U of Z_x, set $K_U := \{v \in \mathrm{Psef}^i(U, X) | \ (v \cdot L^i) \leq 1\}$. These K_U are compact, as $\mathrm{Psef}^i(U, X) \subseteq \mathrm{Psef}^i(Z_x)$ and K_{Z_x} is compact. Hence $\{v \in \mathrm{Psef}^i(x, X) | \ (v \cdot L^i) \leq 1\} = \cup_U K_U$ is compact. Hence $\mathrm{Psef}_i(x, X)$ is closed. By De Jong's alteration theorem [dJ96], there is a smooth projective variety Y with a generically finite and surjective morphism $q := Y \twoheadrightarrow Z_x$. Let $\mathrm{BPF}^{l-i}(Y)$ be the cone in $N^{l-i}(Y)_{\mathbb{R}} = N_i(Y)_{\mathbb{R}}$ as in [Dan20, Definition 3.3.1]. It definition shows that $\mathrm{BPF}^{l-i}(Y) \subseteq \mathrm{Psef}_i(\eta, Y)$ where η is the generic point of Y. Hence we have $q_*(\mathrm{BPF}^{l-i}(Y)) \subseteq \mathrm{Psef}_i(x, X)$. By [Dan20, Theorem 3.3.3 (1)], $\mathrm{BPF}^{l-i}(Y)$ has non-empty interior. As $q_* : N_i(Y)_{\mathbb{R}} \to N_i(X)_{\mathbb{R}}$ is surjective, $q_*(\mathrm{BPF}^{l-i}(Y))$ (hence $\mathrm{Psef}_i(x, X)$) has non-empty interior. As $\mathrm{Psef}_i(x, X) \subseteq \mathrm{Psef}_i(X)$ and $\mathrm{Psef}_i(X)$ is salient, $\mathrm{Psef}_i(x, X)$ is salient. □

For every $v \in \mathrm{Psef}_i(x, X)$, we define a positive generalized cycle $v\delta_x$ of X as the element $(\alpha_Z)_{Z \in \mathcal{Z}(X)} \in \prod_{Z \in \mathcal{Z}(X)} N_i(Z)_{\mathbb{R}}$ such that for every $Z \in \mathcal{Z}(X)$, $\alpha_Z = 0$ if $x \notin Z$ and $\alpha_Z = v$ if $x \in Z$. We now check that $v\delta_x$ is contained in $\mathcal{G}_i^+(X)$. Let $Z_1, \dots, Z_m \in \mathcal{Z}(X)$ with $x \in \cup_{j=1}^m Z_j$. Let J be the set of j such that $x \notin Z_j$. Set $U := Z_x \setminus (\cup_{j \in J} Z_j)$ which is a non-empty Zariski open subset of Z_x. There are effective i-cycles W_n taking form $W_n = \sum_{s=1}^l a_{n,s} Z_{w_{n,s}}$ such that $a_{n,s} > 0$, $w_{n,s} \in U$ and $[W_n] \to v$ in $N_i(Z_x)_{\mathbb{R}}$. Then the image of $v\delta_x$ in $\prod_{j=1}^m N_i(Z_j)_{\mathbb{R}}$ can be approximated by the images of $W_n, n \geq 0$. So $v\delta_x \in \prod_{Z \in \mathcal{Z}(X)} N_i(Z)_{\mathbb{R}}$ is contained in the closure of effective i-cycles of X. Hence we get $v\delta_x \in \mathcal{G}_i^+(X)$. we call such $v\delta_x$ the *atoms* in $\mathcal{G}_i^+(X)$

The following result shows that every positive generalized cycle is a positive combination of at most countably many atoms.

Theorem 5.9. *Every $\alpha \in \mathcal{G}_i^+(X)$ takes form*

$$\alpha = \sum_{j \geq 0} v_j \delta_{x_j}$$

where $v_j \in \mathrm{Psef}_i(x_j, X)$ with $\sum_{j \geq 0} (v_j \cdot L^i) < +\infty$ where L is any ample line bundle on X.

Combing Theorem 5.9 with Corollary 4.4, we indeed showed that $\mathcal{G}_0^+(X) = \mathcal{M}(|X|)$.

Proof. By Theorem 4.3 and Corollary 4.4, write $\nu(L^i, \alpha) = \sum_{j \geq 0} a_j \delta_{x_j}$. Set $Z_j := Z_{x_j}$. For each j, $\alpha_j := \pi_{Z_j} \alpha \in \mathcal{P}_i^+(Z_j)$ defines a vector-valued measure $\nu_{\alpha_j}^{Z_j} \in$

$\mathcal{M}(|Z_j|, \mathrm{Psef}_i(Z_j))$. We may write $\nu^{Z_j}_{\alpha_j} = v_j\delta_{x_j} + \beta_j$ where $\epsilon_j \in \mathcal{M}(|Z_j|, \mathrm{Psef}_i(Z_j))$ with $\beta_j(\{x_j\}) = 0$. We have $(v_j \cdot L^i) = a_j$.

For every $\epsilon > 0$, there is a non-empty Zariski open subset U_ϵ of Z_j such that for every non-empty Zariski open subset W of U_ϵ, we have

$$(L^i \cdot (\pi_W(\alpha) - v_j)) = \int_W (L^i \cdot \beta) < \epsilon.$$

As $R_{Z_j}(\pi_W(\alpha)) \subseteq N_i(W, X)$, we get $v_j \in \mathrm{Psef}^+_i(x_j, X)$.

The above construction shows that for every $Z \in \mathcal{Z}(X)$,

$$\nu^Z_{\pi_Z(\alpha)} = \nu^Z_{\pi_Z(\sum_{j\geq 0} v_j\delta_{x_j})}.$$

Then we have

$$R_Z(\alpha) = \nu^Z_{\pi_Z(\alpha)}(Z) = R_Z(\sum_{j\geq 0} v_j\delta_{x_j}).$$

This concludes the proof. □

Corollary 5.10. *Let L be an ample line bundle on X. Then $K := \{\alpha \in \mathcal{G}_i(X)|\ (L^i \cdot \alpha) \leq 1\}$ is compact and sequentially compact.*

Proof. By Lemma 5.3, K is compact. We now show that K is sequentially compact. Let $\alpha_n \in K, n \geq 0$. Then we have $\nu(L^i, \alpha_n) \in \mathcal{M}^1(|X|)$. By Corollary 4.4, up to taking subsequences, we may assume that $\nu(L^i, \alpha_n)$ converges to a measure

$$\nu = \sum_{j\geq 0} a_j\delta_{x_j} \in \mathcal{M}^1(|X|).$$

We may assume the above x_j are distinct. Set $Z_j := Z_{x_j}$. By diagonal method, after taking subsequences, we may assume that $\nu^{Z_j}_{\pi_{Z_j}\alpha_n}, n \geq 0$ converges.

We want to show that α_n converges. For this, we only need to show that for every $Z \in \mathcal{Z}(X)$, $R_Z(\alpha_n)$ converges. For every $\epsilon > 0$, there is $M \geq 0$ such that $\sum_{j\geq M} a_j < \epsilon/2$. For every $j = 1, \ldots, M$, pick a non-empty Zariski open subset U_j of Z_j such that for every $s \neq j$ with $x_j \notin Z_s$, we have $U_j \cap Z_s = \emptyset$. Then $U_1, \ldots, U_M$ are disjoint constructible subsets. After shrinking U_j, we may assume that $U_j \cap Z = \emptyset$ if and only if $x_j \notin Z$. Set $U_0 := X \setminus (\cup_{j=1}^M U_j)$. We have

$$R_Z(\alpha_n) = \sum_{j=1}^{M} \nu^Z_{\pi_{Z_j}(\alpha_n)}(U_j) + \nu^Z_{\alpha_n}(U_0).$$

For every $j = 1, \ldots, M$, $\nu^Z_{\pi_{Z_j}(\alpha_n)}$ converges, so there is $N \geq 0$ such that for $n \geq m \geq N$, we have $\|\nu^Z_{\pi_{Z_j}(\alpha_n)}(U_j) - \nu^Z_{\pi_{Z_j}(\alpha_m)}(U_j)\|_L \leq \epsilon/2M$, where $\|\cdot\|_L$ is the norm on $N_i(Z)_{\mathbb{R}}$ induced by $L^i \in \mathrm{P}^i(Z)^\circ$. Moreover $\|\nu^Z_{\alpha_n}(U_0)\|_L \leq \sum_{j\geq M} a_j \leq \epsilon/2$. So we get

$$\|R_Z(\alpha_n) - R_Z(\alpha_m)\|_L \leq \epsilon$$

for $n \geq m \geq N$. This concludes the proof. □

5.3. **A dynamical application.** Let $f : X \to X$ be a surjective endomorphism. Let L be an ample line bundle on X.

Proposition 5.11. *Let $x \in X$ and $v \in \mathrm{Psef}_1(x, X)$. Set $\alpha := v\delta_x$. Let D be any effective Cartier divisor on X. Assume that the orbit $O_f(x)$ is Zariski dense in X. Then we have*

$$\liminf_{n\to\infty} \frac{(f_*^n(\alpha)\cdot D)}{(L\cdot f_*^n(\alpha))} \geq 0. \tag{5.7}$$

Moreover, if M is a big line bundle, then there is $\delta > 0$, such that

$$(f_*^n(\alpha)\cdot M) > \delta(f_*^n(\alpha)\cdot L)$$

for $n \gg 0$.

Proof. There is $\beta > 0$ and $s \geq 0$ such that

$$(f_*^n(\alpha)\cdot D) = (\sum_{i=1}^{m} c_i e^{i\theta_i n})\beta^n n^s + O(\beta^n n^{s-1})$$

where $c_i \neq 0$ and $\theta_i \in \mathbb{R}/2\pi\mathbb{Z}$ are distinct. It is clear that

$$\beta^n n^s \leq C(f_*^n(\alpha)\cdot L) \tag{5.8}$$

for some $C > 0$.

Define a continuous function $h : (\mathbb{R}/2\pi\mathbb{Z})^m \to \mathbb{R}$ by

$$(\phi_1,\ldots,\phi_m) \mapsto \sum_{i=1}^{m} c_i e^{i\phi_i}.$$

Define $\Theta : \mathbb{Z} \to (\mathbb{R}/2\pi\mathbb{Z})^m$ by

$$n \mapsto (\theta_1 n,\ldots,\theta_m n).$$

Let Z the closure of $\Theta(\mathbb{Z})$. By Poincaré recurrence theorem, there is a strictly increasing sequence $n_i \in \mathbb{Z}_{\geq 0}, i \geq 0$ such that

$$\Theta(n_i) \to 0.$$

Then for every $m > 0$,

$$\Theta(m) = \lim_{i\to\infty} \Theta(m - n_i)$$

and

$$\Theta(-m) = \lim_{i\to\infty} \Theta(-m + n_i).$$

So Z is also the closure of $\Theta(\mathbb{Z}_{\geq 0})$ and of $\Theta(\mathbb{Z}_{\leq 0})$.

Lemma 5.12. *For every non-empty open subset U of Z, there is $r_U \geq 1$ such that for every $n \geq 0$, there is $m \in \{n, n+1, \ldots, n + r_U\}$ such that $\Phi(m) \in U$.*

We claim that $h \geq 0$ on Z. Otherwise, there is $b < 0$ such that

$$U := \{z \in Z |\ h(z) < b\} \neq \emptyset.$$

Set

$$W := \{n \geq 0 |\ \Phi(n) \in U\}$$

and

$$w(n) := \#(\{0, \dots, n-1\} \cap W).$$

By Lemma 5.12, we have

$$\liminf_{n\to\infty} w(n)/n > 0. \tag{5.9}$$

As

$$(f_*^n(\alpha) \cdot D) = h(\Phi(n))\beta^n n^s + O(\beta^n n^{s-1}), \tag{5.10}$$

for $n \in W$, we have

$$(f_*^n(\alpha) \cdot D) < 0.$$

Then, by Proposition 5.4, we get $f^n(x) \in \operatorname{Supp} D$ for every $n \in W$. As $O_f(x)$ is Zariski dense in X, by the Weak dynamical Mordell-Lang [Xie23, Theorem 1.17] (see also [Fav00, Theorem 2.5.8]), we have

$$\lim \#\{m = 0, \dots, n-1 |\ f^n(x) \in \operatorname{Supp} D\}/n = 0.$$

It follows that

$$\lim_{n\to\infty} w_n/n = 0,$$

which contradicts to (5.9). This proves the claim.

By (5.10), we get

$$(f_*^n(\alpha) \cdot D) \geq -C'\beta^n n^{s-1}$$

for some $C' > 0$. We conclude the proof of (5.7) by (5.8).

As M is big, we may write $M = \delta' L + E$ where $\delta' > 0$ and E is effective. By (5.7), for $n \gg 0$, we have

$$(f_*^n(\alpha) \cdot E) > -\frac{\delta'}{2}(f_*^n(\alpha) \cdot L)$$

Hence we have

$$(f_*^n(\alpha) \cdot M) > \delta'(f_*^n(\alpha) \cdot L) - \frac{\delta'}{2}(f_*^n(\alpha) \cdot L) = \frac{\delta'}{2}(f_*^n(\alpha) \cdot L).$$

This concludes the proof. □

Proof of Lemma 5.12. There is $r \geq 0$ such that $\Phi(r) \in U$. We only need to Lemma 5.12 for the open subset $U - \Phi(r)$. After replacing U by $U - \Phi(r)$, we may assume that $0 \in U$. After replacing U by $U \cap (-U)$, we may assume that $U = -U$. Pick an open neighborhood V of 0 such that $V = -V$ and $V + V \subseteq U$. As Z is compact, there is a finite subset F of Z such that $F + V = Z$. For every $z \in F$, there is $r_z \geq 0$ such that $\Phi(-r_z) \in z + V$. In other words, we have

$$z + \Phi(r_z) \in -V = V.$$

For every $n \geq 0$, there is $z_n \in F$ such that $\Phi(n) \in z_n + V$. Then we have

$$\Phi(n + r_{z_n}) \in z_n + V + \Phi(r_{z_n}) \in V + V \subseteq U.$$

Set $r_U := \max\{r_z |\ z \in F\}$. We conclude the proof. □

6. The spectrum for the ample cone

Let X be a projective variety over $\mathbf{k}$ of dimension d and $f : X \to X$ is a surjective endomorphism. Let V be an irreducible and periodic subvariety of X of dimension $d_V \geq 0$. Recall that

$$\mu_i(V, f) := \mu_i(f|_V^{r_V} : V \to V)^{1/r_V}$$

where $r_V \geq 1$ is a period of V. It does not depend on the choice of r_V.

The aim of this section is to prove the following result.

Theorem 6.1. *(=Theorem 1.5) We have*

$$\mathrm{Sp}\,(f^*, \mathrm{Amp}(X)) = \cup_V \{\mu_i(V, f)|\ i = 1, \ldots, d_V\}$$

where the union taken over all irreducible periodic subvarieties.

In other words, for $\alpha \in \mathbb{R}_{>0}$, f is α-amplified if and only if for every periodic irreducible subvariety V, $f^{r_V}|_V$ is α-amplified, where $r_V \geq 1$ is a period of V.

6.1. **Growth rate.** Let $\|\cdot\|$ be any norm on $N_1(X)_{\mathbb{R}}$. For every $v \in N_1(X)_{\mathbb{R}} \setminus \{0\}$, by (2.1) in Section 2, $\beta_f(v) := \lim_{n\to\infty} \|f_*^n(v)\|^{1/n}$ converges and does not depend on the choice of $\|\cdot\|$. We call it the *growth rate* of v. By (2.2) in Section 2, $\beta_f(v) \in \{|c||\ c \in \mathrm{Sp}\,(f)\}$. So $\beta_f(v)$ has only finitely many possible values. Moreover $\beta_f(\alpha) \leq \lambda_1(f) = \mu_1(f)$.

The aim of this section is to study $\beta_f(v)$ for $v \in \mathrm{Psef}_1(X) \setminus \{0\}$. Let L be any ample line bundle on X. By Remark 4.5, L induces a norm $\|\cdot\|_L$ on $N^1(X)_{\mathbb{R}}$ such that for every $v \in \mathrm{Psef}_1(X)$, $\|v\|_L = (v \cdot L)$. Using this norm, for $v \in \mathrm{Psef}_1(X)$, we get

$$\beta_f(v) = \lim_{n\to\infty} (f_*^n(v) \cdot L)^{1/n}.$$

Lemma 6.2. *Let $v_i \in \mathrm{Psef}_1(X), i \geq 0$ with $\sum_{i\geq 0} \|v_i\| < +\infty$. Set $v := \sum_{i\geq 0} v_i$. Assume that $v \neq 0$, then we have*

$$\beta_f(v) = \max\{\beta_f(v_i)|\ v_i \neq 0\}.$$

Proof. For every $i \geq 0$ with $v_i \neq 0$, we have

$$(f_*^n(v) \cdot L) \geq (f_*^n(v_i) \cdot L).$$

Hence $\rho_f(v) \geq \rho_f(v_i)$. As $\rho_f(v_i)$ has only finitely many possible values, we get

$$\beta_f(v) \geq \max\{\beta_f(v_i)|\ v_i \neq 0\}.$$

Set $\beta := \max\{\beta_f(v_i)|\ v_i \neq 0\}$. By (2.2) in Section 2, we have $v_i \in E_{\overline{\mathbb{D}}(\beta)}$ for every $i \geq 0$. As $E_{\overline{\mathbb{D}}(\beta)}$ is closed, we have $v \in E_{\overline{\mathbb{D}}(\beta)}$. Hence $\rho_f(v) \leq \beta$. This concludes the proof. □

Every $v \in \mathrm{Psef}_1(X)$ can be presented by an positive generalized cycle $\alpha \in \mathcal{G}_1^+(X)$. For the simplicity, we also write $\rho_f(\alpha)$ for $\rho_f(v)$. By Theorem 5.9, α is a positive combination of at most countably many atoms. By Lemma 6.2, we only need to understand the growth rate of atoms in $\mathcal{G}_1^+(X)$.

Proposition 6.3. *Let $x \in X$ and $v \in \mathrm{Psef}_1(x, X)$. Set $\alpha := v\delta_x$. Assume that the orbit $O_f(x)$ is Zariski dense in X. Then we have*

$$\beta_f(\alpha) \in \{\mu_i(f), i = 1, \ldots, d\}.$$

Proof. Assume by contradiction that $\beta_f(\alpha) \notin \{\mu_i(f), i = 1, \ldots, d\}$. Set $\mu_{d+1} = 0$. There is a unique $i = 1, \ldots, d$ such that $\beta_f(\alpha) \in (\mu_i, \mu_{i+1})$.

For every $n \geq 0$, define $L_n := (f^n)^*L$ and $\alpha_n := (f^n)_*\alpha$. By projection formula, we have

$$(L_{n_1} \cdot \alpha_{n_2}) = (L \cdot \alpha_{n_1+n_2}).$$

Pick $\epsilon \in (0, 1)$ such that

$$\epsilon^{-1}\mu_{i+1} < \beta < \epsilon^2\mu_i.$$

There is $m_0 \geq 1$ such that for every $m \geq m_0$,

$$\epsilon^{2m}\mu_i^m + \epsilon^{-m}\mu_{i+1}^m < \epsilon^m\mu_i^m.$$

By [Xie24, Theorem 3.7], there is $m \geq 1$ such that

$$M := L_{2m} + \mu_i^m\mu_{i+1}^m L - \epsilon^m\mu_i^m L_m$$

is big. By Proposition 5.11, there is $N \geq 0$ such that for every $n \geq N$,

$$(\alpha_{mn} \cdot M) \geq 0.$$

Then we get

$$(\alpha_{(n+2)m} \cdot L) + \mu_i^m\mu_{i+1}^m(\alpha_{nm} \cdot L) - \epsilon^m\mu_i^m(\alpha_{(n+1)m} \cdot L) \geq 0$$

for $n \geq N$. It follows that

$$(\alpha_{(n+2)m}\cdot L) - \epsilon^{-m}\mu_{i+1}^m(\alpha_{(n+1)m}\cdot L) \geq \epsilon^{2m}\mu_i^m((\alpha_{(n+1)m}\cdot L) - \epsilon^{-m}\mu_{i+1}^m(\alpha_{nm}\cdot L)) \tag{6.1}$$

for all $n \geq N$. As $\beta_f(\alpha) = \beta > \epsilon^{-1}\mu_{i+1}$, we have

$$(\alpha_{(n+1)m} \cdot L) - \epsilon^{-m}\mu_{i+1}^m(\alpha_{nm} \cdot L) > 0$$

for $n \gg 0$. Then we get

$$\beta^m = \liminf_{n\to\infty}(\alpha_{nm} \cdot L)^{1/n} \geq \epsilon^{-m}\mu_{i+1}^m,$$

which is a contradiction. □

Apply Proposition 6.3 for every periodic irreducible subvarieties, we get the following result for any atom.

Corollary 6.4. *Let $x \in X$ and $v \in \mathrm{Psef}_1(x, X)$. Set $\alpha := v\delta_x$. Then every irreducible component V of $\cap_{m\geq 0}\overline{O_f(f^m(x))}^{zar}$ is f-periodic. Then we have*

$$\beta_f(\alpha) \in \{\mu_i(V, f), i = 1, \ldots, d\}.$$

Combine Lemma 6.2, Corollary 6.4 and Theorem 5.9, we get the following result.

Theorem 6.5. *For every $v \in \mathrm{Psef}_1(X) \setminus \{0\}$, we have*

$$\beta_f(v) \in \cup_V\{\mu_i(V, f)|\ i = 1, \ldots, d_V\}$$

where the union taken over all irreducible periodic subvarieties.

6.2. **The spectrum for the ample cone.**

Proof of Theorem 6.1. For $\alpha \in \mathbb{R}_{>0}$, if f is α-amplified, then for every periodic irreducible subvariety V with period r_V, $f^{r_V}|_V$ is α^{r_V}-amplified. Hence we get

$$\cup_V \mathrm{Sp}\left((f^{r_V}|_V)^*, \mathrm{Amp}(V)\right)^{1/r_V} \subseteq \mathrm{Sp}\left(f^*, \mathrm{Amp}(X)\right),$$

here $\mathrm{Sp}\left((f^{r_V}|_V)^*, \mathrm{Amp}(V)\right)^{1/r_V} := \{\beta^{1/r_V} |\ \beta \in \mathrm{Sp}\left(f^{r_V}|_V, \mathrm{Amp}(V)\right)\}$. As the big cone contains the ample cone, we have

$$\{\mu_i(V, f)|\ i = 1, \dots, d_V\} \subseteq \mathrm{Sp}\left((f^{r_V}|_V)^*, \mathrm{Amp}(V)\right)^{1/r_V}.$$

So we get

$$\cup_V \{\mu_i(V, f)|\ i = 1, \dots, d_V\} \subseteq \mathrm{Sp}\left(f^*, \mathrm{Amp}(X)\right).$$

Set $S := \cup_V \{\mu_i(V, f)|\ i = 1, \dots, d_V\}$. By contradiction, assume that

$$\mathrm{Sp}\left(f^*, \mathrm{Amp}(X)\right) \not\subseteq S.$$

By Theorem 2.10, $E_S \cap \mathrm{Amp}(X) = \emptyset$. By Hahn-Banach theorem, there is $Z \in \mathrm{Psef}_1(X) \setminus \{0\}$ such that $E_S \subseteq Z^{\perp}$. In particular, we have $\beta_f(Z) \notin S$. This contradicts to Theorem 6.5. □

Corollary 6.6. *Let Y be a projective variety over $\mathbf{k}$ and $g : Y \to Y$ be an endomorphism. Let $\pi : X \to Y$ be a surjective morphism such that $\pi \circ f = g \circ \pi$. If f is α-amplified for some $\alpha \in \mathbb{R}_{>0}$, then g is α-amplified.*

Proof. Assume that f is α-amplified. By contradiction, assume that g is not α-amplified. By Theorem 6.1, there is an irreducible g-periodic subvariety V of Y such that $\alpha \in \{\mu_i(V, g)|\ i = 1, \dots, d_V\}$. After replacing f, g by a suitable iterate, we may assume that $g(V) = V$. There is an irreducible component W of $\pi^{-1}(V)$ which is f-periodic and satisfies $\pi(W) = V$. After replacing f, g by a suitable iterate, we may assume that $f(W) = W$. The product formula for relative dynamical degrees (c.f. [DN11], [Dan20] and [Tru20, Theorem 1.3]) shows that $\{\mu_i(V, g)|\ i = 1, \dots, d_V\} \subseteq \{\mu_i(W, f)|\ i = 1, \dots, d_W\}$. By Theorem 6.1, we have

$$\alpha \in \{\mu_i(W, f)|\ i = 1, \dots, d_W\} \subseteq \mathrm{Sp}\left(f^*, \mathrm{Amp}(X)\right).$$

Then f is not α-amplified, which is a contradiction. □

6.3. **The answer of Krieger-Reschke's question.** In this section $\mathbf{k}$ is algebraically closed field and X is a normal projective variety over $\mathbf{k}$. Let f be a surjective endomorphism of X. The aim of this section is to prove Corollary 1.8.

We first recall the definition of Albanese variety. See [FGI$^+$05, Remark 9.5.25], [Lan83, Chapter II.3] and [CMZ20, Section 5] for detial. The Albanese variety of X is the abelian variety

$$\mathrm{Alb}(X) := \mathrm{Pic}^0(\mathrm{Pic}^0(X)_{\mathrm{red}}).$$

There is a canonical morphism $\mathrm{alb}_X : X \to \mathrm{Alb}(X)$ called the *Albanese morphism* satisfying the following universal property: for every morphism

$$\phi : X \to A$$

from X to an abelian variety A, there exists a unique morphism

$$\psi : \mathrm{Alb}(X) \to A$$

such that $\phi = \psi \circ \mathrm{alb}_X$. The universal property shows that alb_X is unique up to composing an automorphism of $\mathrm{Alb}(X)$ and $\mathrm{alb}_X(X)$ is not contained in any translation of proper abelian subvariety of $\mathrm{Alb}(X)$. Denote by

$$f_{\mathrm{Alb}(X)} : \mathrm{Alb}(X) \to \mathrm{Alb}(X)$$

on $\mathrm{Alb}(X)$ the endomorphism induced by $f : X \to X$ i.e. the unique endomorphism on $\mathrm{Alb}(X)$ such that

$$\mathrm{alb}_X \circ f = f_{\mathrm{Alb}(X)} \circ \mathrm{alb}_X.$$

We note that for different choices of alb_X, the induced $f_{\mathrm{Alb}(X)}$ are conjugate.

Corollary 6.7 (=Corollary 1.8)**.** *If $f : X \to X$ is quasi-amplified. Then $f_{\mathrm{Alb}(X)}$ is amplified.*

Proof. For every $m \geq 1$, define $f^{\times 2m} : X^{2m} \to X^{2m}$ by

$$f^{\times 2m} : (x_1, \dots, x_m, y_1, \dots, y_m) \mapsto (f(x_1), \dots, f(x_m), f(y_1), \dots, f(y_m)).$$

As f is quasi-amplified, $f^{\times 2m}$ is also quasi-amplified.

Define a morphism $s_m : X^{2m} \to \mathrm{Alb}(X)$ by

$$s_m : (x_1, \dots, x_m, y_1, \dots, y_m) \mapsto \sum_{i=1}^{m} (\mathrm{alb}_X(x_i) - \mathrm{alb}_X(y_i)).$$

Define $F : \mathrm{Alb}(X) \to \mathrm{Alb}(X)$ by

$$y \mapsto f_{\mathrm{Alb}(X)}(y) - f_{\mathrm{Alb}(X)}(0).$$

It is clear that

$$F \circ s_m = s_m \circ f^{\times 2m}.$$

So we only need to show that F is amplified.

We note that $0 \in s_m(X^{2m})$ for every $m \geq 0$. As $s_m(X^{2m})$ is an increasing sequence of irreducible Zariski closed subsets of $\mathrm{Alb}(X)$, there is $l \geq 1$ such that $s_m(X^{2m}) = s_l(X^{2l})$ for every $m \geq l$. Then we have

$$s_l(X^{2l}) + s_l(X^{2l}) \subseteq s_{2l}(X^{4l}) = s_l(X^{2l}).$$

So $s_l(X^{2l})$ is a abelian subvariety of $\mathrm{Alb}(X)$. As $s_l(X^{2l})$ contains a translation of $\mathrm{alb}_X(X)$, we get $s_l(X^{2l}) = \mathrm{Alb}(X)$ i.e. s_l is surjective. By Corollary 1.6, F is quasi-amplified. On abelian varieties, the big cone and the ample cone are the same. Then F is amplified. As the action of F and $f_{\mathrm{Alb}(X)}$ on $N^1(\mathrm{Alb}(X))$ are the same, $f_{\mathrm{Alb}(X)}$ is amplified. □

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