DYNAMICAL MORDELL–LANG CONJECTURE FOR SPLIT SELF-MAPS OF AFFINE CURVE TIMES PROJECTIVE CURVE

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Abstract. We prove the dynamical Mordell–Lang conjecture for product of endomorphisms of an affine curve and a projective curve over $\overline{\mathbb{Q}}$.

1. Introduction

The dynamical Mordell–Lang conjecture is one of the core problems in the field of arithmetic dynamics. It was proposed by Ghioca and Tucker in [GT09] and can be stated as follows:

Dynamical Mordell–Lang Conjecture (DML Conjecture). *Let $f : X \to X$ be an endomorphism of a quasi-projective variety over a field K of characteristic 0, and V be a closed subvariety of X. Then for every $x \in X(K)$, the return set $\{n \in \mathbb{N} |\ f^n(x) \in V(K)\}$ is a finite union of arithmetic progressions.*

There is an extensive literature on various cases of the DML conjecture. Two significant cases are as follows:

(1) If X is a quasi-projective variety over $\mathbb{C}$, and f is an étale endomorphism of X, then the DML conjecture holds for (X, f). See [Bel06] and [BGT10, Theorem 1.3].
(2) If $X = \mathbb{A}^2_{\mathbb{C}}$, and f is an endomorphism of X, then the DML conjecture holds for (X, f). See [Xie17] and [Xie, Theorem 3.2].

One can consult [BGT16, Xie] and the references therein for further known results.

In this article, we investigate the DML conjecture for certain types of split endomorphisms.

Theorem 1.1. *Let X be an affine curve and Y be a projective curve over $\overline{\mathbb{Q}}$. Let $f : X \to X$ and $g : Y \to Y$ be endomorphisms. Then DML conjecture holds for $(X \times Y, f \times g)$.*

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It turns out that major case is about $\mathbb{A}^1 \times \mathbb{P}^1$. Hence we state this as a proposition and will mainly deal with it later.

Proposition 1.2. *Let $f : \mathbb{A}^1 \to \mathbb{A}^1$ and $g : \mathbb{P}^1 \to \mathbb{P}^1$ be endomorphisms over $\overline{\mathbb{Q}}$. Then DML conjecture holds for $(\mathbb{A}^1 \times \mathbb{P}^1, f \times g)$.*

In the proof of Proposition 1.2, we first apply the results in [BGT10, Xie17] and [XY] to make some reductions and further assumptions about the endomorphisms. Namely, we may assume that $\deg(f) = \deg(g) > 1$ and no iterate of g can conjugate to a polynomial. Then the key observation is that for an appropriate place, the $\mathbb{A}^1$ coordinate of a non-preperiodic orbit tends to infinity with the maximal speed, while the $\mathbb{P}^1$ coordinate of any subsequence of that orbit cannot tend to any point with such a speed. This forces that orbit to have a finite intersection with every (non-horizontal and non-vertical) curve.

We can deduce from Theorem 1.1 that the DML conjecture holds in more general settings.

We recall a concept following [Xie, Definition 1.3]. For a quasi-projective variety X over a field K and an endomorphism f, we say (X, f) satisfies the *DML(1) property*, if for any curve $C \subseteq X$ and any point $x \in X(K)$, the return set $\{n \in \mathbb{N} | \ f^n(x) \in C(K)\}$ is a finite union of arithmetic progressions. Here "1" stands for the dimension of the closed subvariety.

Corollary 1.3. *Let X be an affine variety and Y be a projective variety over $\overline{\mathbb{Q}}$. Let $f : X \to X$ and $g : Y \to Y$ be dominant endomorphisms. Assume that (X, f) and (Y, g) satisfy the DML(1) property. Then $(X \times Y, f \times g)$ satisfies the DML(1) property.*

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2. PROOF OF THE MAIN RESULTS

Firstly, we recall a lemma which plays a key role in the proof. It guarantees that for a rational function which has no exceptional points, any subsequence of a non-preperiodic orbit cannot tend to any point with the maximal speed. It was proved in [Sil93, Theorem E]. See also [Mat23, Theorem 1.11] and [Mat25, Theorem 1.8] for some generalizations.

Lemma 2.1. *Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be an endomorphism of degree d over a number field K, and $p \in \mathbb{P}^1(K)$ be a non-exceptional point. Fix a coordinate of $\mathbb{P}^1$ such that p is not the infinity. Let $v \in M_K$ be a place. Then for any non-preperiodic $x \in \mathbb{P}^1(K)$, we have*

$$\lim_{n\to+\infty} \frac{-\log \min\{|f^n(x) - p|_v, 1\}}{d^n} = 0.$$

Now we can prove Proposition 1.2.

Proof of Proposition 1.2. First we make some assumptions and reductions.

In view of [BGT10] and [XY, Corollary 1.9], we only need to treat the case when $\deg f = \deg g = d > 1$.

If g is conjugated to a polynomial map, then we reduce to the case of endomorphisms of $\mathbb{A}^2$ [Xie17]. Since the DML property is invariant under iteration, we may assume that any iteration of g is not conjugated to a polynomial map. Let (x_0, y_0) be the starting point and let $C \subseteq \mathbb{A}^1 \times \mathbb{P}^1$ is an irreducible curve. In order to prove the DML conjecture, we may assume that x_0 and y_0 are neither preperiodic points for f and g, and that C is not a fiber of $\mathbb{A}^1$ or $\mathbb{P}^1$. In this case, we prove that $C(\overline{\mathbb{Q}}) \cap O_{f\times g}(x_0, y_0)$ is finite.

Extend f to ∞. Let $\overline{C}$ be the closure of C in $\mathbb{P}^1 \times \mathbb{P}^1$ and $l_\infty = \{\infty\} \times \mathbb{P}^1$. Let $b_1, \ldots, b_k$ be the second factor of the intersection points in $\overline{C} \cap l_\infty$. Applying a suitable conjugation by an element in $\mathrm{Aut}(\mathbb{P}^1)$ on the second $\mathbb{P}^1$-factor, we may assume that $\infty \notin \{b_1, \ldots, b_k\}$. Let K be a number field so that all data above are defined over K.

Lemma 2.2. *If $x_0 \in \mathbb{A}^1(K)$ is not a preperiodic point of f, then there exist a place $v \in M_K$, constants $c_1, c_2 > 0$, and a positive integer N, such that for $n > N$, we have*

$$c_1 d^n < \log |f^n(x_0)|_v < c_2 d^n.$$

Proof. Write $f = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$, where $a_d \neq 0$. Denote $M_{K,\infty}$ as the set of archimedean places of K. Let $S = M_{K,\infty} \cup \{v \in M_K |\ |a_d|_v \neq 1\} \cup \bigcup_{i=0}^{d-1} \{v \in M_K |\ |a_i|_v > 1\}$. Note that S is a finite set. For $v \in S$, we denote $C_v = \frac{2}{|a_d|_v}(1 + \sum_{i=0}^{d-1} |a_i|_v) + 1$.

Since $O_f(x_0)$ is infinite, we know $\{h(f^n(x_0))|\ n \in \mathbb{N}\}$ is unbounded by the Northcott property. Here h is the height function. Then we can find either a place $v \notin S$ together with an integer N such that $|f^N(x_0)|_v > 1$, or a place $v \in S$ together with an integer N such that $|f^N(x_0)|_v > C_v$.

In the previous case, we have $|f^{n+1}(x_0)|_v = |f^n(x_0)|_v^d$ when $n \geq N$. Hence the lemma follows.

In the latter case, the inequalities $\frac{1}{2}|a_d|_v < \frac{|f^{n+1}(x_0)|_v}{|f^n(x_0)|_v^d} < \frac{3}{2}|a_d|_v$ and $|f^n(x_0)|_v > C_v$ hold for every $n \geq N$. Taking logarithm, we get

$$\log(\frac{1}{2}|a_d|_v) < \log|f^{n+1}(x_0)|_v - d\log|f^n(x_0)| < \log(\frac{3}{2}|a_d|_v).$$

Then for $n \geq N$, by taking summation in the standard way, we get

$$(\log|f^N(x_0)|_v + \frac{\log(|a_d|_v/2)}{d-1})d^{n-N} - \frac{\log(|a_d|_v/2)}{d-1} < \log|f^n(x_0)|_v$$

$$< (\log|f^N(x_0)|_v + \frac{\log(3|a_d|_v/2)}{d-1})d^{n-N} - \frac{\log(3|a_d|_v/2)}{d-1}.$$

Thus we finish the proof. □

Now assume that $C \cap O_{f\times g}((x_0, y_0))$ is infinite. By Lemma 2.2, we find a place $v \in M_K$ where $|f^n(x_0)|_v \to \infty$. Let $(n_l)_{l\geq 1}$ be the return set $\{n \in \mathbb{N} | \ (f^n(x_0), g^n(y_0)) \in C(K)\}$, φ be the defining function of C, and $\varphi_\infty = \varphi|_{l_\infty}$. Let x and y indicate the standard coordinates on $\mathbb{A}^1$ and $\mathbb{P}^1$, respectively. Write $\varphi = \sum_{i=0}^m x^i \sum_{j=0}^n a_{ij}y^j$, then $\varphi_\infty = \sum_{j=0}^n a_{mj}y^j$. Then $a_{mn} \neq 0$ as we have assumed that $\overline{C}$ does not intersect l_∞ at (∞, ∞).

For a point $(x, y) \in C(K)$ such that $x \neq 0$, by rearranging the terms of the defining equation φ and by dividing both sides by x^m, we have

$$\sum_{i=0}^{m-1} x^{i-m} \sum_{j=0}^{n} a_{ij}y^j = -\sum_{j=0}^{n} a_{mj}y^j = -a_{mn}\prod_{s=1}^{k}(y - b_s)^{l_s} \tag{2.1}$$

where $l_1, \ldots, l_k$ are the multiplicities of the roots $b_1, \ldots, b_k$ in φ_∞.

Write $(x_l, y_l) = (f^{n_l}(x_0), g^{n_l}(y_0))$. We claim that $|x_l|_v \to \infty$ forces $\min_{1\leq s\leq k} |y_l - b_s|_v \to 0$. Otherwise, by extracting subsequence, we can assume $|y_l - b_s|_v > \varepsilon$ for some $\varepsilon > 0$, every s and every $l \geq 1$. If $\{|y_l|_v\}_{l\geq 1}$ is bounded, then when $l \to \infty$, the LHS of (2.1) tends to 0 while the RHS has a positive lower bound, which is impossible. If $\{|y_l|_v\}_{l\geq 1}$ is unbounded, by extracting subsequence, we assume $|y_l|_v \to \infty$. Divide by y_l^n in the both sides of (2.1), then when $k \to \infty$, the LHS of (2.1) tends to 0 while the RHS tends to $a_{mn} \neq 0$, a contradiction. Therefore, we get $\min_{1\leq s\leq k} |y_l - b_s|_v \to 0$.

Passing to subsequence, we may assume $|y_l - b_1|_v \to 0$. Then there is a constant c_0 such that $|y_l - b_1|_v^{l_1} < \frac{c_0}{|x_l|_v}$ for l sufficiently large. So we get $-\log|y_l - b_1|_v > cd^{n_l}$ for some constant $c > 0$ when l is large.

Now we apply Lemma 2.1 to get a contradiction. It only remains to verify that b_1 is not an exceptional point for g. But as we have assumed that no iterate of g can conjugate to a polynomial map, in fact g has no exceptional point. Otherwise, the iteration g^2 will have an invariant exceptional point. Applying a suitable conjugation by an element in $\mathrm{Aut}(\mathbb{P}^1)$, we may send that point to ∞ and then g^2 is conjugated to a polynomial map. Thus we finish the proof. $\square$

Now we prove Theorem 1.1 and Corollary 1.3.

Proof of Theorem 1.1. We may assume that f and g are dominant. By taking normalization, we assume that X and Y are smooth. Take a smooth projective closure $\bar{X}$ of X. Then we can extend f to an endomorphism $\bar{f} : \bar{X} \to \bar{X}$.

If the genus of $\bar{X}$ is greater than 1, then some iteration of $\bar{f}$ is the identity, and the DML conjecture holds trivially in this case. The same is true if the genus of Y is greater than 1.

If the genus of $\bar{X}$ and Y are both 1, then $\bar{f}$ and g are both étale. Hence $\bar{f} \times g$ is also étale. Then the DML conjecture holds by [BGT10].

If the genus of $\bar{X}$ is 1 and the genus of Y is 0, then $\bar{f}$ is étale and the DML conjecture holds by [BZ23, Corollary 1.2]. The same is true if genus of $\bar{X}$ is 0 and the genus of Y is 1.

If the genus of $\bar{X}$ and Y are both 0, then $Y \cong \mathbb{P}^1$ and $X \cong \mathbb{P}^1 \setminus E$, where E is a non-empty finite set. Since $\bar{f}$ is surjective, we have $\bar{f}(E) = E$, which implies that every point in E is periodic. After iteration, we may assume that they are all fixed points. Then we can extend f to $\bar{X} \setminus \{\text{one point}\} \cong \mathbb{A}^1$, and the result follows from Proposition 1.2. $\square$

Proof of Corollary 1.3. Let $C \subset X \times Y$ be a curve, $(x, y) \in (X \times Y)(\overline{\mathbb{Q}})$ be a point, and $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ be projections. In order to verify the DML(1) property, we may assume that both x and y are not preperiodic. Suppose $\# O_{f\times g}((x, y)) \cap C(\overline{\mathbb{Q}}) = \infty$. Let $C_1 := p_1(C)$ and $C_2 := p_2(C)$. Then $C_1 \subset X$ is a curve, and $\# O_f(x) \cap C_1(\overline{\mathbb{Q}}) = \infty$. By the DML(1) property for f, there are positive integers n_0 and m such that the infinite sequence $\{f^{n_0+km}(x) |\ k \in \mathbb{N}\} \subset C_1(\overline{\mathbb{Q}})$. Hence $f^m(C_1) = C_1$. Similarly, C_2 is a periodic curve for g.

After iteration, we may assume that C_1 is invariant for f and C_2 is invariant for g, i.e. $f(C_1) = C_1$ and $g(C_2) = C_2$. Then it suffices to verify the DML conjecture for $f|_{C_1} \times g|_{C_2}$ and $C \subset C_1 \times C_2$, which follows from Theorem 1.1. $\square$

REFERENCES

[Bel06] J. P. Bell. A generalized Skolem–Mahler–Lech theorem for affine varieties. *J. London Math. Soc. (2)*, **73**(2):367–379, 2006.

[BGT10] J. P. Bell, D. Ghioca, and T. J. Tucker. The dynamical Mordell–Lang problem for étale maps. *Amer. J. Math.*, **132**(6):1655–1675, 2010.

[BGT16] J. P. Bell, D. Ghioca, and T. J. Tucker. *The Dynamical Mordell–Lang Conjecture*, volume **210** of *Mathematics Surveys and Monographs*. American Mathematical Society, Providence, RI, 2016.

[BZ23] J. P. Bell and X. Zhong. p-adic interpolation of orbits under rational maps. *Proc. Amer. Math. Soc.*, **151**(11):4661–4672, 2023.

[GT09] D. Ghioca and T.J. Tucker. Periodic points, linearizing maps, and the dynamical Mordell–Lang problem. *J. Number Theory*, **129**(6):1392–1403, 2009.

[Mat23] Y. Matsuzawa. Growth of local height functions along orbits of self-morphisms on projective varieties. *Int. Math. Res. Not. IMRN*, (4):3533–3575, 2023.

[Mat25] Y. Matsuzawa. Existence of arithmetic degrees for generic orbits and dynamical Lang–Siegel problem. *J. Reine Angew. Math.*, **2025**(825):305–335, 2025.

[Sil93] J. H. Silverman. Integer points, Diophantine approximation, and iteration of rational maps. *Duke Math. J.*, **71**(3):793–829, 1993.

[Xie] J. Xie. Around the dynamical Mordell–Lang conjecture. Text available at http://scholar.pku.edu.cn/sites/default/files/xiejunyi/files/arounddml20230701fu_ben_.pdf.

[Xie17] J. Xie. The dynamical Mordell–Lang conjecture for polynomial endomorphisms of the affine plane. *Astérisque*, **394**:vi+110, 2017.

[XY] J. Xie and S. Yang. Height arguments toward the dynamical Mordell–Lang problem in arbitrary characteristic. arXiv:2504.01563v2.

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