QCQI: Assignment #1

1 Vector Bases

Suppose $|\alpha_i\rangle$ $(i = 0, ..., n-1)$ are n vectors in an n dimensional Hilbert space, we call $\{|\alpha_i\rangle\}$ to be a basis when $\langle\alpha_i|\alpha_j\rangle = \delta_{ij}$ $(i, j = 0, ..., n-1)$. (1) Prove that $\{|\alpha_i\rangle\}$ is a basis iff $\sum_{i=0}^{n-1} |\alpha_i\rangle\langle\alpha_i| = \boldsymbol{I}$; (2) Find the operator $\boldsymbol{M}$ such that $\sum_{i=0}^{n-1} i^2 |\alpha_i\rangle\langle\alpha_i| \boldsymbol{M} = \boldsymbol{M} \sum_{i=0}^{n-1} i^2 |\alpha_i\rangle\langle\alpha_i|$; (3) Prove that any n dimensional vector can be expressed as $|\psi\rangle = \sum_i \psi_{\alpha_i} |\alpha_i\rangle$; (4) Suppose there is another basis $\{|\beta_i\rangle\}$, prove that there exists a unitary such that $|\alpha_i\rangle = U|\beta_i\rangle$, $\forall i$.

(Hint: use ket bra notations)

2 Direct sums and tensor products

(1) Consider vectors $|+\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ and $|\Psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, calculate $|\psi_1\rangle = |+\rangle \oplus |\Psi\rangle$ and $|\psi_2\rangle = |+\rangle \otimes |\Psi\rangle$.

(2) Consider the matrix

$$\boldsymbol{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

calculate

$$A = \langle\psi_1| \boldsymbol{X} \oplus (\boldsymbol{X} \otimes \boldsymbol{X}) |\psi_1\rangle, \quad B = \langle\psi_2| \boldsymbol{X} \otimes \boldsymbol{X} \otimes \boldsymbol{X} |\psi_2\rangle. \tag{1}$$

3 Norms

Norms on a vector space can be defined as functions satisfying three conditions: 1. absolute scalability: $||a\boldsymbol{v}|| = |a| \cdot ||\boldsymbol{v}||$; 2. if $||\boldsymbol{v}|| = 0$, then $\boldsymbol{v}$ is a zero vector; 3. triangle inequality: $||\boldsymbol{u}|| + ||\boldsymbol{v}|| \geq ||\boldsymbol{u} + \boldsymbol{v}||$

(1) prove that norm is a positive definite function.

(2) recall the definition of p-norm, is $\frac{1}{2}$-norm a well defined norm? and why?

4 SVD and spectral decomposition

(1) Suppose $\boldsymbol{A}$ is a $m \times n$ matrix $\boldsymbol{A} = [\boldsymbol{w}_1, ..., \boldsymbol{w}_n]$ where $\boldsymbol{w}_i$ $(i = 1, ...n)$ are m dimensional column vectors satisfying $\boldsymbol{w}_i^T \boldsymbol{w}_j = 0 (i \neq j)$. Do SVD decomposition for $\mathbf{A}$.

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(2) Suppose the $\boldsymbol{A}$ matrix is defined as follows

$$\boldsymbol{A} = \begin{bmatrix} \sqrt{2} & 2 & 2 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

(2a) Do SVD decomposition for $\mathbf{A}$.

(2b) Solve the minimum problem $\underset{x\in\mathbb{R}^3}{min}||\boldsymbol{A}x - b||_2$ where

$$b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

5 Special matrices

(1) Prove that the eigenvalue of a unitary operator is $e^{i\theta}$.

(2) Prove that the eigenvalue of a projection operator is either 1 or 0.

(3) Prove that the eigenvalue of a Hermitian operator is real.

6 Functions of normal matrices

Suppose the spectral decomposition of a normal matrix M is $M = \sum_i \lambda_i \Pi_i$ with $\Pi = |\psi_i\rangle\langle\psi_i|$.

(1) Show that Π is a projection operator;

(2) Show that $M^n = \sum_i \lambda_i^n \Pi_i$ for any positive integer n.

(3) Consider the $\boldsymbol{X}$ matrix of question 2, calculate $e^{i\boldsymbol{X}\theta}$.

(4) Use the spectral decomposition to show that matrix $K = -i\log(U)$ is Hermitian for any unitary matrix U.

7 Schmidt decomposition

A tensor product state $|\psi\rangle$ can be written in the form of $|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$. Given three states:

$$|\Psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \; |\Psi_2\rangle = \frac{|00\rangle + |11\rangle + |01\rangle + |10\rangle}{\sqrt{2}}, \; |\Psi_3\rangle = \frac{|00\rangle + |01\rangle + |10\rangle}{\sqrt{2}}.$$

(1) Determine whether the above states are tensor product states.

(2) Compute the rank of matrix $A = \sum_{i,j} a_{i,j} |i\rangle\langle j|$ with $a_{i,j} = \langle ij|\Psi\rangle$ $(i, j = 0, 1)$.

(3) Find the Schmidt decomposition of the above states, i.e., find two bases $\{|\psi_i\rangle\}$ and $\{|\phi_i\rangle\}$ such that $|\Psi\rangle = \sum_i \lambda_i |\psi_i\rangle |\phi_i\rangle$.

8 Matrix-vector duality

For any matrix $M = \sum_{ij} M_{ij} |i\rangle_B \langle j|_A$, we can map it to a vector $|M\rangle = \sum_{ij} M_{ij} |j\rangle_A |i\rangle_B$.

(1) Prove that $\mathrm{tr}[M^\dagger N] = \langle M|N\rangle$.

(2) Suppose $|\Psi\rangle = \sum_j |j\rangle_A |j\rangle_A$, prove that $|M\rangle = (I_A \otimes M) |\Psi\rangle$.

(3) Given the a SVD of M, show the Schmidt decomposition of $|M\rangle$.