# Lecture 14 Quantum Communication

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In this lecture, we study the three famous quantum communication protocols — quantum teleportation, quantum dense coding, and entanglement swapping. They are all based on manipulating quantum entanglement.

## 1 Quantum teleportation

Imagine in the future that we have a quantum Internet, where each user possesses a quantum computer and two users can communicate quantum information. A very basic feature of a quantum Internet is of course to transfer a quantum state from one user to another. While the most straightforward way is just sending the quantum state stored in the quantum computer. This may be very hard to implement practically, since quantum computers may be realized with different solid states, such as trapped ions and superconducting qubits, and sending such solid states are practically hard at least for the current technology. Even in the classical world, solid states, such as the CPU of our laptop, are more likely to be a local machine, which of course could be moved, but generally not used to transmit information for distant users. Instead, we generally use light or photons to carry and transmit information, simply because they are cheap to realize and photon has relatively weak interaction with the environment. We would expect a similar situation in the quantum world, where we realize our quantum computer using solid states and transfer information using photons for most scenarios.

Now imagine that we have prepared a qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ in a local quantum computer of Alice, can we send it to a distant party Bob using photons? One strategy is to prepare the state again using the photon, but then it requires full information of the state and hence costly state tomography[1]. Can we do it more smartly so that we can transfer the state but without knowing it? This seemly impossible task is actually realizable with the so-called quantum teleportation protocol. The basic idea is to use quantum entanglement and apply non-destructive measurements. In the following, we first introduce the protocol and see how it works for any qubit state. Then we briefly discuss how it works for multi-qubit states. At last, we show it from a mathematical and graphical perspective to explain why it works.

### 1.1 Teleporting a qubit state

Suppose Alice prepares a qubit $|\psi\rangle_A = a|0\rangle_A + b|1\rangle_A$ and Alice and Bob share some entangled state $|\Phi^+\rangle_{A'B} = \frac{1}{\sqrt{2}}(|00\rangle_{A'B} + |11\rangle_{A'B})$ [2]. The quantum teleportation protocol works as follows.

1. Alice performs a Bell state measurement (BSM) on $AA'$ and obtain a classical measurement outcome of two bits [3].

[1] State tomography is simply a procedure to learn the density matrix of the state. It requires to prepare many-copies of the state and then measure the state in the complete Pauli basis. State tomography becomes exponentially costly with the system size, i.e., the number of qubits.

[2] The entangled state is generally distributed using photons and then stored in a solid state quantum memory. For example, Alice can generate a Bell pair and send one half to Bob. Then Alice and Bob store the Bell pair using a quantum memory, which can input with the photon state and output it when needed.

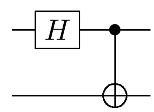
[3] We note that it is possible to interact solid states and photons. For example, photons can excite orbitals of atoms and such a process could be exploited to implement the interaction and hence the BSM.

A BSM is defined by the projective measurement on one of the four Bell state

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \;\; |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \tag{1}$$

It is easy to verify that $\{|\Phi^{\pm}\rangle, |\Psi^{\pm}\rangle\}$ forms an orthonormal basis for two-qubit states.

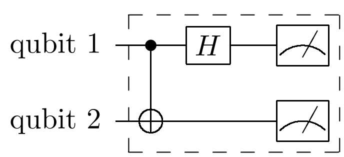
How can we realize BSM? Remember that a normal projective measurement on the computational basis is a projection to one of the four states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, which could be realized by measuring each qubit in the $\{|0\rangle, |1\rangle\}$ basis. Secondly, we can always use a unitary to rotate a basis to another one. In this case, we can rotate $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ to the Bell basis $\{|\Phi^{\pm}\rangle, |\Psi^{\pm}\rangle\}$ using the following circuit




Denote the circuit by $U$, then we have

$$U|00\rangle = |\Phi^+\rangle, \; U|01\rangle = |\Psi^+\rangle, \; U|10\rangle = |\Phi^-\rangle, \; U|11\rangle = |\Psi^-\rangle. \tag{2}$$

To implement the BSM, we can thus apply $U^\dagger$ first and measure in the computational basis, as follows.




Then the four computational basis measurement outcomes $00, 01, 10, 11$ corresponds to projection to $|\Phi^+\rangle, |\Psi^+\rangle, |\Phi^-\rangle, |\Psi^-\rangle$, respectively.

2. Alice send the measurement outcomes to Bob.

3. Bob applies $I$, $\sigma_X$, $\sigma_Z$, $\sigma_Z\sigma_X$ if the measurement outcome is respectively $00, 01, 10, 11$.

Let's see why the above protocol works. At the beginning, the initial state is

$$\begin{aligned}
|\psi\rangle_{AA'B} &= \frac{1}{\sqrt{2}}(a|0\rangle_A + b|1\rangle_A)(|00\rangle_{A'B} + |11\rangle_{A'B}), \\
&= \frac{1}{\sqrt{2}}(a|000\rangle_{AA'B} + a|011\rangle_{AA'B} + b|100\rangle_{AA'B} + b|111\rangle_{AA'B}), \\
&= \frac{1}{2}\Big[a(|\Phi^+\rangle_{AA'} + |\Phi^-\rangle_{AA'})|0\rangle_B + a(|\Psi^+\rangle_{AA'} + |\Psi^-\rangle_{AA'})|1\rangle_B \\
&\quad + b(|\Psi^+\rangle_{AA'} - |\Psi^-\rangle_{AA'})|0\rangle_B + b(|\Phi^+\rangle_{AA'} - |\Phi^-\rangle_{AA'})|1\rangle_B\Big] \\
&= \frac{1}{2}\Big[|\Phi^+\rangle_{AA'}\big(a|0\rangle_B + b|1\rangle_B\big) + |\Phi^-\rangle_{AA'}\big(a|0\rangle_B - b|1\rangle_B\big) \\
&\quad + |\Psi^+\rangle_{AA'}\big(a|1\rangle_B + b|0\rangle_B\big) + |\Psi^-\rangle_{AA'}\big(a|1\rangle_B - b|0\rangle_B\big)\Big]
\end{aligned} \tag{3}$$

Here we have used $|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)$, $|11\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle - |\Phi^-\rangle)$, $|01\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle + |\Psi^-\rangle)$, and $|10\rangle = \frac{1}{\sqrt{2}}(|\Psi^+\rangle - |\Psi^-\rangle)$.

After Alice's BSM measurement, the whole state collapse to one of the following four cases with equal 1/4 probability

$$\begin{aligned} 00: &\quad |\Phi^+\rangle_{AA'}\left(a|0\rangle_B + b|1\rangle_B\right),\\ 01: &\quad |\Psi^+\rangle_{AA'}\left(a|1\rangle_B + b|0\rangle_B\right),\\ 10: &\quad |\Phi^-\rangle_{AA'}\left(a|0\rangle_B - b|1\rangle_B\right),\\ 11: &\quad |\Psi^-\rangle_{AA'}\left(a|1\rangle_B - b|0\rangle_B\right). \end{aligned} \tag{4}$$

Note that the state on Bob's side is almost the initial state of Alice up to some local unitary transformation. After applying the corresponding unitary based on the measurement outcome, Bob recovers Alice's state with 100% probability.

We remark that the classical measurement outcome is crucial to recover Alice's state. Without it, one can easily verify that Bob's quantum state is a maximally mixed state (think why).

## 1.2 Teleporting a multi-qubit state

Can we adapt the teleportation protocol to teleport an arbitrary $n$-qubit state, which is what we need to do in reality. The answer is surprisingly simple — we just apply quantum teleportation to every qubit of the multi-qubit state. To see why this works, we only need to be aware that quantum teleportation keeps entanglement.

Specifically, imagine that Alice has a multi-qubit quantum state $|\psi_{A_1A_{2\ldots n}}\rangle = a|0\rangle_{A_1}|\psi_0\rangle_{A_{2\ldots n}} + b|1\rangle_{A_1}|\psi_1\rangle_{A_{2\ldots n}}$. Then the initial state becomes

$$\begin{aligned} |\psi\rangle_{A_1A_{2\ldots n}A'B} = \frac{1}{2}\Big[ &|\Phi^+\rangle_{AA'}\left(a|0\rangle_B|\psi_0\rangle_{A_{2\ldots n}} + b|1\rangle_B|\psi_1\rangle_{A_{2\ldots n}}\right) + |\Phi^-\rangle_{AA'}\left(a|0\rangle_B|\psi_0\rangle_{A_{2\ldots n}} - b|1\rangle_B|\psi_1\rangle_{A_{2\ldots n}}\right)\\ &+ |\Psi^+\rangle_{AA'}\left(a|1\rangle_B|\psi_0\rangle_{A_{2\ldots n}} + b|0\rangle_B|\psi_1\rangle_{A_{2\ldots n}}\right) + |\Psi^-\rangle_{AA'}\left(a|1\rangle_B|\psi_0\rangle_{A_{2\ldots n}} - b|0\rangle_B|\psi_1\rangle_{A_{2\ldots n}}\right)\Big]. \end{aligned} \tag{5}$$

After applying the BSM on $AA'$ as well as the local unitary on $B$ based on the measurement outcome, the final state becomes

$$a|0\rangle_B|\psi_0\rangle_{A_{2\ldots n}} + b|1\rangle_B|\psi_1\rangle_{A_{2\ldots n}}, \tag{6}$$

which is very similar to the initial state $|\psi_{A_1A_{2\ldots n}}\rangle = a|0\rangle_{A_1}|\psi_0\rangle_{A_{2\ldots n}} + b|1\rangle_{A_1}|\psi_1\rangle_{A_{2\ldots n}}$ and the only difference is that system $A_1$ is now replaced with system $B$. Therefore, when we apply quantum teleportation, we really teleport the *quantum* state, which keeps its entanglement with the rest system. Now, if we want to teleport the whole state $|\psi_{A_1A_{2\ldots n}}\rangle$, we only need to sequentially apply quantum teleportation to every qubit system $A_1$ and $A_{2\ldots n}$. Since each teleportation needs one initially shared Bell pair and two classical bits of information, the protocol for any $n$-qubit state requires $n$ pre-shared Bell pair and $2n$-bits of classical information.

We remark that teleportation does not violate the no-cloning theorem. This is because we only move the state from one party to another instead of copying it. When teleportation finishes, we still only have one copy of the state.

## 1.3 Some intuition why teleportation works (optional)

We first note that the four Bell states are related via local unitary

$$|\Phi^+\rangle_{AA'} = |\Phi^+\rangle_{AA'}, \ |\Psi^+\rangle_{AA'} = \sigma_X^A|\Phi^+\rangle_{AA'}, \ |\Phi^-\rangle_{AA'} = \sigma_Z^A|\Phi^+\rangle_{AA'}, \ |\Psi^-\rangle_{AA'} = \sigma_Z^A\sigma_X^A|\Phi^+\rangle_{AA'}. \tag{7}$$

One may found that these four unitary operators are actually just the four unitary operators Bob needs apply at the third step. We denote these unitary operators by $U_{ij}$, i.e., $U_{00} = \mathbb{I}^A$, $U_{01} = \sigma_X^A$, $U_{10} = \sigma_Z^A$, $U_{11} = \sigma_Z^A\sigma_Z^A$, and relabel the four Bell states as

$$|\Phi^+\rangle_{AA'} \equiv |\Phi_{00}\rangle_{AA'}, \ |\Psi^+\rangle_{AA'} \equiv |\Phi_{01}\rangle_{AA'}, \ |\Phi^-\rangle_{AA'} \equiv |\Phi_{10}\rangle_{AA'}, \ |\Psi^-\rangle_{AA'} \equiv |\Phi_{11}\rangle_{AA'}, \tag{8}$$

then we have

$$|\Phi_{ij}\rangle_{AA'} = U_{ij}\,|\Phi^+\rangle_{AA'}\,. \tag{9}$$

Now we consider the projective measurement of $|\Phi_{ij}\rangle_{AA'}$ on the initial state $|\psi\rangle_{AA'B} = \frac{1}{\sqrt{2}}(a\,|0\rangle_A + b\,|1\rangle_A)(|00\rangle_{A'B} + |11\rangle_{A'B})$. Ignoring the measurement probability, the output state of system $B$ is

$$\langle\Phi_{ij}|_{AA'}\,|\psi\rangle_{AA'B} = \langle\Phi^+|_{AA'}\,U_{ij}^\dagger(a\,|0\rangle_A + b\,|1\rangle_A)(|00\rangle_{A'B} + |11\rangle_{A'B}). \tag{10}$$

We note that this equation is equivalent to the following one with a partial transpose on system $A'$,

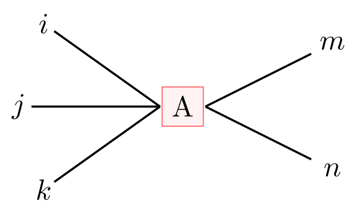
$$\begin{aligned}
\langle\Phi_{ij}|_{AA'}\,|\psi\rangle_{AA'B} &= \mathrm{Tr}_{AA'}[\langle\Phi_{ij}|_{AA'}\,|\psi\rangle_{AA'B}]^{T_{A'}},\\
&= \mathrm{Tr}_{AA'}\left[\left(\langle\Phi^+|_{AA'}\right)^{T_{A'}} U_{ij}^\dagger(a\,|0\rangle_A + b\,|1\rangle_A)(|00\rangle_{A'B} + |11\rangle_{A'B})^{T_{A'}}\right],\\
&\propto \mathrm{Tr}_{AA'}\left[\mathbb{I}_{A'A}U_{ij}^\dagger(a\,|0\rangle_A + b\,|1\rangle_A)\mathbb{I}_{BA'}\right],\\
&= \mathrm{Tr}_{A}\left[\mathbb{I}_{BA}U_{ij}^\dagger(a\,|0\rangle_A + b\,|1\rangle_A)\right],\\
&= U_{ij}^\dagger(a\,|0\rangle_B + b\,|1\rangle_B).
\end{aligned} \tag{11}$$

Here the key observation is that $\langle\Phi^+|_{AA'}^{T_{A'}} = \mathbb{I}_{A'A}$. Therefore, the output state of system $B$ is actually proportional to $U_{ij}^\dagger(a\,|0\rangle_B + b\,|1\rangle_B)$. We can thus apply $U_{ij}$ to invert the unnecessary unitary and obtain the target state $a\,|0\rangle_B + b\,|1\rangle_B$.

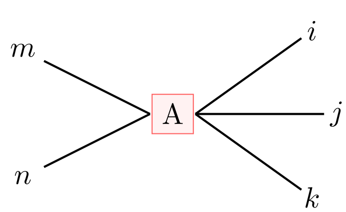
### 1.4 A graphical description — tensor networks (optional)

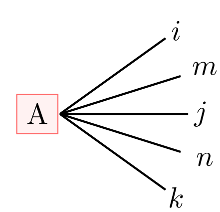
While the above mathematical derivation still seems abstract, there is actually an equivalent but very intuitive graphical description using the powerful tool of tensor networks.

First a tensor could be understood as a general matrix with multiple indices. For example, we consider a tensor $A^{ijk}_{mn}$ and use the following graph to represent it. Here we simply use lines toward left (right) to indicate subscript (superscript).




How the lines are arranged and directed actually does not matter very much. For example, the above tensor $A^{ijk}_{mn}$ is isomorphic to $A^{mn}_{ijk}$, $A_{ijkmn}$, which are denoted as follows.

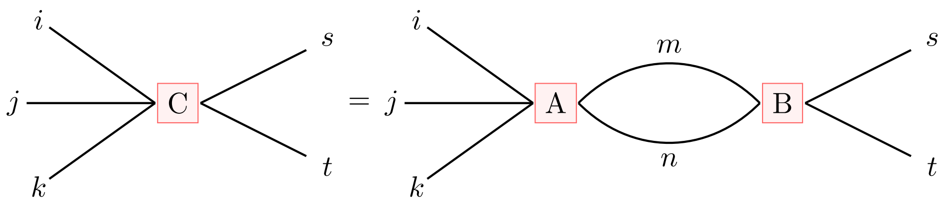



A key concept in tensor network is contraction. Imagine we have two tensors $A^{ijk}_{mn}$ and $B^{mn}_{st}$, then we can contract the same indices to get a new tensor

$$C^{ijk}_{st} = \sum_{mn} A^{ijk}_{mn}B^{mn}_{st}, \tag{12}$$

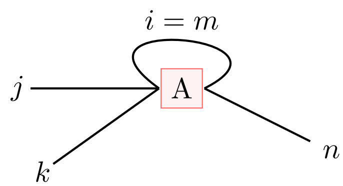
and graphically as




We can also contract two indices of one tensor, similar to how we apply trace or partial trace. For example, we can contract the $i$ and $m$ indices to get

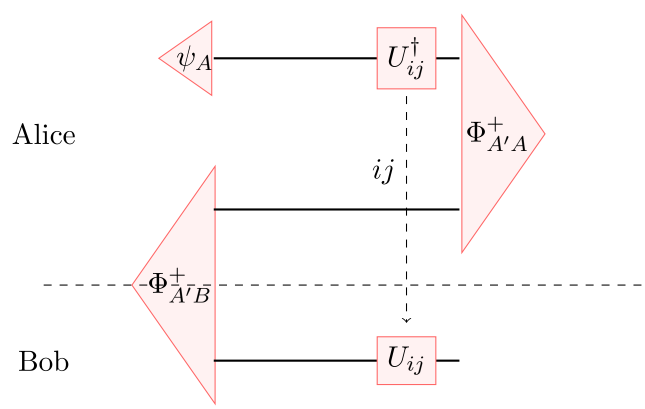
$$\tilde{A}_{n}^{jk} = \sum_{i=m} A_{mn}^{ijk} \tag{13}$$

And graphically we have

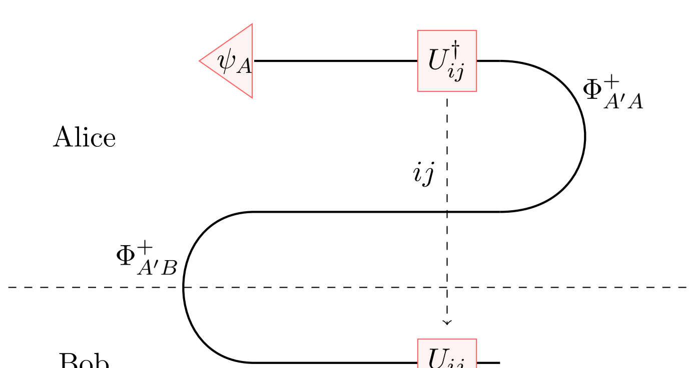



With the above definition, we can use tensor network to describe a quantum process. Consider in the computational basis, any quantum state $|\psi\rangle = \sum_j \psi_j |i\rangle$ defines a tensor $\psi_j$, and a projective measurement of $|\psi\rangle$ (using only the bra part) also defines a tensor $\bar{\psi}^{j}$[4] . A unitary $U = \sum_{ij} U_i^j |i\rangle \langle j|$ defines a tensor $U_i^j$.

Using the above tensor network language, we can represent the teleportation protocol graphically as follows.

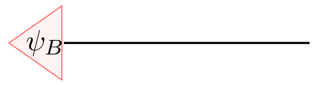



The magic is that the tensor corresponding to Bell state and measurement is actually the same tensor for the identity matrix. Therefore, the above tensor network is equivalent to the following one




[4]Here we used $\langle\psi|$ to denote the complex conjugate operation.

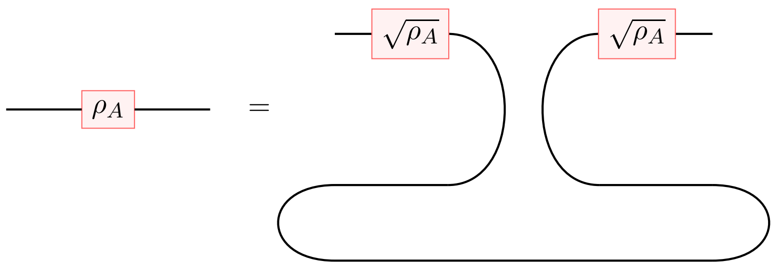
which is equivalent to a quantum state of Bob



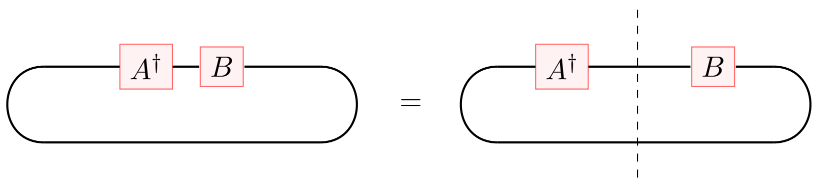

### 1.4.1 Other applications

The tensor network approach is extremely useful when it involves the Bell state, the swap gate, trace or partial trace. Here we give several examples.

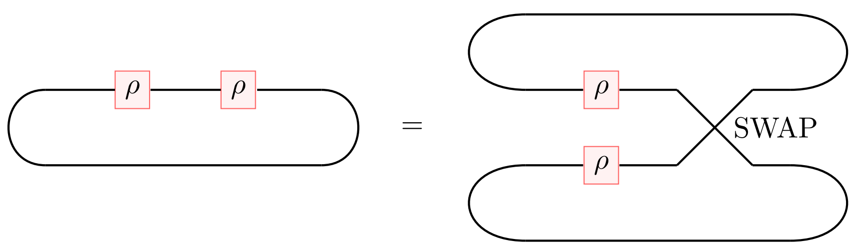
- Purification — a purification of $\rho_A$ is $|\psi\rangle_{AB} = \sqrt{\rho_A} \otimes \mathbb{I}_A |\Phi^+\rangle_{AB}$. That is $\rho_A = \mathrm{Tr}_B[\psi_{AB}]$.




- Matrix inner product $\mathrm{Tr}[A^\dagger B] = \langle\Phi^+|A \otimes B|\Phi^+\rangle$.




- Swap test — $\mathrm{Tr}[\rho^2] = \mathrm{Tr}[(\rho \otimes \rho)\mathrm{SWAP}]$, here SWAP is the swap unitary operator.




## 2 Dense coding

The second problem is that by sending quantum states, for example, using photons, can we send more information compared to classical communication. The answer is yet. By sending a qubit state, we can effectively send two classical bits of information. The protocol works as follows.

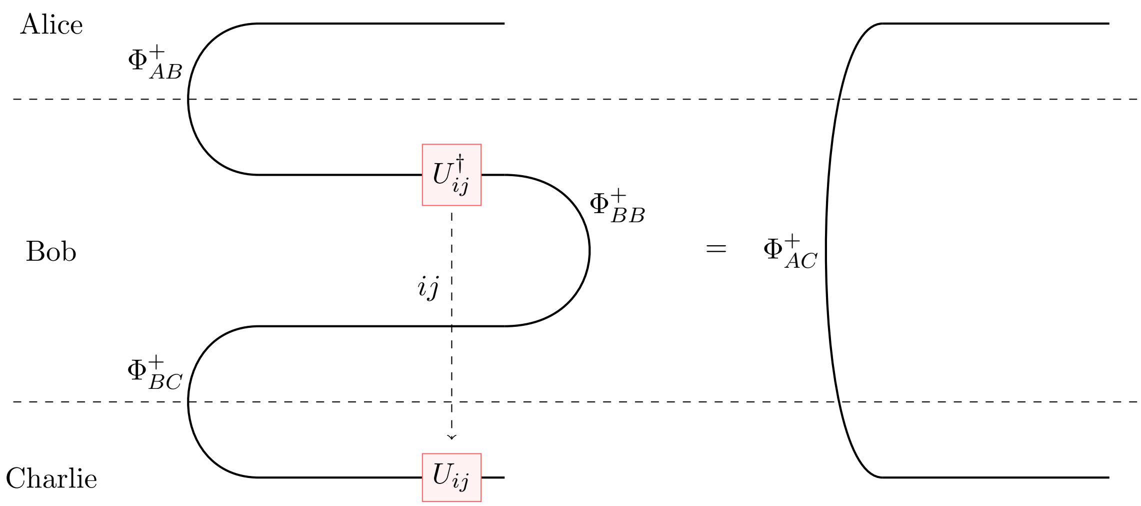
1. Alice and Bob share an entangled state $|\Phi^+\rangle_{AB}$.
2. Alice apply $\mathbb{I}$, $\sigma_X$, $\sigma_Z$, $\sigma_Z\sigma_X$ to her system if she wants to send 00, 01, 10, 11.
3. Alice send her system to Bob.
4. Bob perform a BSM on the state and obtain the two classical bit based on the four BSM measurement outcomes.

The proof of the protocol is actually very straightforward. We only need to recall the transformation of Eq. (7) from $|\Phi^+\rangle_{AB}$ to the other three Bell states.

With entanglement swapping and dense coding, we can actually show that there does not better protocols. Specifically, given shared Bell pair, we have to transfer two classical bits when we want to teleport a quantum state; likewise, we have to send a qubit if we want to send two classical bits. Suppose that there exists a better teleportation protocol that requires less than two bits of classical information. Then we can run that teleportation protocol $n$ times (by sending $2n' < 2n$ classical bits) to send $n$ qubits. We can next use these $n$ qubits to send $2n$ classical bits. Therefore, we can effectively send $2n > 2n'$ classical bits by physically sending $2n'$ bits, which leads to a contradiction. Similarly, we can show that we need at least a qubit to send two bit of information.

## 3 Entanglement swapping

The protocol of entanglement swapping is to extend entanglement for distant parties. Suppose Alice and Bob shares a Bell pair $|\Phi^+\rangle_{AB}$, and Bob and Charlie shares a Bell pair $|\Phi^+\rangle_{BC}$. Then we can apply entanglement swapping to build a Bell pair $|\Phi^+\rangle_{AC}$ between Alice and Charlie. The protocol is just apply quantum teleportation to teleport Bob's state of $|\Phi^+\rangle_{AB}$ to Charlie, which consumes $|\Phi^+\rangle_{BC}$. The entanglement swapping protocol can be summarized using tensor network as follows.




Entanglement swapping is especially important for building entanglement of two separated parties. Together with error correction, it allows us to share an almost perfect Bell pair between any two parties.