QCQI: Assignment #4

1 Transformation of Kraus Operators

(1) Suppose two Kraus channels $\{K_j\}$ and $\left\{\tilde{K}_k\right\}$ describe a same channel, prove that there exists a unitary U such that $K_j = \sum_k U_{jk}\tilde{K}_k$.(Theorem 3 in lecture notes)

(2) Prove that all quantum operations $\mathcal{E}$ on a system of Hilbert space dimension d can be generated by an operator-sum representation containing at most d^2 elements, i.e.

$$\mathcal{E}(\rho) = \sum_{k=1}^{M} E_k \rho E_k^\dagger$$

where $1 \leq M \leq d^2$.

(hint1: this result is non-trivial because general map $d^2 \to d^2$ should be d^4 dimensional)

(hint2: define a matrix $W_{jk} \equiv \operatorname{tr}\left(E_j^\dagger E_k\right)$, find its rank and try to diagonalize it)

(3) Suppose $\mathcal{E}$ is a quantum operation mapping a d-dimensional input space to a d'-dimensional output space. Show that $\mathcal{E}$ can be described using a set of at most dd' operation elements $\{E_k\}$.

2 Special Channels

(1) Prove $(\rho + X\rho X + Y\rho Y + Z\rho Z) = I/2$ for qubit systems.

(2) Show that the circuit in Figure 1 can be used to model the phase damping quantum operation, provided θ is chosen appropriately.

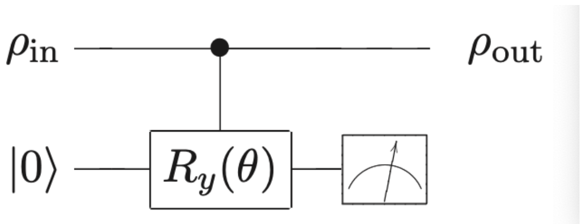

图 1:

(3) A quantum process $\mathcal{E}$ is unital if $\mathcal{E}(I) = I$. Show that the depolarizing and phase damping channels are unital, while amplitude damping is not.

3 Fixed points and contractive map

Prove that trace-preserving quantum operations are strictly contractive, i.e.

(1) for any trace-preserving quantum operation $\mathcal{E}$, density operators ρ, σ, prove that trace-preserving quantum operations are contractive, i.e.

$$D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma) \tag{1}$$

(2) Schauder's fixed point theorem is a classic result from mathematics that implies that any continuous map on a convex, compact subset of a Hilbert space has a fixed point. Use Schauder's fixed point theorem to prove that any trace-preserving quantum operation $\mathcal{E}$ has a fixed point, that is, ρ such that $\mathcal{E}(\rho) = \rho$.

(3) Suppose $\mathcal{E}$ is a strictly contractive trace-preserving quantum operation, that is, for any ρ and $\sigma, D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$. Show that $\mathcal{E}$ has a unique fixed point.

4 Fidelity

(1) Prove that the fidelity is jointly concave,

$$F\left(\sum_i p_i \rho_i, \sum_i p_i \sigma_i\right) \geq \sum_i p_i F\left(\rho_i, \sigma_i\right)$$

(2) Define angle between two states $A(\rho, \sigma) \equiv \arccos F(\rho, \sigma)$, prove that the angle is contractive

$$A(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq A(\rho, \sigma) \tag{2}$$

where $\mathcal{E}$ is a trace-preserving quantum operation.