QCQI: Assignment #2

Due Sunday, October 17 24:00.

1

(1) Suppose A and B are commuting Hermitian operators. Prove that

$$\exp(A)\exp(B) = \exp(A+B).$$

(2) If A and B are not commuting, does the equation in (1) still hold? If not, give a counterexample.

2

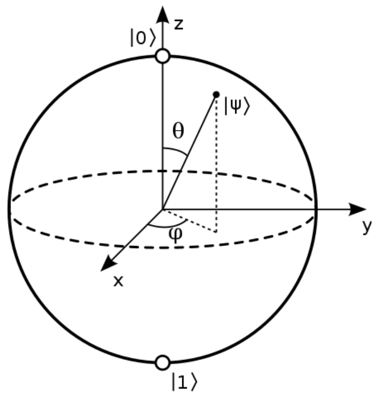

Fig 1: Bloch sphere

(1) The quantum state

$$\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

corresponds to the point on the Bloch sphere

$$\hat{j} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

(You can verify it on Fig 1.)

Show that the operator

$$\sigma_{\hat{j}} = j_x\sigma_x + j_y\sigma_y + j_z\sigma_z$$

has eigenvectors (some quantum states) which correspond to the points $\pm\hat{j}$ on the Bloch sphere.

(2) Show that antipodal points on the Bloch sphere are orthogonal quantum states.

(3) Show that if the vectors $\hat{j}$ and $\hat{k}$ are perpendicular, then $\sigma_{\hat{j}}$ and $\sigma_{\hat{k}}$ anticommute.

3

Suppose Alice and Bob initially share a pair of qubits in the entangled state (Alice holds the first qubit)

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

(1) If Alice measures her qubit in basis $\{|0\rangle, |1\rangle\}$, show the corresponding projective measurement on the whole system (in the form $\{\Pi_i\}$), and what state the whole system will be with what probability after this measurement.

(2) If Alice measures her qubit in basis $\{|+\rangle, |-\rangle\}$, answer the same question in (1).

(3) After Alice measures her own qubit, Bob measures his qubit in an arbitrary orthogonal basis $\{|v_1\rangle, |v_2\rangle\}$. Compare the probability of Bob getting $|v_1\rangle$ and $|v_2\rangle$ in his measurement if Alice chooses different measurement basis in (1) and (2).

4

Given a set of two bits gate $\{AND, NAND, OR, NOR, XOR, NXOR\}$, show all complete subsets with elements less than or equal two.

("Complete subset" means all two bits gate can be implemented using gates in this subset.)

(You can use auxiliary bits.)

5

(1) Let x be a real number and A a matrix such that $A^2 = I$. Show that

$$\exp(iAx) = \cos(x)I + i\sin(x)A.$$

(hint: how to prove $e^{ix} = \cos(x) + i\sin(x)$?)

(2) If $\hat{n} = (n_x, n_y, n_z)$ is a real unit vector in three dimensions then we generalize the definitions of $U_x(\theta), U_y(\theta), U_x(\theta)$ by defining

$$U_{\hat{n}}(\theta) \equiv \exp(-i\theta\hat{n}\cdot\vec{\sigma}/2),$$

where $\vec{\sigma}$ denotes the three component vector (X, Y, Z) of Pauli matrices.

- Prove that
$$U_{\hat{n}}(\theta) = \cos\left(\frac{\theta}{2}\right) I - i\sin\left(\frac{\theta}{2}\right)(n_x X + n_y Y + n_z Z).$$

- Prove that all single-qubit unitary operator can be written in the form
$$U = \exp(i\alpha)U_{\hat{n}}(\theta).$$

(hint: You may find the conclusion in problem 6(4) of assignment 1 helpful.)

6

Let $R_{\hat{n}}(\theta)$ be a 3×3 matrix which represents a rotation by θ about the $\hat{n}$ axis ($\hat{n}$ is defined in problem 5(2)).

(1) Give a geometric interpretation of the equation

$$R_{\hat{n}}(\alpha) = R_z(\phi)R_y(\theta)R_z(\alpha)R_y(-\theta)R_z(-\phi),$$

and determine $\hat{n} = (n_x, n_y, n_z)$ from ϕ and θ.

(R_x, R_y, R_z are defined in lecture note 4.)

(2) In lecture 4, we have proved that

$$\begin{aligned} U_x(\theta)^\dagger \vec{\sigma} U_x(\theta) &= R_x(\theta)\vec{\sigma} \\ U_y(\theta)^\dagger \vec{\sigma} U_y(\theta) &= R_y(\theta)\vec{\sigma} \\ U_z(\theta)^\dagger \vec{\sigma} U_z(\theta) &= R_z(\theta)\vec{\sigma}. \end{aligned}$$

In problem 5(2), we define $U_{\hat{n}}(\theta)$ as a generalization of U_x, U_y and U_z.

Prove that

$$U_{\hat{n}}(\theta)^\dagger \vec{\sigma} U_{\hat{n}}(\theta) = R_{\hat{n}}(\theta)\vec{\sigma}.$$

(When $R_{\hat{n}}(\theta)$ is applied to $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, σ_i is considered to be an element of a vector rather than a matrix.)

(hint: You may find the equation in problem 5(2) helpful.)

7

(1) Show that any single-qubit unitary operator can be written in the form

$$U = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2}. \end{bmatrix}$$

(α, β, γ and δ are real numbers.)

(2) Prove that for any unitary operator U on a single qubit, there exist real numbers $\theta_1, \theta_2, \theta_3, \theta_4$ such that

$$U = e^{i\theta_1}U_z(\theta_2)U_y(\theta_3)U_z(\theta_4).$$

8

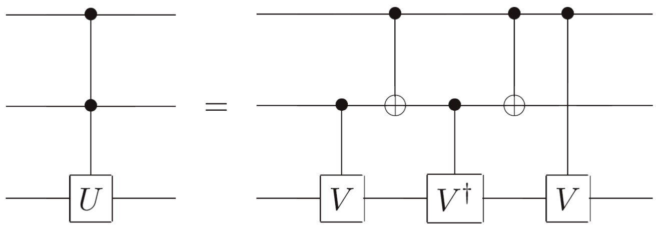

Fig 2: Circuit for Controlled-Controlled-U

(1) Suppose U is a single-qubit unitary operator, and V is a unitary operator chosen so that $V^2 = U$. Then the operation $C^2(U)$ (i.e. Controlled-Controlled-U) may be implemented using the circuit shown in fig 2.

Verify this construction works.

(2) **Suppose you can implement any single-qubit operation using one single-qubit gate**, prove that a Controlled-Controlled-U gate (for a single-qubit unitary U) can be constructed using at most eight single-qubit gates and six $CNOT$ gates.

(hint: In lecture 4, we give a construction of Controlled-U gate using single-qubit gate and $CNOT$ gate.)