HOMEWORK SET 1
SPRING 2021

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* *Due Monday March 29, 2021.*

1. Recall that a group G is said to be *residually finite* if for every nontrivial $g \in G$, there exists a finite group quotient $G \to Q$, such that the image $\bar{g} \in Q$ of g is nontrivial. Show that every finitely generated abelian group is residually finite.

2. For any positive integer n, show that every finitely generated group G has at most finitely many subgroups of index n.

3. If G is finitely generated and residually finite, show that every surjective homomorphism $G \to G$ is an isomorphism. How about injective homomorphisms?

4. Let $F = \Sigma_{1,1}$ be an oriented one-holed torus. Let $J_\pm = F \times S^1$ be two copies of the product 3-manifold of F with an oriented circle. Construct a closed oriented 3–manifold M as $J_+ \cup_h J_-$, such that $h\colon \partial J_+ \to \partial J_-$ is an orientation-reversing homeomorphism, identifying $\partial F \times *$ with $* \times S^1$, and $* \times S^1$ with $\partial F \times *$.
   (1) Compute the abelianization of the fundamental group $\pi_1(M)$.
   (2) Use Poincaré duality to deduce the homology groups $H_*(M;\mathbb{Z})$.
   (3) Does $\pi_1(M)$ have nontrivial center?

5. Let $\Sigma = \Sigma_g$ be an orientable closed surface of genus g. Let $M = PT\Sigma$ be the projectivized tangent bundle over Σ, with fibers isomorphic to real projective lines. (The points in M are the 1–dimensional linear subspaces of the tangent spaces $T_x\Sigma$, where x ranges over Σ.)
   (1) Compute $H_*(M;\mathbb{Z})$.
   (2) Show that the universal covering space of M is homeomorphic to $\mathbb{R}^3$ or S^3.