Reading Assignments and Notes: Introduction to Hyperbolic Geometry

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Spring 2020
Reading Assignment: Weeks 15 and 16

To get a better idea of rigidity you need to understand flexibility. In general, this can be a very large topic involving a lot of important results. However, in this course you are only asked to learn some basics of them, as the reading assignment for Weeks 15 and 16 below:

▶ [Thu, Chapter 5, Sections 5.2–5.5]

You should focus on Theorem 5.6 in [Thu, Section 5.5]. With this theorem, in principle you should be able to build up a complete proof of the Jørgensen–Thurston theorem, based on all the components you have in hand. (See Neumann–Zagier’s paper we mentioned before or [BP, Chapter E].)
Space of deformations

One way to motivate space of deformations is to look at the figure-eight knot example again. In general, suppose you have a complete hyperbolic 3–manifold $M$ with a geodesic ideal triangulation $\mathcal{T}$. Then you obtain a solution of the parameters $z_i$ for the (marked) tetrahedra, which satisfies the equations imposed by the edge cycles. If $M$ has cusps, then there should also be some nearby solutions which correspond to complete hyperbolic structures on the hyperbolic Dehn fillings $M_\gamma$ of $M$.

Intuitively, it is tempting to say that those $M_\gamma$ are obtained from “deforming the hyperbolic structure” of $M$, and that those $M_\gamma$ “converge to” $M$ as the length of the filling slopes tends to infinity. We would like to invent suitable terminology and mathematically make sense of the above words.
Denote by $\mathcal{W}$ the space of solutions to the edge-cycle equations with unknowns $z_i$, such that $\text{Im}(z_i) > 0$.

For each point $z \in \mathcal{W}$, the developing map gives rise to a holonomy representation $\rho_z : \pi_1(M) \to \text{PSL}(2, \mathbb{C})$. So one way to make sense of the convergence is to say (and to prove) that $\rho_{z_\gamma}(g) \to \rho_{z_{\text{cusp}}}(g)$ in $\text{PSL}(2, \mathbb{C})$ for every $g \in \pi_1(M)$. This is what we call **algebraic convergence** of (sequences of) $\text{PSL}(2, \mathbb{C})$–representations of $\pi_1(M)$.

However, we may also say that $M_\gamma$ converge to $M$ in the sense that $\rho_{z_\gamma}(\pi_1(M))$ converge to $\rho_{z_{\text{cusp}}}(\pi_1(M))$ in $\text{PSL}(2, \mathbb{C})$ as subsets. This means every $\rho_{z_{\text{cusp}}}(g)$ is the limit of some sequence $\rho_{z_\gamma}(g_\gamma)$. This is what we call **geometric convergence** of (sequences of) subgroups of $\text{PSL}(2, \mathbb{C})$. 
In the case of hyperbolic Dehn fillings, $M_\gamma$ converge to $M$ both in the algebraic sense and the geometric sense, and we say that they strongly converge to $M$. After choosing base points of $M$ and $M_\gamma$ (suitably in some $\epsilon$–thick part), the strong convergence is equivalent to the Gromov–Hausdorff convergence for metric spaces.

So you see that one may come up with various notions of flexibility. In the literature, they lead to several different topics that got developed independently. One that relates to geometric convergence is the Chabauty topology (see BP Section E.1). Algebraic convergence gives rise to the analytic topology (the “usual” point-set topology) on the representation variety (see M. Culler and P. Shalen “Varieties of group representations and splittings of 3-manifolds” for a good introduction).
What Thurston calls the space of deformation is roughly “a small neighborhood of $\mathcal{W}$ near a complete solution $z_{\text{cusp}}$”, and what interests us is the dimension of $\mathcal{W}$ near that point. This dimension is figured out in Theorem 5.6.

In the literature, however, the notion $\text{Def}(M)$ is not so commonly used, except in 3-dimensional hyperbolic geometry. Thurston didn’t introduce it with a formal definition, and you may consult BP E.6-iii for more rigorous treatment.