Reading Assignments and Notes: Introduction to Hyperbolic Geometry

Yi Liu

BICMR, Peking University

Spring 2020

Reading Assignment: Weeks 7 and 8

The reading assignment for Weeks 7 and 8 is about the metric completion of those incomplete hyperbolic structures on the figure-eight knot complement.

- [Thu, Chapter 4, Sections 4.4–4.6]
- [Rat, Section 10.5 (the rest from last time)]

You will see that all but finitely many Dehn fillings of the figure-eight knot complement admit complete hyperbolic structures. [Thu, Sections 4.8 and 4.10] are more of a topological flavor. You may need some knowledge of 3–manifold topology to understand them. So you are encouraged to read these sections, but not required.

While you read...

Keep in mind that the figure-eight knot complement is our basic example of a finite-volume hyperbolic 3–manifold. You see through the very concrete computations that hyperbolic structures on the Dehn fillings are intimately related to the deformation of the complete hyperbolic structure. Make sure that you understand all the details of the computation in this example.

Then, ask yourself the following questions:

- Can you generalize the computations to other 3–manifolds with an ideal triangulation? Which steps always work, and which steps rely on specific features of the example?
- What generalization can you reasonably conjecture from the figure-eight knot example?

While you read...

A satisfactory answer to the above questions will lead to the Jørgensen–Thurston picture of finite-volume hyperbolic 3–manifolds.

The following paper might look a bit technical to you at this point, but it essentially covers all the important points in the generalization:

- W. Neumann, D. Zagier: Volumes of hyperbolic three-manifolds. Topology 24 (1985), no. 3, 307–332.

The treatment of Benedetti and Petronio actually follows the approach of that paper.

Try to see if you can locate the answers to the above questions in that paper.

Extra reading

In general, it is possible to glue up finitely many n–dimensional ordinary hyperbolic polyhedra (that is, without ideal vertices). How to make sure that the resulting n–dimensional polyhedral complex is a closed hyperbolic n–manifold? Moreover, can one read off a finite presentation of the fundamental group in this case?

These questions are answered by the Poincaré polyhedron theorem.

Roughly speaking, you need the link of any k–dimensional face to be isomorphic to $\mathbb{S}^{n-k-1}$, so it involves a combinatorial topological condition together with a geometric condition analogous to "the sum of dihedral angles $= 2\pi$". The fundamental group is generated by the side-pairing transformations. The relators are listed by the edge-cycles around the edges (i.e. codimension–2 faces).

In particular, the gluing in codimension ≥ 3 does not affect the fundamental group.

Extra reading

In this course, we will not go much into the details of the theorem. You will need some combinatorial topological terminology to set it up. The proof is again of a combinatorial flavor, but not extremely hard.

If you have time after reading the assigned materials, read the following part of Ratcliffe's book about the Poincaré polyhedron theorem:

- [Rat, Theorems 11.2.1 and 11.2.2]

You may want to unwrap the cases $n = 2, 3$ to understand the idea of the proof.