Reading Assignments and Notes: Introduction to Hyperbolic Geometry

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Reading Assignment: Weeks 5 and 6

In Weeks 5 and 6, we aim at understanding the complete hyperbolic structure on the figure-eight knot complement. The reading assignment is as follows.

- [Thu, Chapter 4, Sections 4.1–4.3]
- [Rat, Sections 8.5 and part of 10.5 (up to the uniqueness of complete hyperbolic structures)]

The first three sections of [Thu, Chapter 4] constructs the hyperbolic structure on a figure-eight knot complement, and establishes the uniqueness. The same treatment is elaborated in [Rat, Section 10.5]. However, to understand the main concern of this part, that is, completeness, you need to read [Rat, Section 8.5], which translates the usual (Riemannian) metric geometric notion into a convenient form for transformation geometries.

Hyperbolic structures on the figure-eight knot complement

There is a basic remark that you need to keep in mind: When people say that a compact 3–manifold is hyperbolic, they usually means that the interior of the 3–manifold admits a complete hyperbolic structure. However, this is not the case when people talk about hyperbolic structures on a 3–manifold with an ideal triangulation.

In the latter context, a hyperbolic structure is often allowed to be incomplete, and in particular, this applies to the materials that you are going to read in the coming two weeks.

Hyperbolic structures on the figure-eight knot complement

You have already seen that the figure-eight knot complement admits an ideal triangulation. Or, see [Rat, Section 10.3] if you have not done so.

To obtain a possibly incomplete hyperbolic structure, it is natural to try to hyperbolize each of the ideal tetrahedra, which means to identify it with a geodesic ideal hyperbolic tetrahedron. We must require that for any edge of the glued-up manifold, the dihedral angles around that edge should sum up to be 2π. In this way, at least any point on the edge will have a neighborhood isometric to a hyperbolic 3–ball of some small radius.

The above condition will be translated into a system of polynomial equations, with unknowns each corresponds to a tetrahedron.

You will see that the shape (=isometric type) of any ideal hyperbolic geodesic tetrahedron can be described with a complex number z above the real axis. There are some redundancy, for example, $(z-1)/z$ and $1/(1-z)$ describe the same shape with permuted vertex labelings. The equations are then written down to describe the dihedral angle sum conditions, for the edges of the glue-up manifold.

Finding a hyperbolic structure therefore amounts to finding a solution to these equations. The first part of [Rat, Section 10.5] studies the complete hyperbolic structure obtained from its corresponding solution.

How about the incomplete hyperbolic structures?

You will also obtain a space of solutions to the gluing equations, and all the other solutions describe an incomplete hyperbolic structure on the figure-eight knot complement.

It is a natural next step to consider metric completion of those incomplete structures. Then several possibilities may occur: The metric completion may be closed hyperbolic 3–manifolds, or 3–orbifolds, or some more nasty spaces that you don't want to study topologically. The first case will be the most interesting, leading to the so-called Dehn surgery invariants. This will be the next topic of our course, but we will spend another two weeks on it and investigate more general situations.