HOMEWORK SET 7
SPRING 2016

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* *Due Tuesday May 31, 2016.*

1. For matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$, prove the following trace identities $\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1})$.

2. For matrices $A, B \in \mathrm{SL}_2(\mathbb{C})$, prove the following statements:
   (1) If $\mathrm{tr}(A) = \mathrm{tr}(B) = 2$, and A, B do not have a common eigenvector, then $\mathrm{tr}(AB) \neq 2$.
   (2) If $\mathrm{tr}(A) \neq 2$, and A, B do not have a common eigenvector, then they can be conjugated so that A is diagonal, and B has an entry 1 at the upper-right place.

3. Let π be a group generated by two elements. If $\rho, \rho' : \pi \to \mathrm{SL}_2(\mathbb{C})$ are two irreducible representations such that $\chi_\rho(g) = \chi_\rho(g')$ for all $g \in \pi$, show that ρ is conjugate to ρ'. (*Remark*. In fact, this holds for any group π.)

4. Let $\rho : \pi \to \mathrm{SL}_2(\mathbb{C})$ be a representation of a finitely generated group π. Show that ρ is reducible if and only if for every element c of the commutator subgroup $[\pi, \pi]$, $\chi_\rho(c) = 2$. (*Hint*: For the 'if' direction, show that some 1-dimensional subspace L of $\mathbb{C}^2$ is preserved by all $c \in [\pi, \pi]$, then it follows that L is preserved by π under ρ.)