HOMEWORK SET 2
SPRING 2017

INSTRUCTOR: YI LIU

* *Due Wednesday March 22, 2017.*

1. Suppose that X is a locally compact Hausdorff topological space and G is a group which acts properly discontinuously on X by homeomorphisms. (Recall that the action being *properly discontinuous* means that every compact subset $K \subset X$ meets only finitely many conjugates $g.K$ for all $g \in G$.) Prove the following statements:

(1) For every point $x \in X$, the stabilizer $\mathrm{Stab}_G(x)$ is a finite subgroup of G.
(2) Every point $x \in X$ has an open neighborhood U such that $g.U = U$ for every $g \in \mathrm{Stab}_G(x)$, and that $g.U \cap U = \emptyset$ for all other $g \in G$.
(3) The quotient space X/G is locally compact and Hausdorff.

2. The *modular group* of fractional linear transformations is defined to be:

$$\Gamma = \left\{ z \mapsto \frac{az+b}{cz+d} \colon a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

which acts on the upper half plane $U^2 = \{z \in \mathbb{C} \colon \mathrm{Im}(z) > 0\}$.

(1) Show that Γ is generated by the transformations $S : z \mapsto z + 1$ and $T : z \mapsto -1/z$. (Hint: Use the idea that any matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ can be reduced to the row echelon form
$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$
by elementary row operations.)
(2) Show that Γ acts properly discontinuously on U^2.
(3) Show that the following region
$$F = \{z \in U^2 \colon -1/2 < \mathrm{Re}(z) \le 1/2, |z| > 1\} \cup \{z \in U^2 \colon 0 \le \mathrm{Re}(z) \le 1/2, |z| = 1\}$$
is a fundamental polygon of Γ, namely, every point $z \in U^2$ which is not the fixed point of a nontrivial elliptic element is contained in $g.F$ for exactly one $g \in \Gamma$.

3. Consider the following surface defined for $(x, y, z) \in \mathbb{R}^3$ and $z > 0$:

$$S \colon 3(x^2 + y^2) = z.$$

Let S be equipped with the path-induced metric from the standard Euclidean metric of $\mathbb{R}^3$. Explicitly construct a developing map $D \colon \widetilde{S} \to \mathbb{E}^2$. Is D is injective? Sujective?