HOMEWORK SET 2
SPRING 2016

INSTRUCTOR: YI LIU

* *Due Tuesday March 22, 2016.*

1. Suppose that X is a locally compact Hausdorff topological space and G is a group which acts properly discontinuously on X by homeomorphisms. (Recall that the action being *properly discontinuous* means that every compact subset $K \subset X$ meets only finitely many conjugates $g.K$ for all $g \in G$.) Prove the following statements:

   (1) For every point $x \in X$, the stabilizer $\mathrm{Stab}_G(x)$ is a finite subgroup of G.
   (2) Every point $x \in X$ has an open neighborhood U such that $g.U = U$ for every $g \in \mathrm{Stab}_G(x)$, and that $g.U \cap U = \emptyset$ for all other $g \in G$.
   (3) The quotient space X/G is locally compact and Hausdorff.

2. The *modular group* of fractional linear transformations is defined to be:

$$\Gamma = \left\{ z \mapsto \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{Z}, ad-bc=1 \right\},$$

which acts on the upper half plane $U^2 = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

   (1) Show that Γ is generated by the transformations $S : z \mapsto z+1$ and $T : z \mapsto -1/z$. (Hint: Use the idea that any matrix $A \in \mathrm{SL}(2,\mathbb{Z})$ can be reduced to the row echelon form

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

   by elementary row operations.)
   (2) Show that Γ acts properly discontinuously on U^2.
   (3) Show that the following region

$$F = \{z \in U^2 : -1/2 < \mathrm{Re}(z) \leq 1/2, |z| > 1\} \cup \{z \in U^2 : 0 \leq \mathrm{Re}(z) \leq 1/2, |z| = 1\}$$

   is a fundamental polygon of Γ, namely, every point $z \in U^2$ which is not the fixed point of a nontrivial elliptic element is contained in $g.F$ for exactly one $g \in \Gamma$.

3. Continue with the notations of Problem 2. The level 2 *principal congruence subgroup* of Γ is defined to be:

$$\Gamma(2) = \left\{ \left(z \mapsto \frac{az+b}{cz+d} \right) \in \Gamma \,:\, a, d \text{ odd}, \, b, c \text{ even} \right\}.$$

   (1) Show that $\Gamma(2)$ acts freely on U^2.
   (2) Show that $\Gamma(2)$ is a normal subgroup of Γ finite index.
   (3) Find the area of the hyperbolic surface $U^2 \,/\, \Gamma(2)$. Can you identify its homeomorphism type?

4. A *limit point* of a transformation $g \in \mathrm{Isom}_+(\mathbf{H}^2)$ is a fixed point of its induced action on $\partial_\infty \mathbf{H}^2$. Writing g as a fractional linear transformation of the upper half plane U^2, a limit point of g is some $r \in \mathbb{R} \cup \{\infty\}$ that is fixed under its formula. Observe that every nontrivial transformation g has exactly 0, 1, or 2 limit points if it is elliptic, parabolic, or hyperbolic, respectively.

   Show that if a hyperbolic g and a parabolic h have a common limit point, then the group $\langle g, h \rangle$ cannot act properly discontinuously on U^2. (Hint: Equivalently, show that $\langle g, h \rangle$ is not discrete in $\mathrm{PSL}(2,\mathbb{R})$. By conjugation, one can reduce to the concrete case that $g : z \mapsto z+1$, and $h : z \mapsto \lambda z$ for some $\lambda > 1$.)