HOMEWORK SET 1
SPRING 2017

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* *Due Wednesday March 8, 2017.*

1. Using the distance formula

$$\mathrm{d}_B(u,v) = \operatorname{arccosh}\left(1 + \frac{2|u-v|^2}{(1-|u|^2)(1-|v|^2)}\right)$$

for the conformal disk model $B^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < 1\}$ with the Riemannian metric $\mathrm{d}s_B = 2|\mathrm{d}x|/(1-|x|^2)$, derive the distance formula for the upper half space model $U^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ and the upper half space model of the hyperbolic 3-space $\mathbb{H}^3$ with the Riemannian metric $\mathrm{d}s_U = |\mathrm{d}x|/x_3$.

2. Denote by $B(r)$ the disk of radius r in $\mathbb{H}^2$. Compute the area of $B(r)$ and the length of $\partial B(r)$. Conclude that

$$\mathrm{Area}_{\mathbb{H}^2}(B(r)) \,<\, \mathrm{Length}_{\mathbb{H}^2}(\partial B(r)).$$

3. Consider the following fractional linear transformation of the extended complex plane $\widehat{\mathbb{C}}$:

$$f(z) = \frac{2iz+1}{z-i}.$$

Explicitly describe the isometry ϕ of $\mathbb{H}^3$ induced by f via the Poincaré extension. Namely, identify enough geometric features of ϕ (such as the type, the axis, the rotation angle or translation distance, and so on) so that ϕ can be completely determined.

4. Using the upper half space model $\mathbf{U}^3$, show that every element $g \in \mathrm{Isom}_+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$ can be written as a product of factors

$$g = kan$$

where k is a rotation about $(0,0,1)$, and a is a loxodromic transformation that fixes 0 and ∞ on $\partial_\infty \mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$, and n is a parabolic transformation that fixes ∞. (This is called the *Iwasawa Decomposition* of $\mathrm{Isom}_+(\mathbb{H}^3)$.)