HOMEWORK SET 1
SPRING 2016

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* *Due Tuesday March 8, 2016.*

1. Denote by $\mathbf{U}^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ the upper half space model of the hyperbolic 3-space $\mathbb{H}^3$ with the metric $\mathrm{d}s_{\mathbf{U}} = |\mathrm{d}x|/x_3$, and by $\mathbf{B}^3 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| < 1\}$ the unit ball model where the metric $\mathrm{d}s_{\mathbf{B}} = 2|\mathrm{d}x|/(1 - |x|^2)$. Find an explicit isometry between $\mathbf{U}^3$ and $\mathbf{B}^3$.

2. Denote by $B(r)$ the disk of radius r in $\mathbb{H}^2$. Compute the area of $B(r)$ and the length of $\partial B(r)$. Conclude that

$$\mathrm{Area}_{\mathbb{H}^2}(B(r)) < \mathrm{Length}_{\mathbb{H}^2}(\partial B(r)).$$

3. Prove that all horospheres of $\mathbb{H}^3$ are conjugate. That is, for any two horospheres S and S' of $\mathbb{H}^3$, there exists an isometry of $\mathbb{H}^3$ that takes S to S'.

4. For any isometric transformation $g \in \mathrm{Isom}_+(\mathbb{H}^3)$, define the *translation distance* of g to be

$$\mathrm{td}(g) = \inf_{x \in \mathbb{H}^3} d_{\mathbb{H}^3}(x, g.x).$$

Show that the translation distance is 0 if g is elliptic or parabolic, and strictly positive if g is loxodromic. For $g \in \mathrm{PSL}(2, \mathbb{R})$ with $\mathrm{tr}^2(g) > 4$, compute $\mathrm{td}(g)$ in terms of $\mathrm{tr}^2(g)$.

5. Using the upper half space model $\mathbf{U}^3$, show that every element $g \in \mathrm{Isom}_+(\mathbb{H}^3) \cong \mathrm{PSL}(2, \mathbb{C})$ can be written as a product of factors

$$g = kan$$

where k is a rotation about $(0, 0, 1)$, and a is a loxodromic transformation that fixes 0 and ∞ on $\partial_\infty \mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$, and n is a parabolic transformation that fixes ∞. This is known as the *Iwasawa Decomposition* of $\mathrm{Isom}_+(\mathbb{H}^3)$.