Erratum to “A characterization of virtually embedded subsurfaces in 3-manifolds”

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The purpose of this note is to report an incorrect formula in the author's paper [L]: The asserted Formula 7.2 is not true in general and there is a gap in its argument. Consequently, the statement of Corollary 1.3 is in fact false, and the construction of Example 8.1 has to be modified to meet some different requirements, (see [Sun] for a more general treatment). Apart from those damages, the results of [L] remain true as they were stated. The error has been recently pointed out to the author by Hongbin Sun.

The error can be pinned down at the sentence that “$\ell(l'_\delta)$ depends only on the carrying vertex v” [L, page 22], where Formula 7.2 is derived. We notice that throughout Section 7 there, the flow-time length ℓ is supposed to be understood with respect to the prescribed Nielsen–Thurston suspension flow structure. However, for the quoted sentence to be true, the length should be taken rather with respect to the suspension flow associated with S'_v. In general, the two suspension flows are isotopic as 1-foliations (unparametrized flows), but they are not necessarily the same flow in the parametrized sense (even up to rescaling).

The following proposition describes the exact situation when the quoted sentence holds true.

Proposition 1. *Let F_v be a (connected) compact oriented surface and $\theta_v : F_v \to F_v$ be a pseudo-Anosov or periodic automorphism. Denote by J_v the mapping torus of θ_v equipped with the suspension flow of θ_v. Suppose that $S_v \looparrowright J_v$ is an oriented properly immersed virtual fiber transverse to the θ_v–suspension flow. Let J'_v be a finite cover of J_v which contains an embedded lift S'_v of S_v, such that the monodromy is isotopic to a pseudo-Anosov or periodic automorphism of S'_v which fixes all the periodic points on $\partial S'_v$. Then the following statements are equivalent:*

1. *For all degeneracy slope l'_δ on $\partial J'_v$, the flow-time length $\ell(l'_\delta)$ with respect to the θ_v–suspension flow is constant independent of δ.*

2. *There exist a nonzero integer m_v and a properly embedded taut subsurface K'_v of J'_v with every component of $\partial K'_v$ parallel to degeneracy slopes, such that the equality*

$$m_v \cdot [S'_v] = [F'_v] + [K'_v]$$

   holds in $H_2(J'_v, \partial J'_v; \mathbb{Z})$, where F'_v is any elevation of F_v in J'_v.

3. *For any finite cover J''_v of J_v into which S_v and F_v can be elevated to be embedded fibers S''_v and F''_v, the class $[S''_v] \in H_2(J''_v, \partial J''_v; \mathbb{Q})$ is a nonzero rational multiple of $[F''_v]$ plus some class η with $\partial\eta \in H_1(\partial J''_v; \mathbb{Q})$ contained in the subspace spanned by the degeneracy slopes of J''_v.*

Proof. To see that the first statement implies the second, observe that the preimage F'_v of F_v in J'_v is a union of parallel copies of F'_v. By definition,

$$\ell(l'_\delta) = \mathbf{i}(l'_\delta, \partial\mathsf{F}'_v) = \frac{[J'_v : J_v]}{[F'_v : F_v]} \cdot \mathbf{i}(l'_\delta, \partial F'_v) = \frac{[J'_v : J_v]}{[F'_v : F_v]} \cdot \frac{\mathbf{i}(l'_\delta, \partial F'_v)}{\mathbf{i}(l'_\delta, \partial S'_v)}.$$

We may assume without loss of generality that S_v is oriented by the positive direction of the θ_v–flow, then the geometric intersection numbers agree with the algebraic intersection numbers on $H_1(\partial J'_v; \mathbb{Z})$. As $\ell(l'_\delta)$

is constant independent of δ, there exists a constant m_v such that $\partial([F'_v] - m_v \cdot [S'_v])$ intersects every $[l'_\delta]$ algebraically zero times. In fact, m_v is a nontrivial integer since the monodromy assumption for S'_v implies that $\mathbf{i}(S'_v, l'_\delta) = 1$. The claim equality holds for any taut subsurface K'_v that realizes $[F'_v] - m_v \cdot [S'_v]$.

To see that the second statement implies the third, denote by V'' the preimage of the subspace of $H_1(\partial J''_v; \mathbb{Q})$ generated by degeneracy slopes, for the homomorphism $\partial : H_2(J''_v, \partial J''_v; \mathbb{Q}) \to H_1(\partial J''_v; \mathbb{Q})$. We must show that $[S''_v] \in \mathbb{Q} \cdot [F''_v] + V''$. When J''_v is a regular finite cover of J_v factoring through J'_v, the property follows directly from the second statement. In general, we take a regular finite cover $\tilde{J}_v$ of J_v that factors through both J''_v and J'_v, then the transfer homomorphism gives rise to an embedding of $H_2(J''_v, \partial J''_v; \mathbb{Q})$ into $H_2(\tilde{J}_v, \partial \tilde{J}_v; \mathbb{Q})$, so that the image intersects $\mathbb{Q} \cdot [\tilde{F}_v] + \tilde{V}$ in exactly $\mathbb{Q} \cdot [F''_v] + V''$. As S''_v is a fiber, the transferred image of $[S''_v]$ is a multiple of $[\tilde{S}_v]$, which lies in $\mathbb{Q} \cdot [\tilde{F}_v] + \tilde{V}$ as we have argued. So $[S''_v] \in \mathbb{Q} \cdot [F''_v] + V''$.

To see that the third statement implies the first, we may take J''_v to be J'_v. Then ratio between the geometric intersection number $\mathbf{i}(l'_\delta, \partial F'_v)/\mathbf{i}(l'_\delta, \partial S'_v)$ is the constant m_v, independent of δ, so the calculation in the first part shows that $\ell(l'_\delta)$ is constant independent of l'_δ. □

For a periodic θ_v, the necessary and sufficient conditions of Proposition 1 are always satisfied: Indeed, K_v can be taken to be a collection of vertical annuli. In this case, [L, Formula 7.2] recovers the formula of Rubinstein–Wang [RW]. For a pseudo-Anosov θ_v,[L, Formula 7.2] only works for a more restricted class of flow-transverse subsurfaces: If every vertex subsurface of the almost fiber part is known to be constructed by virtually homologically perturbing the prescribing fiber class in a “pseudo-vertical direction” (the third condition of Proposition 1), Formula 7.2 remains valid. However, in the pseudo-Anosov case using methods such as [PW, Proposition 3.11], it is always possible to find some properly immersed flow-transverse subsurface S_v that violates the homological requirement of Proposition 1. We conclude that it is generally impossible to determine the spirality character for flow-transverse subsurfaces merely from the intersection pattern of curves on the JSJ tori.

References

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