Separability of surface subgroups for 3–manifold groups

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BICMR

Osaka, January 2016

“ The **topology** of 3–manifolds is largely determined by their fundamental **groups**.”

Dictionary

- A **3–manifold group** is the fundamental group of a connected 3–manifold. It determines the homeomorphism type of the 3–manifold if the 3–manifold is **orientable, closed, and aspherical**.
- A **surface subgroup** of a 3–manifold group is a subgroup isomorphic to the fundamental group of a connected closed surface. The inclusion determines a map (or an **immersion**) of the surface into the 3–manifold up to homotopy.
- The surface subgroup being **separable** roughly means that the immersion is **virtually embedded**.

We wonder that ...

Given an **immersion**

$$S \looparrowright M$$

provided

- M — a 3–manifold &
- S — a closed surface

whether there exists

- $\tilde{M}$ — a **finite cover** of M

such that the immersion lifts to be an **embedding**

$$S \hookrightarrow \tilde{M}.$$

We actually wonder that ...

Given a π_1–**injective immersion**

$$S \looparrowright M$$

provided

- M — an **orientable, closed, aspherical** 3–manifold &
- S — a **possibly nonorientable** closed surface

whether there exists

- $\tilde{M}$ — a **finite cover** of M

such that the immersion lifts to be an **embedding**

$$S \hookrightarrow \tilde{M}$$

up to homotopy.

We actually wonder that ...

Given an **essential immersion**

$$S \looparrowright M$$

provided

- M — an **orientable, closed, aspherical** 3–manifold &
- S — a **possibly nonorientable** closed surface

whether there exists

- $\tilde{M}$ — a **finite cover** of M

such that the immersion lifts to be an **embedding**

$$S \hookrightarrow \tilde{M}$$

up to homotopy.

“ When is an **essentially immersed** subsurface of a 3–manifold **virtually embedded**? ”

* Where does the question come from?

Particularly useful are **essentially embedded** subsurfaces, cutting along which induces nice decompositions of the 3–manifold and its fundamental group.

A 3–manifold which contains an essentially embedded subsurface is called a **Haken manifold**.

Virtual Haken Conjecture (VHC)

" Every closed 3-manifold with an infinite fundamental group has a finite cover which is Haken. "

- proposed by F. Waldhausen in 1970s
- proved by I. Agol in 2012

* The story about VHC

The hard case of VHC is for **hyperbolic** closed 3–manifolds.

Kahn–Markovic (2009): **“** There are **essentially immersed** closed subsurfaces in such 3-manifolds. **”**

VHC is implied if we know that essentially immersed closed subsurfaces are **virtually embedded** in **this** case.

Agol (2012): **“** Yes, indeed. **”**

* The botany of 3–manifolds

Closed 3–manifolds are connected sum of prime manifolds. And prime manifolds are geometric or non-geometric.

The Geometric–JSJ Decomposition

“ Non-geometric orientable closed aspherical 3–manifold can be cut into **hyperbolic** or **Seifert fibered** pieces along a canonical minimal collection essentially embedded **tori**.**”**

A non-geometric 3–manifold is a **graph manifold** if all the pieces are Seifert fibered, otherwise it is a **mixed manifold**.

Samples

For **geometric** 3–manifolds, all essentially immersed subsurfaces are virtually embedded. These include

- all hyperbolic 3–manifolds
- all Seifert fibered spaces
- torus bundles or semi-bundles with Anosov monodromy

Rubinstein–Wang (1998): **“** There are **graph manifolds** which contain essentially immersed subsurfaces that are **not** virtually embedded! **”**

Separability in a nutshell

Given a group G, a subgroup H is said to be **separable** if

- for any element $g \notin H$ of G,
  there exists a **finite quotient** $\phi : G \to Q$
  under which $\phi(g) \notin \phi(H)$.

For surface subgroups of 3–manifold groups,

Scott (1978): “ **Separability** implies **virtual embedded-ness**. ”

Przytycki–Wise (2013): “ And the **converse** is true, too. ”

Now we have a satisfactory dictionary.

" When is a **surface subgroup** of a 3–manifold group **separable**? "

A canonical decomposition

For an essentially immersed subsurface of a non-geometric 3–manifold, there exists a **canonically induced decomposition**,

- cutting the surface along a minimal collection of simple closed curves into pieces which are properly immersed in the JSJ pieces of the 3–manifold.

A canonical decomposition

For an essentially immersed subsurface of a non-geometric 3–manifold, there exists a **canonically induced decomposition**,

- cutting the surface along a minimal collection of **edge curves** into **vertex subsurfaces** which are **carried by** the JSJ pieces of the 3–manifold.

Types of vertex subsurfaces

A vertex subsurface carried by a **Seifert fibered** piece is

- **vertical** or
- **horizontal**

A vertex subsurface carried by a **hyperbolic** piece is

- **quasi-Fuchsian** or
- **geometrically infinite**

Morally, vertical and quasi-Fuchsian subsurfaces share the feature of being **quasi-convex**; horizontal and geometrically infinite subsurfaces share the feature of being **virtually fibered**.

Clues

Vertex subsurfaces are all virtually embedded in the carrier pieces.

The surface is virtually embedded if no horizontal or geometrically infinite vertex subsurfaces present.

Introduce the **almost fiber part** of the immersed surface to be the union of all **horizontal** and **geometrically infinite** vertex subsurfaces glued along all edge curves between them.

A nontrivial lemma shows that the surface must be virtually embedded as long as the almost fiber part is virtually embedded.

“ **How** exactly does an almost fiber fail to be virtually embedded? ”

Picture in the cover correpsonding to $\pi_1(S)$

Denote

- $S \looparrowright M$ — an essential immersion
- $\Phi(S) \subset S$ — the almost fiber part

The immersion lifts to be an embedding into the covering space $\tilde{M}_S$ corresponding to the image of $\pi_1(S)$:

$$S \hookrightarrow \tilde{M}_S.$$

Now each lifted vertex subsurface S_v is carried by a $\tilde{J}_v \subset \tilde{M}_S$.

Gluing the $\tilde{J}_v$ accordingly, the lifted $\Phi(S)$ is carried by a canonical submanifold:

$$\Phi(S) \hookrightarrow \tilde{M}_{\Phi(S)}.$$

What restricts the gluing

There exists a **flow** on

$$\tilde{J}_v \cong S_v \times \mathbb{R}$$

which is **unique** up to rescaling the flow speed.

If $\Phi(S)$ is virtually fibered, then $\tilde{M}_{\Phi(S)}$ must have a compact subsurface parallel to the lifted $\Phi(S)$, which is obtained by gluing other slices of those $\tilde{J}_v$.

Suppose we decide to match the time-1 slices to produce the desired parallel subsurface, then we have to **rescale the flow speed** to make the slices match along common boundary. It works only if a cocycle condition of compatibility is satisfied.

Theorem (L. 2015)

Let S be an essentially immersed closed subsurface of a closed orientable aspherical 3-manifold M. Denote by $\Phi(S)$ the almost fiber part of S.
Then there exists a naturally induced homomorphism:

$$\sigma : H^1(\Phi(S); \mathbb{Z}) \longrightarrow \mathbb{Q}^\times,$$

which depends only on the immersion. Moreover, S is virtually embedded if and only if the invariant σ takes values in $\{\pm 1\}$.

“ Is the invariant **reasonably** calculable? ”

The first answer

The invariant is **algorithmically computable**.
With appropriate input, separability of surface subgroup for 3-manifold groups is **decidable**.

A more delicate answer

If we know **how** the vertex subsurfaces of the almost fiber part **are virtually fibered**, then the invariant can be **efficiently** calculated.

For example, if M is the mapping torus of a surface automorphism with **the suspension flow** that is "standard" in each JSJ piece, and if S is **transverse** to the flow. Then the invariant can be computed using the degeneracy slopes on the boundary of the JSJ pieces, in a form closely similar to Rubinstein–Wang's original formula.

Corollary

Let $\phi : F \to F$ be an autormorphism of a closed orientable surface. If some fractional Dehn twist coefficient along a Nielsen–Thurston reduction curve does not vanish, then the mapping torus M_ϕ contains an essentially immersed subsurface which is not virtually embedded.

Thank **Y**ou