Bounded quasi-Fuchsian subsurfaces in closed hyperbolic 3-manifolds

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Background

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Every closed hyperbolic 3-manifold contains a closed π_1-injectively immersed quasi-Fuchsian subsurface.

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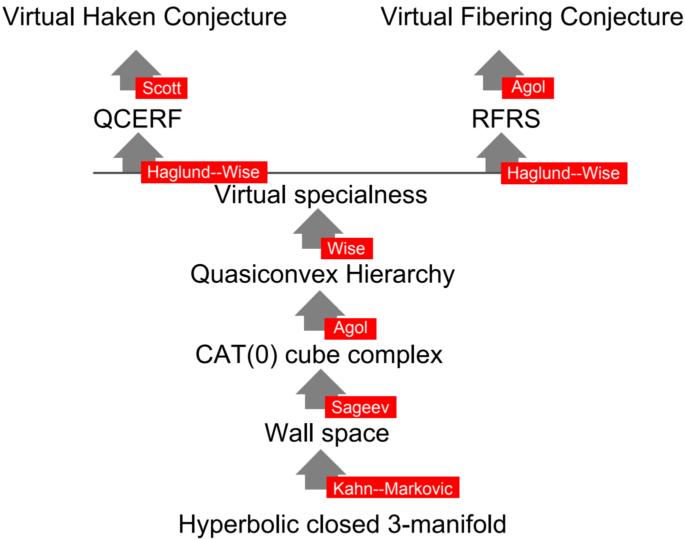
Relation to VHC and VFC



Relation to Ehrenpreis Conjecture

Theorem (Calegari)

Every rationally null-homologous, π_1-injectively immersed oriented closed 1-submanifold of a closed hyperbolic surface has an equidegree finite cover which bounds a π_1-injective immersed subsurface.

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- Calegari's result can be derived from Kahn–Markovic's good correction theory for closed hyperbolic surfaces with rational coefficients.
- The good pants construction of Kahn and Markovic together with their good correction theory imply a proof of the Ehrenpreis Conjecture.

Motivation

- Separating dynamical, geometrical, and topological ingredients of Kahn–Markovic construction.
- Generalizing the Good Correction Theory to closed hyperbolic 3-manifolds.
- Understanding the integral coefficient case.
- Addressing connectedness of the resulting subsurface.

Main result in special cases

Theorem (L.–Markovic)

Every rationally null-homologous, π_1-injectively immersed oriented closed 1-submanifold of a closed hyperbolic 3-manifold has an equidegree finite cover which bounds an oriented connected compact π_1-injective immersed quasi-Fuchsian subsurface.

Theorem (L.–Markovic)

Every rational second homology class of a closed hyperbolic 3-manifold has a positve integral multiple represented by an oriented connected closed π_1-injectively immersed quasi-Fuchsian subsurface.

Dichotomy

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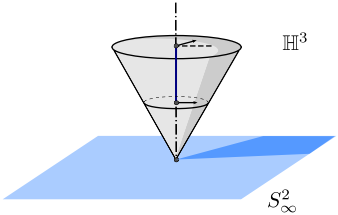
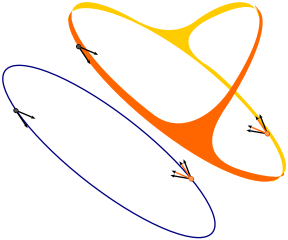
In fact, Kahn–Markovic subsurfaces are not only quasi-Fuchsian, but also nearly totally geodesic.

Good curves and good pants

In an orientable closed hyperbolic 3-manifold,

- An (R, ϵ)-good curve is a geodesic closed curve of approximately length R with approximately trivial monodromy up to error ϵ.
- An (R, ϵ)-good pair of pants is an immersed pair of pants which is approximately regular totally geodesic with cuff lengths R up to error ϵ.





Question

Why should there be good curves or pants?

Connection Principle

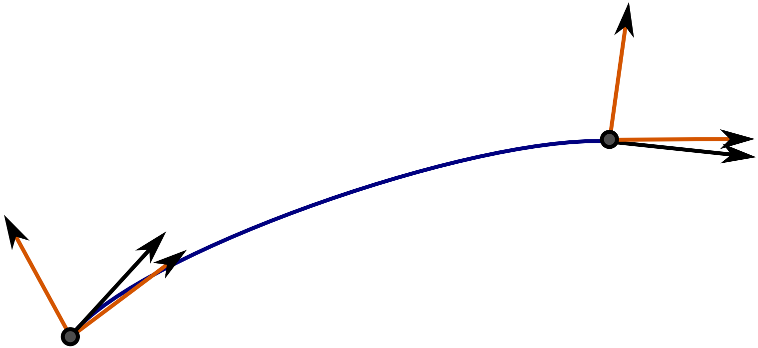
For any sufficiently small positive δ, and any sufficiently large prescibed length L depending on δ and M, the following Connection Principle is true:

- For any pair of precribed points p, q, and any prescribed directions and framings at these points, there exists a geodesic segment from p to q approximately having the prescribed geometric parameters up to error δ.

This follows from the mixing property of the frame flow of closed hyperbolic 3-manifolds.

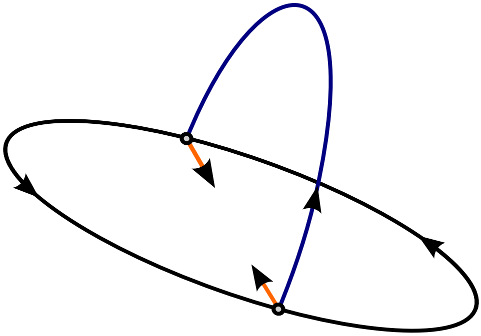
Illustration

- Drawing a segment with endpoint framings



Illustration

- Drawing a good pair of pants by splitting



Question

How to construct a quasi-Fuchsian subsurface?

Kahn–Markovic's recipe

The idea of the construction is

to glue up a collection of nicely distributed geometrically good pants along good curves in a well controlled way.

1. Take a finite collection $\mathcal{F}_0$ of good pants.
2. Make sure that the feet of cuffs are nearly evenly distributed along each good curve.
3. Match the cuffs up in a nearly 1-shearing fashion along common cuffs.

Discussion

- In fact, Kahn and Markovic have constructed $\mathcal{F}_0$ by constructing an integral measure μ_0 on the discrete finite set of oriented good pants $\mathbf{\Pi}_{R,\epsilon}$, where ϵ is sufficiently small depending on M, and R is sufficiently large depending on ϵ and M.

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- If $\mathcal{E}$ is a collection of good pants whose cuffs match up in oppositely oriented pairs, then $\mathcal{E} \sqcup \mathcal{F}_0$ is also such a collection.
- There is a reasonable generalization for bounded subsurfaces.

Good panted subsurfaces

In an orientable closed hyperbolic 3-manifold, an (R, ϵ)-panted subsurface is an immersed oriented compact subsurface with a decomposition into (R, ϵ)-pants. An (R, ϵ)-panted subsurface

- is not necessarily connected, or π_1-injective
- but will be after merging with $\mathcal{F}_0$

Question

Does every null-homologous (R, ϵ)-multicurve bound an (R, ϵ)-panted subsurface?

Good Panted Cobordism Group

$$\mathbf{\Omega}_{R,\epsilon}(M) = \frac{\{\text{oriented } (R,\epsilon)\text{-multicurves}\}}{\{\text{cobordism by } (R,\epsilon)\text{-panted subsurfaces}\}}$$

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- Kahn–Markovic's Good Correction Theory shows in dimension 2 that

  $$\mathbf{\Omega}_{R,\epsilon}(M) \otimes \mathbb{Q} \xrightarrow{\cong} H_1(M;\mathbb{Q})$$

Structure of good panted cobordism group

Theorem (L.–Markovic)

Let M be an oriented closed hyperbolic 3-manifold. For any universally small positive ϵ and for any sufficiently large positive R depending only on M and ϵ, there exists a canonical isomorphism

$$\mathbf{\Omega}_{R,\epsilon}(M) \cong H_1(\mathrm{SO}(M); \mathbb{Z}).$$

Here $\mathrm{SO}(M)$ denotes the total space of the bundle of special orthonormal frames of M.

- Moreover, for any (R, ϵ)-multicurve L in M, $[L]_{R,\epsilon}$ is sent to $[L]$ in $H_1(M; \mathbf{Z})$ under the natural projection $H_1(\mathrm{SO}(M); \mathbb{Z}) \to H_1(M; \mathbb{Z})$.

Canonical lifts

The positive direction

$$\Phi : \mathbf{\Omega}_{R,\epsilon}(M) \to H_1(\mathrm{SO}(M); \mathbb{Z})$$

is constructed by specifying the image for the good panted cobordism class of any good curve, which is represented by a field of frames over γ.

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- Start with any frame at a point p of γ. The field of frames $\hat{\gamma}$ over γ differs from the nearly parallel transportation lift by an extra element of $\mathrm{SO}(M)|_p$.

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- The field of frames $\hat{\gamma}$ defined in this way is canonical up to homotopy as $\pi_1(\mathrm{SO}(M)|_p) \cong \mathbb{Z}_2$.
- The canonical lift $\hat{\gamma}$ induces a well defined homomorphism of $\mathbf{\Omega}_{R,\epsilon}(M)$ to $H_1(\mathrm{SO}(M); \mathbb{Z})$.

Explanation

- $0 \to \mathbb{Z}_2 \to H_1(\mathrm{SO}(M); \mathbb{Z}) \to H_1(M; \mathbb{Z}) \to 0$

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- split because $\mathrm{SO}(M)$ is trivial
- the lift by nearly parallel transportation not working

The inverse of Φ

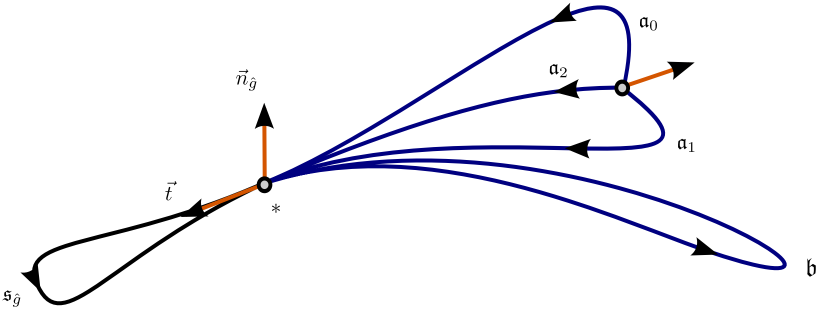
- Take a base frame $\mathbf{e}$ at a base point $*$ of M. Take a triangular finite presentation of $\pi_1(\mathrm{SO}(M), \mathbf{e})$.
- Conjugate so that the presentation complex satisfies certain nice geometrical conditions.
- Assign to each generator g a suitably chosen good multicurve $L(g)$.
- Assign to each simplicial relator σ a suitable chosen compatible good panted subsurface $F(\sigma)$.

As a consequence, there is a homomorphism

$$\Psi : \pi_1(\mathrm{SO}(M), \mathbf{e}) \to \mathbf{\Omega}_{R,\epsilon},$$

which factors through $H_1(\mathrm{SO}(M), \mathbf{e})$.

Illustration



- $\Psi(\mathfrak{s}_{\hat{g}}) = [\mathfrak{s}_{\hat{g}}\mathfrak{a}_{01}]_{R,\epsilon} + [\mathfrak{s}_{\hat{g}}\mathfrak{a}_{12}]_{R,\epsilon} + [\mathfrak{s}_{\hat{g}}\mathfrak{a}_{20}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}}\mathfrak{b}]_{R,\epsilon} - [\mathfrak{s}_{\hat{g}}\bar{\mathfrak{b}}]_{R,\epsilon}.$

Pantifying an integral second homology class

Similar idea applies to pantification of integral second homology classes. Suppose we have an incompressible subsurface S.

- Take a base point $*$ of M on the given incompressible subsurface S. Take a one-point triangulation of S.
- Conjugate so that S satisfies certain nice geometrical conditions.
- Assign to each 1-cell g a suitably chosen good multicurve $L(g)$.
- Assign to each simplicial 2-cell σ a suitable chosen compatible good panted subsurface $F(\sigma)$.

The second homology class does not change heuristically because we have only drawn auxiliary segments during the proof.

Further problems

- Quasi-Fuchsian subsurfaces in the cusped case
- Quasi-convex subgroups in word hyperbolic groups
- Good cobordism of quasi-Fuchsian subsurfaces
- Virtual cubing of hyperbolic manifolds

Thank you