Presentation Length and Simon's Conjecture

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Simon's Conjecture

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Conjecture (Problem 1.12.D in Kirby's List)

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In other words, every knot group surjects at most finitely many distinct knot groups.

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- **Boileau-Rubinstein-Wang 2005**: Restricting to non-zero degree maps;
- **Silver-Whitten 2006**: From a fibered knot complement;
- **Boileau-Boyer-Reid-Wang 2009**: From any two-bridge knot complement;
- **Horie-Kitano-Matsumoto-Suzuki 2009**: For knots up to 11 crossings; etc.

The Main Result

Theorem (Agol-L. 2010)

Suppose G is a finitely generated group of $b_1(G) = 1$. Then G surjects at most finitely many distinct knot groups.

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Note one may reduce to the case when G is finitely presented. Furthermore, after such reduction, the number of admissible targets is bounded, in fact, in terms of the presentation length of G.

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Q: Anything special about knot complements here?
A: They have very simple types of Seifert fibered pieces; and their JSJ tori are all separating; and there are desatellite maps for satellite knot complements.

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More specifically,

- 'finite presentation' $\rightsquigarrow$ the presentation length $\ell(G)$; and
- 'tinyness' $\rightsquigarrow$ that $b_1(G) = 1$.

The Strategy

When M is a knot complement, its homeomorphism type is determined the following data:

- The type of the rooted JSJ tree;
- The homeomorphism types of JSJ pieces;
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To prove Simon's conjecture, one must show the finiteness for each of the above. The key ingredients are the simplicial volume bound, and the factorization through extended drillings, which together allows us to make a drilling argument.

Presentation Length

Definition

Let G be a finitely presented group. For any finite presentation $\mathcal{P} = (x_1, \cdots, x_n; r_1, \cdots, r_m)$ of G with the word length $|r_j| \geq 2$, for $1 \leq j \leq m$, define:

$$\ell(\mathcal{P}) = \sum_{j=1}^{m} (|r_j| - 2).$$

We define the presentation length of G to be the minimum of $\ell(\mathcal{P})$ among all such presentations $\mathcal{P}$.

Roughly speaking, $\ell(G)$ counts the minimal number of triangles needed to construct a presentation 2-complex of G.

Motivational Results

The presentation length of the fundamental group bounds the geometry of a closed hyperbolic manifold:

Theorem (Cooper 1999)

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Cooper's proof used a drilling argument which inspires many interesting applications.

Simplicial Volume Bound

Theorem (Agol-L. 2010)

Let G be a finitely presented group with $b_1(G) = 1$, and M be a compact orientable aspherical 3-manifold. Suppose G surjects $\pi_1(M)$, then:

$$v_3\|M\| \leq \pi\,\ell(G),$$

where $v_3 \approx 1.01494$ is the volume of the regular ideal tetrahedron in $\mathbb{H}^3$, and $\|\cdot\|$ denotes the Gromov norm.

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This is the first ingredient in the proof of Simon's conjecture.

Drilling Argument: A Toy Example

Suppose M a hyperbolic knot complement, and $\gamma \subset M$ a simple closed geodesic. Let $N = M - \gamma$. Then the inclusion $i : N \hookrightarrow M$ induces:

$$i_\sharp : \pi_1(N) \twoheadrightarrow \pi_1(M).$$

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Exercise

Let M, N as above. Let G be a f.g. group with $b_1(G) = 1$, and $\phi : G \to \pi_1(M)$ be a homomorphism. Show that if ϕ factors through $\pi_1(N)$, i.e. $\phi = i_\sharp \circ \psi$ for some $\psi : G \to \pi_1(N)$, then ϕ is not surjective.

A Sketched Solution

Solution

1. Because $b_1(G) = 1$ and $b_1(N) = 2$, the covering $\kappa : \tilde{N} \to N$ corresponding to $\psi(G) < \pi_1(N)$ is infinite.

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This is a prototype of a drilling argument, motivating considerations of factorizations.

Factorization in Hyperbolic Piece

As we are already able to bound the simplicial volume, to bound the type of hyperbolic pieces, we expect something like this:

"Theorem"

If G is a finitely presented group, and M is a knot complement which has a hyperbolic piece containing a sufficiently short simple closed geodesic γ. Let $N = M - \gamma$. Then for any $\phi : G \to \pi_1(M)$, ϕ factors through $\pi_1(N)$.

Factorization in Seifert Fibered Piece

Similarly, to bound the Seifert fibered pieces, we expect:

"Theorem"

If G is a finitely presented group, and M is a knot complement which has a Seifert fibered piece containing an exceptional fiber γ over a sufficiently sharp cone point. Let $N = M - \gamma$. Then for any $\phi : G \to \pi_1(M)$, ϕ factors through $\pi_1(N)$.

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Unfortunately, these naive versions are not true as they stand. We need to introduce the notion of Dehn extensions N^e, which are "ridged manifolds".

Dehn Extensions

Let N be an aspherical orientable compact 3-manifold, and ζ be a slope on an incompressible torus boundary component $T \subset \partial N$. Identify $P = \pi_1(T)$ as a peripheral subgroup of $\pi_1(N)$, thus $\zeta \in P$ primitive. Also identify $P \cong \mathbb{Z} \oplus \mathbb{Z}$ as the integral lattice in $\mathbb{Q} \oplus \mathbb{Q}$.

Definition

For any integer $m > 1$, we define the Dehn extension of $\pi_1(N)$ along a slope ζ with denominator m as the amalgamated product:

$$\pi_1(N)^{e(\zeta,m)} = \pi_1(N) *_P \left(P + \mathbb{Z}\frac{\zeta}{m}\right).$$

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$$\pi_1(N)^{e(\zeta,m)} = \pi_1(N) *_P \left(P + \mathbb{Z}\frac{\zeta}{m}\right).$$

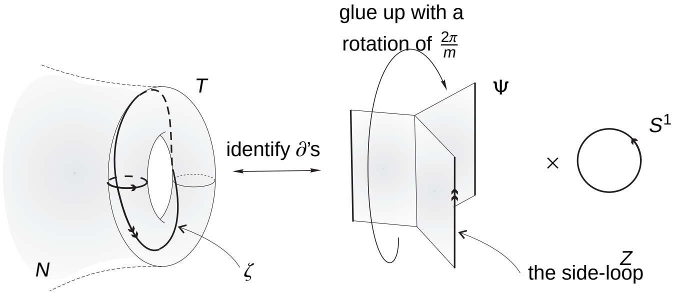
Heuristically, $\pi_1(N)^e$ is obtained from $\pi_1(N)$ by "adjoining roots of ζ".

Topological Viewpoint

From a topological point of view, $\pi_1(N)^e$ is the fundamental group of:

$$N^e = N \cup_T Z,$$

where $Z = Z(m)$ is a mapping cylinder of the covering between tori induced by $P \hookrightarrow P + \mathbb{Z}\frac{\zeta}{m}$.



Properties of Dehn Extensions

Despite some mild singularity, Dehn extensions behave like 3-manifolds in many ways:

- N^e is aspherical;
- N^e has an analogous JSJ decomposition;
- N^e has the same rational homology as that of N;
- $\pi_1(N^e)$ is coherent, indeed, every finitely generated covering has a Scott core.

Sketched Proof

Suppose G is finitely presented with $b_1(G) = 1$, and suppose M is a knot complement so that G surjects $\pi_1(M)$.

1. By a theorem of Weidmann, M has at most $4n - 3$ JSJ pieces, where n is the rank of G, so there are at most finitely many types of admissible JSJ trees.
2. The simplicial volume bound and the factorization imply the finiteness of homeomorphism types of admissible JSJ pieces.
3. Using desatellite maps, the finiteness of admissible choices of meridian-longitudes is reduced to the finiteness in the previous case.

Finally, we conclude that there are at most finitely many admissible homeomorphism types of M.

Thank You!