

A Jørgensen-Thurston Theorem for Homomorphisms

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- 5 volume decreasing under hyperbolic Dehn fillings.

The Jørgensen-Thurston Theorem

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For any $V > 0$, every orientable hyperbolic 3-manifold of volume at most V is homeomorphic to a Dehn filling of one of finitely many orientable hyperbolic 3-manifolds of volume at most V .

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This indicates the structure of the family of hyperbolic 3-manifolds with uniformly bounded volume should be easy to describe.

Our goal

Given $V > 0$, uniformly describe the structure of:

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Is it true for any N_ζ sufficiently close to N that any $\phi : G \rightarrow \pi_1(N_\zeta)$ factors through the Dehn filling $\iota_\zeta : \pi_1(N) \twoheadrightarrow \pi_1(N_\zeta)$?

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Answer: **no**, but **almost yes**.

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The question is motivated by the study of maps between 3-manifolds:

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Given a 3-manifold M , is it possible to describe the structure of maps (up to homotopy) from M to any 3-manifold N ?

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- 1 (Simon)** Does every knot group surject only finitely many distinct knot groups?
- 2 (Rong)** Does every orientable closed 3-manifold **1-dominate** at most finitely many homeomorphically distinct 3-manifolds?
- 3 (Waldhausen)** Is the Heegaard genus of a closed orientable 3-manifold bounded in terms of the rank of its fundamental group?

Another motivation

Rips, Sela, Groves, et al: the **Makanin-Razborov diagram** of:

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Our context is similar to theirs except the target becoming a family of groups. However, this makes some essential difference which gives rise to **Dehn extensions**.

Almost yes

Factorization through extended-fillings:

Theorem (L.)

Suppose G is a finitely generated group, and N is an orientable hyperbolic 3-manifold of finite volume. Then there are finitely many **Dehn extensions** of N along distinct slope-tuples, such that if a slope-tuple $\zeta = (\zeta^1, \dots, \zeta^q)$ avoids finitely many choices on each component, then any homomorphism $\phi : G \rightarrow \pi_1(N_\zeta)$ factors through one of the Dehn extensions via the **extended Dehn filling**.

An example of Dehn extensions

M : hyperbolic with a deep Margulis tube V .

N : $M - V$.

γ : a simple closed braid in V ,

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$$M' = M - \gamma = N \cup_{\partial V} Q.$$

Let $P = \pi_1(\partial V)$ and $P^e = H_1(Q)$, then:

$$\pi_1(N)^e = \pi_1(N) *_P P^e,$$

is a **Dehn extension** of $\pi_1(N)$.

Note Q is homeomorphic to the complement of a 2-component link, and $P^e = P + \mathbb{Z} \frac{\zeta}{m}$ for the meridian $\zeta \in P$.

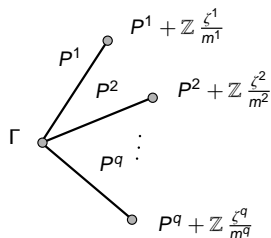
Dehn Extensions

In general, $\Gamma = \pi_1(N)$, and $P^j = \pi_1(T^j)$, $1 \leq j \leq q$.

$m = (m^1, \dots, m^q)$: a q -tuple of positive integers, and

$\zeta = (\zeta^1, \dots, \zeta^q)$: a slope-tuple.

The **Dehn extension** $\Gamma^{e(\zeta, m)}$ of Γ along ζ with denominator-tuple m is:



There is a canonical **extended-filling** epimorphism: $\iota^e : \Gamma^e \twoheadrightarrow \Gamma_\zeta$.



Factorization revisited

Theorem (L.)

Suppose G f.g., and N hyperbolic of finite volume. Then there are finitely many Dehn extensions of N , such that if $\zeta = (\zeta^1, \dots, \zeta^q)$ avoids finitely many choices on each component, then any $\phi : G \rightarrow \pi_1(N_\zeta)$ factors through one of these Dehn extensions.

Heuristically, if N_ζ has a sufficiently short geodesic depending on G , homomorphisms to $\pi_1(N_\zeta)$ factors homologically **as if** through a drilling of a braid homotopically covering the short geodesic.

A homomorphism Jørgensen-Thurston

Theorem (L.)

Suppose G f.g., and $V > 0$. Then:

$$\exists R_1, \dots, R_k,$$

each a Dehn extension of a torsion-free Kleinian group of covolume at most V , such that for any torsion-free Kleinian group H of covolume at most V , every $\phi : G \rightarrow H$ factors through some R_i .

Outline of the proof

- 1 Use the presentation length techniques to show the finitely presented case, (cf. Agol-L. 2010).

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- 2 Use hyperbolic geometry to show Dehn extensions are stable limits, (also cf. Agol-Groves-Manning 2008).
- 3 Use algebraic-geometric properties of representation varieties to show Dehn extensions embed in $SL(2, \mathbb{C})$, and deduce the f.g. case from the f.p. case.

Thank You!