A Jørgensen-Thurston Theorem for Homomorphisms

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5. volume decreasing under hyperbolic Dehn fillings.

The Jørgensen-Thurston Theorem

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This indicates the structure of the family of hyperbolic 3-manifolds with uniformly bounded volume should be easy to describe.

Our goal

Given $V > 0$, uniformly describe the structure of:

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Answer: no, but almost yes.

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The question is motivated by the study of maps between 3-manifolds:

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1. (**Simon**) Does every knot group surject only finitely many distinct knot groups?
2. (**Rong**) Does every orientable closed 3-manifold 1-dominate at most finitely many homeomorphically distinct 3-manifolds?
3. (**Waldhausen**) Is the Heegaard genus of a closed orientable 3-manifold bounded in terms of the rank of its fundamental group?

Another motivation

Rips, Sela, Groves, et al: the Makanin-Razborov diagram of:

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Our context is similar to theirs except the target becoming a family of groups. However, this makes some essential difference which gives rise to Dehn extensions.

Almost yes

Factorization through extended-fillings:

Theorem (L.)

Suppose G is a finitely generated group, and N is an orientable hyperbolic 3-manifold of finite volume. Then there are finitely many Dehn extensions of N along distinct slope-tuples, such that if a slope-tuple $\zeta = (\zeta^1, \cdots, \zeta^q)$ avoids finitely many choices on each component, then any homomorphism $\phi : G \to \pi_1(N_\zeta)$ factors through one of the Dehn extensions via the extended Dehn filling.

An example of Dehn extensions

M: hyperbolic with a deep Margulis tube V.
N: $M - V$.
γ: a simple closed braid in V,

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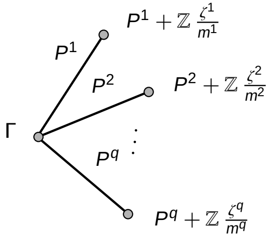
Let $P = \pi_1(\partial V)$ and $P^e = H_1(Q)$, then:

$$\pi_1(N)^e = \pi_1(N) *_P P^e,$$

is a Dehn extension of $\pi_1(N)$.
Note Q is homeomorphic to the complement of a 2-component link, and $P^e = P + \mathbb{Z}\frac{\zeta}{m}$ for the meridian $\zeta \in P$.

Dehn Extensions

In general, $\Gamma = \pi_1(N)$, and $P^j = \pi_1(T^j)$, $1 \leq j \leq q$.
$m = (m^1, \cdots, m^q)$: a q-tuple of positive integers, and
$\zeta = (\zeta^1, \cdots, \zeta^q)$: a slope-tuple.
The Dehn extension $\Gamma^{e(\zeta,m)}$ of Γ along ζ with denominator-tuple m is:



There is a canonical extended-filling epimorphism: $\iota^e : \Gamma^e \twoheadrightarrow \Gamma_\zeta$.

Factorization revisited

Theorem (L.)

Suppose G f.g., and N hyperbolic of finite volume. Then there are finitely many Dehn extensions of N, such that if $\zeta = (\zeta^1, \cdots, \zeta^q)$ *avoids finitely many choices on each component, then any* $\phi : G \to \pi_1(N_\zeta)$ *factors through one of these Dehn extensions.*

Heuristically, if N_ζ has a sufficiently short geodesic depending on G, homomorphisms to $\pi_1(N_\zeta)$ factors homologically as if through a drilling of a braid homotopically covering the short geodesic.

A homomorphism Jørgensen-Thurston

Theorem (L.)

Suppose G f.g., and $V > 0$. Then:

$$\exists R_1, \cdots, R_k,$$

each a Dehn extension of a torsion-free Kleinian group of covolume at most V, such that for any torsion-free Kleinian group H of covolume at most V, every $\phi : G \to H$ factors through some R_i.

Outline of the proof

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2. Use hyperbolic geometry to show Dehn extensions are stable limits, (also cf. Agol-Groves-Manning 2008).
3. Use algebraic-geometric properties of representation varieties to show Dehn extensions embed in $\mathrm{SL}(2, \mathbb{C})$, and deduce the f.g. case from the f.p. case.

Thank You!