Is the Composite Fermion a Dirac particle?

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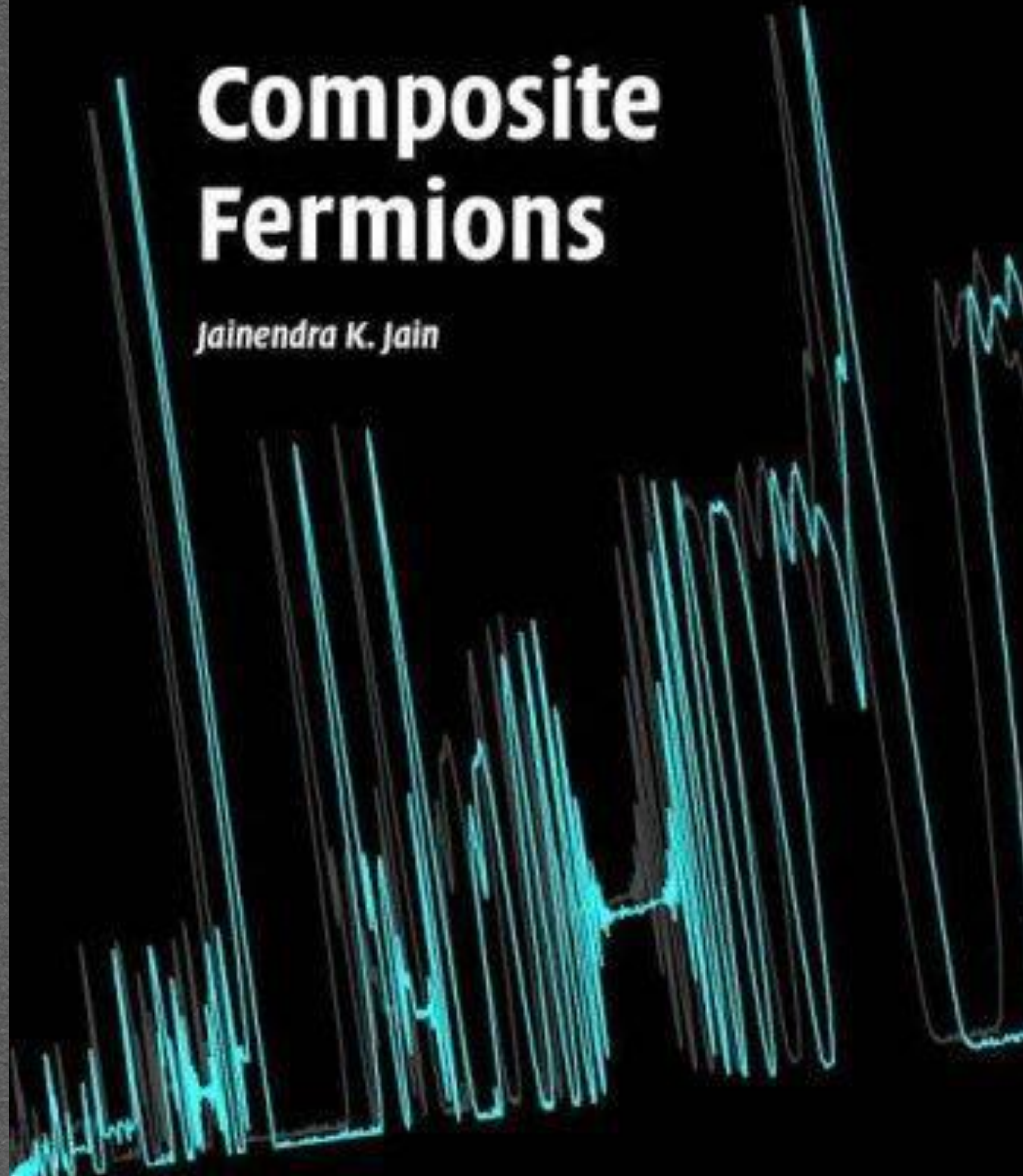
HKU, 1/25/2019

Composite Fermions

Jainendra K. Jain



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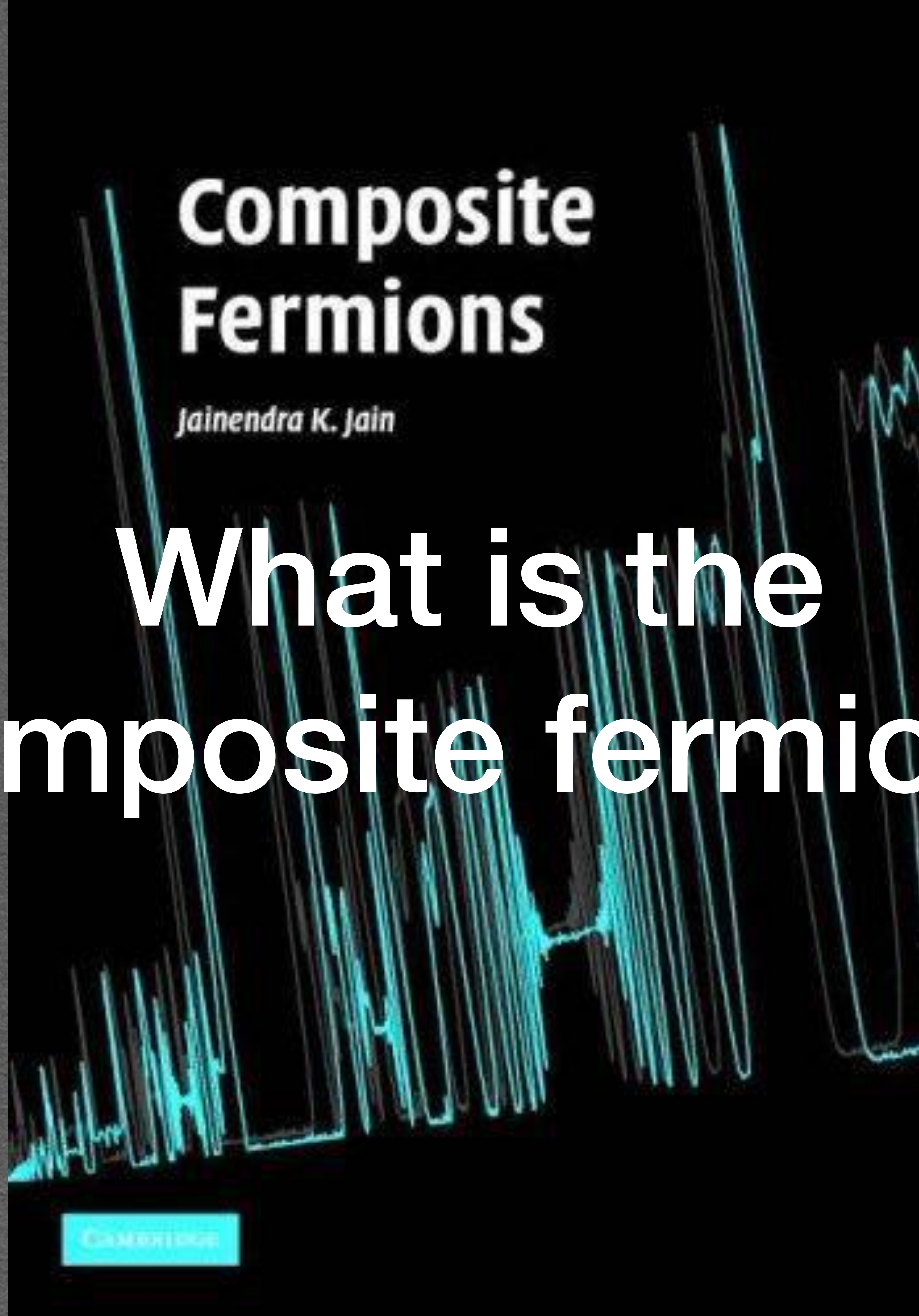


What is the composite fermion?

Outline

- Introduction
- Interpretations of the CF: Newtonian, Dirac or “Niuqian”?
- Evidences from microscopic wave functions
- Proper definition/evaluation of the Berry phase
- Segal-Bargmann transform and CF wave functions
- Conclusion and Outlook

JS and Wencheng Ji, PRB **97**, 125133 (2018); JS, arXiv:1704.07712 (2017); Guangyue Ji and JS, arXiv:1901.00321 (2019).

Composite Fermion

Physics of a partially filled Landau level

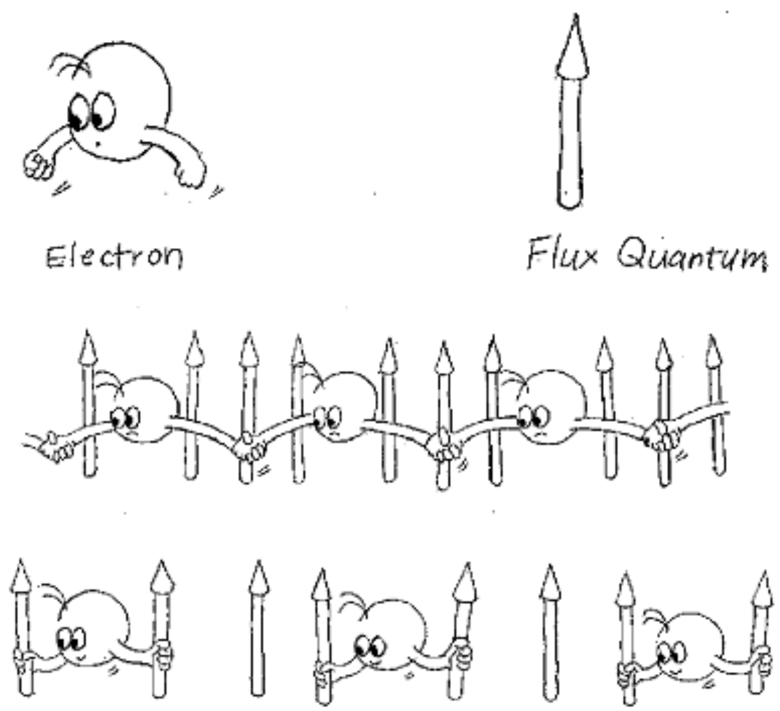
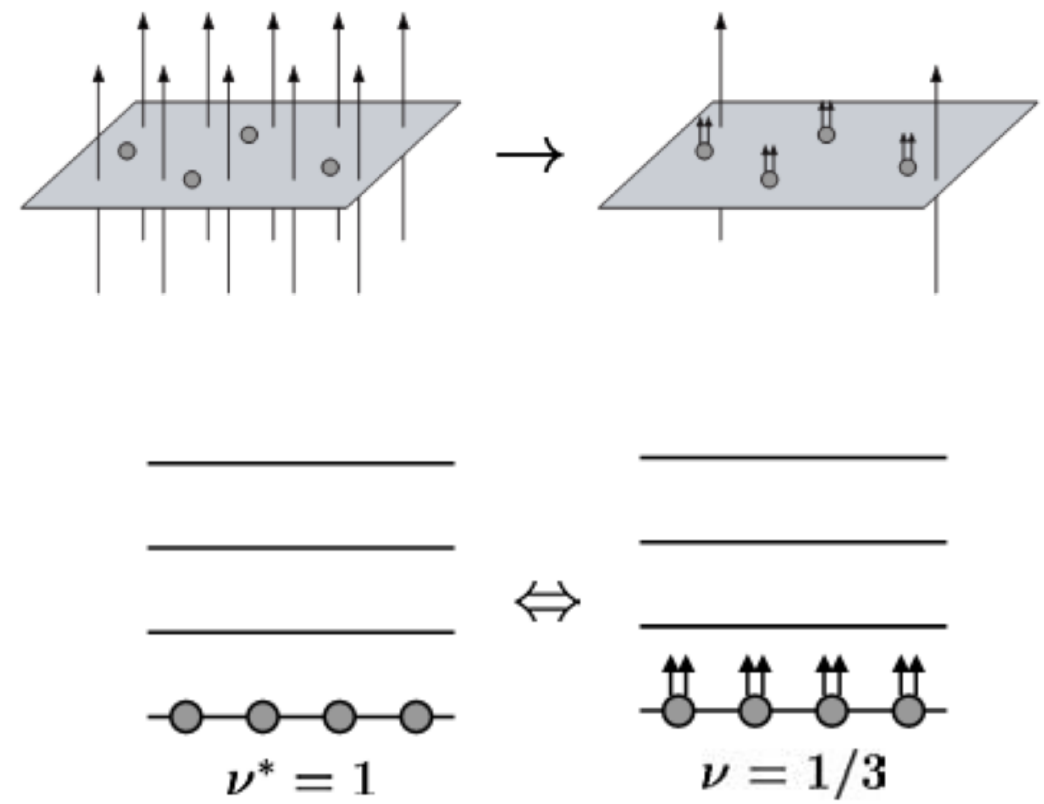


Fig. 5.7. A humorous view of composite fermions. Source: Kwon Park.



From J. K. Jain, *Composite Fermions* (Cambridge, 2007).

$$B_{\mathrm{CF}} = B - 2p\phi_0\rho_e \qquad \frac{1}{\nu} = \pm\frac{1}{\nu_{\mathrm{CF}}} + 2p$$

Hidden Hilbert space

$$\boxed{\Psi(z)} = \boxed{\hat{P}_{\rm LLL} J} \boxed{\psi(z)}$$

Physical Hilbert space ⟵ **Hidden Hilbert space**

$z = x + \mathrm{i}y$

CF Mapping

Bijl-Jastrow factor: $J(z) = \prod_{i<j} \left(z_i - z_j\right)^{2p}$

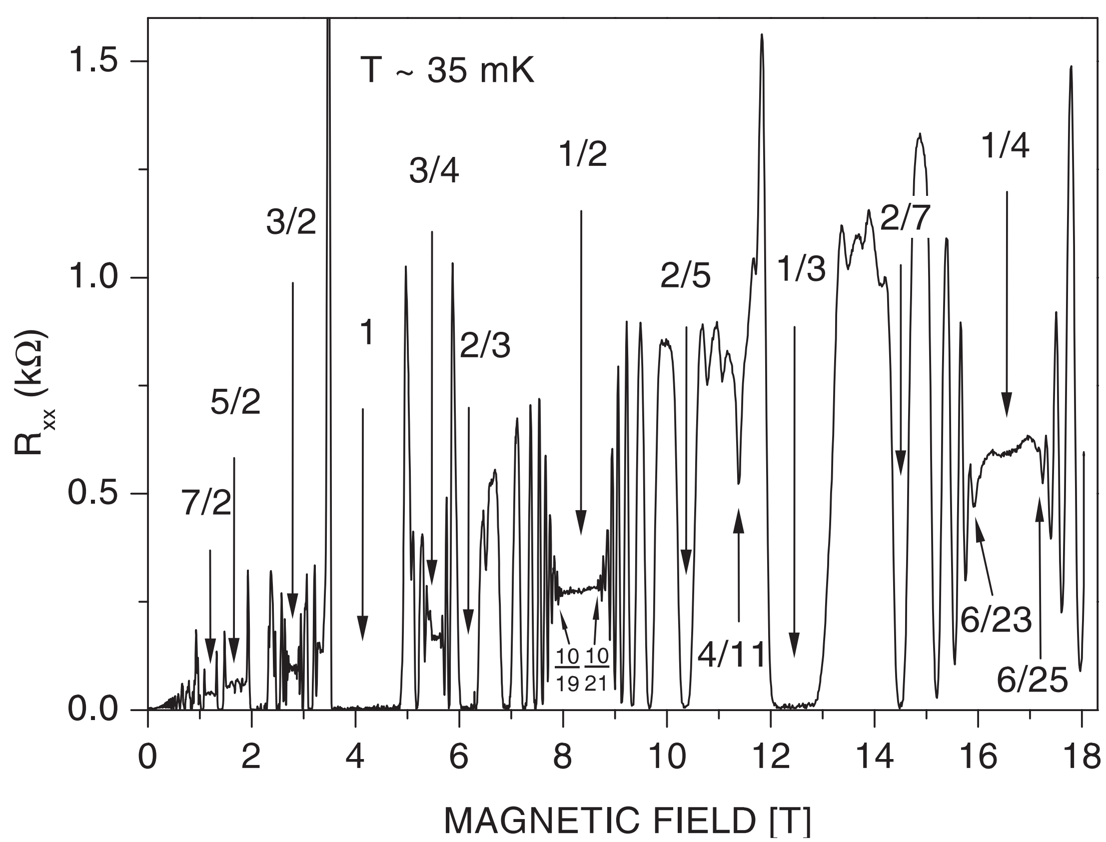
Projection to the LLL: $\hat{P}_{\rm LLL}: z^* \to 2\partial_z$

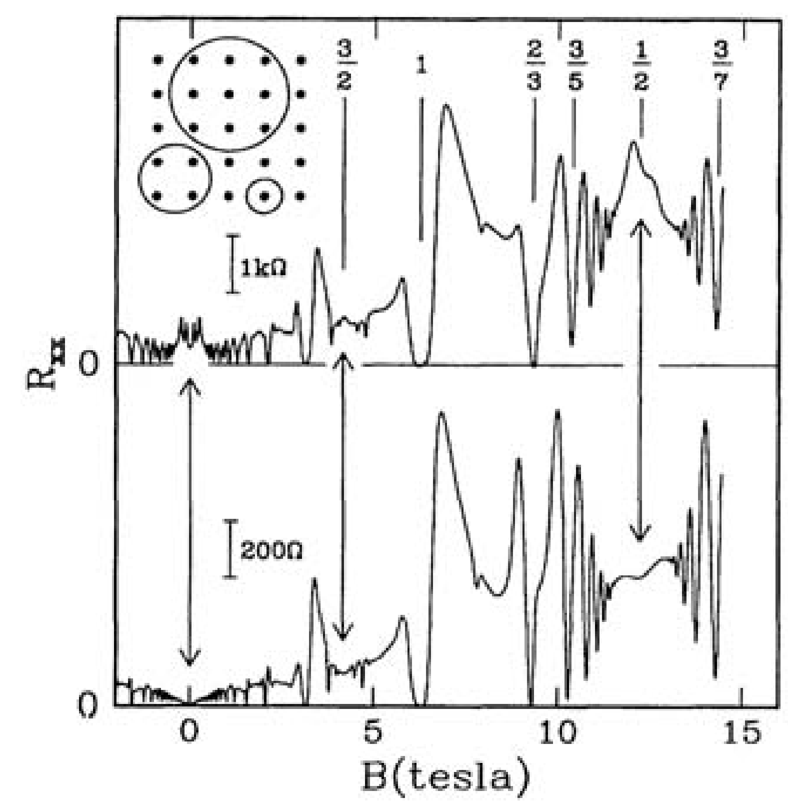
J. K. Jain and P. W. Anderson, PNAS **106**, 9131 (2009): The structure could be *universal* for strongly correlated systems (e.g., spin liquid).

CF Fermi-Liquid

$\nu = 1/2p \longrightarrow B_{\rm CF} = 0 \longrightarrow$ A Fermi liquid of CFs

CF: an ordinary Newtonian particle?





Kalmeyer-Zhang: Phys. Rev. B **46**, 9889 (1992).
Halperin-Lee-Read: Phys. Rev. B **47**, 7312 (1993).

“Two clouds” of the CF Theory

1. **CF Hall conductance at the half-filling**
2. **Asymmetry of the CF mapping for a filling fraction and its hole conjugate**

“Cloud” #1

CF Hall conductance at half filling

The particle-hole symmetry dictates:

$$\sigma_{xy} = \frac{1}{2}\frac{e^2}{h}$$

and the corresponding CF Hall conductance:

$$\sigma_{xy}^{\mathrm{CF}} = -\frac{1}{2}\frac{e^2}{h}$$

Lee, Krotov, and Gan, Kivelson, Phys. Rev. B **55**, 15552 (1997).

However, the effective magnetic field is zero!

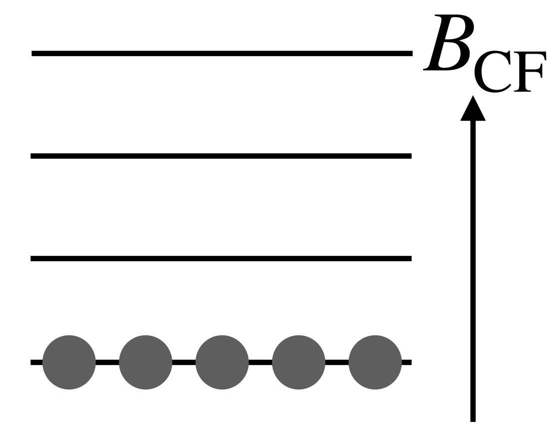
“Cloud” #2

Asymmetry of the CF mapping

$$\nu = 1/3$$

$$\rho_{\mathrm{e}} = \frac{1}{3}\frac{eB}{h} \quad B_{\mathrm{CF}} = \frac{1}{3}B \longrightarrow \nu_{\mathrm{CF}} = 1$$

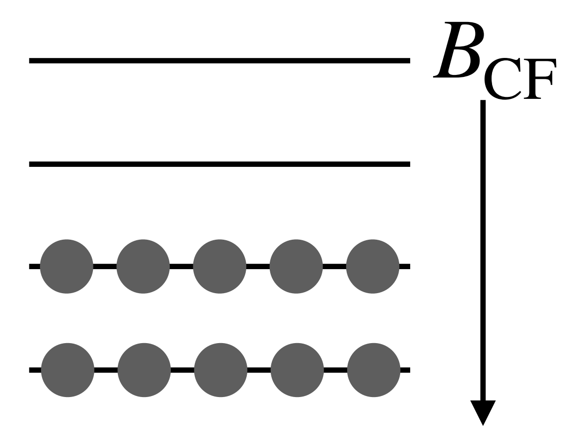
One fully occupied CF Landau level



$$\nu = 2/3$$

$$\rho_{\mathrm{e}} = \frac{2}{3}\frac{eB}{h} \quad B_{\mathrm{CF}} = -\frac{1}{3}B \longrightarrow \nu_{\mathrm{CF}} = 2$$

Two fully occupied CF Landau levels



PHYSICAL REVIEW X **5,** 031027 (2015)

Is the Composite Fermion a Dirac Particle?

Dam Thanh Son

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(Received 19 February 2015; published 2 September 2015)

We propose a particle-hole symmetric theory of the Fermi-liquid ground state of a half-filled Landau level. This theory should be applicable for a Dirac fermion in the magnetic field at charge neutrality, as well as for the $\nu = \frac{1}{2}$ quantum Hall ground state of nonrelativistic fermions in the limit of negligible inter-Landau-level mixing. We argue that when particle-hole symmetry is exact, the composite fermion is a massless Dirac fermion, characterized by a Berry phase of π around the Fermi circle. We write down a tentative effective field theory of such a fermion and discuss the discrete symmetries, in particular, $\mathcal{CP}$. The Dirac composite fermions interact through a gauge, but non-Chern-Simons, interaction. The particle-hole conjugate pair of Jain-sequence states at filling factors $n/(2n+1)$ and $(n+1)/(2n+1)$, which in the conventional composite fermion picture corresponds to integer quantum Hall states with different filling factors, n and $n+1$, is now mapped to the same half-integer filling factor $n+\frac{1}{2}$ of the Dirac composite fermion. The Pfaffian and anti-Pfaffian states are interpreted as d-wave Bardeen-Cooper-Schrieffer paired states of the Dirac fermion with orbital angular momentum of opposite signs, while s-wave pairing would give rise to a particle-hole symmetric non-Abelian gapped phase. When particle-hole symmetry is not exact, the Dirac fermion has a $\mathcal{CP}$-breaking mass. The conventional fermionic Chern-Simons theory is shown to emerge in the nonrelativistic limit of the massive theory.

Dirac interpretation

- A Fermi-sea of Dirac CFs with a density half of the magnetic flux density:

$$\rho_{\mathrm{CF}} = B/2\phi_0$$

- Dirac point contributes a half-quantized (anomalous) Hall conductance [Jackiw, PRD **29**, 2375 (1984)].

- Dirac CFs experience an effective magnetic field:

$$B_{\mathrm{CF}} = -2\phi_0(\rho - \rho_{1/2})$$

- Quantization rule of Dirac Fermions:

$$\rho_{\mathrm{CF}} = \frac{e\,|B_{\mathrm{CF}}|}{h}\left(n + \frac{1}{2}\right)$$

Can we have an “half Landau level” in the real world?

Chu, JS, and S.-Q. Shen, PRB **84**, 085312 (2011).

Alternative interpretation

- The composite fermion is neither an ordinary Newtonian particle nor a Dirac particle, but a “NiuQian” particle subject to **a uniform Berry curvature** and the **Sundaram-Niu dynamics**.

- The uniform Berry curvature gives rise to a π-Berry phase around the Fermi circle and the half-quantized **anomalous Hall conductance** of CFs at the half filling.

- The apparent asymmetry of the CF mapping is actually how the **symmetry is manifested** in a system subject to the Berry curvature.

JS, arXiv:1704.07712 (2017).

Sundaram-Niu Dynamics

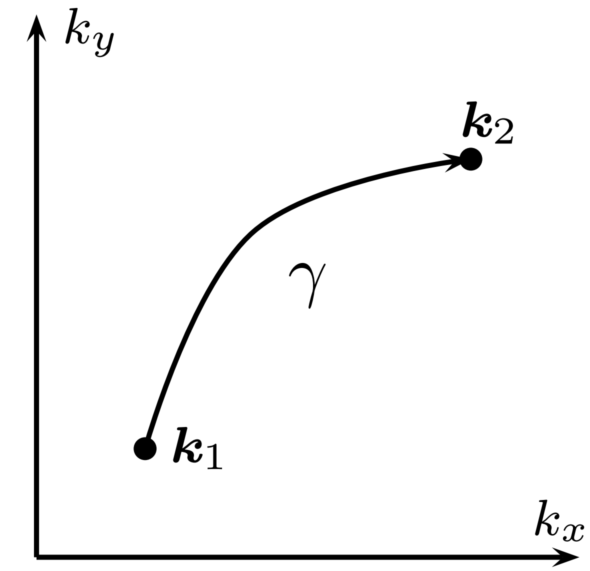
for systems breaking the time-reversal symmetry

$$\dot{\boldsymbol{x}} = \frac{1}{\hbar}\frac{\partial \mathscr{E}(\boldsymbol{k})}{\partial \boldsymbol{k}} - \dot{\boldsymbol{k}} \times \boldsymbol{\Omega}(\boldsymbol{k})$$

$$\hbar\dot{\boldsymbol{k}} = q\boldsymbol{E} + q\dot{\boldsymbol{x}} \times \boldsymbol{B}$$

Berry curvature: “magnetic field” in k-space:

adiabatic transport of a particle in the k-space gives rise to a Berry phase — the counterpart of the Aharonov-Bohm phase in the real space



$$\phi_\gamma = \int_\gamma \boldsymbol{A}(\boldsymbol{k}) \cdot \mathrm{d}\boldsymbol{k} \qquad \boldsymbol{A}(\boldsymbol{k}) = \mathrm{i}\langle u_{n\boldsymbol{k}} \,|\, \partial_{\boldsymbol{k}} u_{n\boldsymbol{k}} \rangle$$

$$\boldsymbol{\Omega}(\boldsymbol{k}) = \nabla_{\boldsymbol{k}} \times \boldsymbol{A}(\boldsymbol{k})$$

Xiao, Chang, Niu, RMP **82**, 1959 (2010).

Anomalous Hall Effect

$$\sigma_{xy}^{\mathrm{AH}} = -\frac{q^2}{\hbar}\int_{FS} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^d}\Omega_z(\boldsymbol{k})$$

For a 2D metallic system

$$\sigma_{xy}^{\mathrm{AH}} = -\frac{q^2}{2\pi h}\phi_{\mathrm{F}}$$

ϕ_{F} : Berry phase around the Fermi-circle

Haldane, PRL **93**, 206602 (2004).

CF Hall Conductance

At the half filling, the CF has an **anomalous Hall conductance**

$$\sigma_{xy}^{\mathrm{CF}} = -\frac{e^2}{2\pi h}\pi = -\frac{e^2}{2h}$$

with respect to the Berry phase

$$\phi_{\mathrm{F}} = +\pi$$

(A)symmetry of CF Mapping

The quantization rule of cyclotron orbits

$$\frac{S_{\boldsymbol{k}}}{2\pi} = \frac{e\,|B_{\mathrm{CF}}|}{\hbar}\left(n + \frac{1}{2} + \frac{\phi_{S_{\boldsymbol{k}}}}{2\pi}\right)$$

The highest occupied cyclotron orbit is always around the half-filled CFL Fermi-circle

$$\nu = \frac{n}{2n+1}$$

$$B_{\mathrm{CF}} = B/(2n+1)$$

$$\phi_{S_{\mathrm{F}}} = \phi_{\mathrm{F}} = \pi$$

$$\frac{S_{\mathrm{F}}}{2\pi} = \frac{e\,|B_{\mathrm{CF}}|}{\hbar}\left(\nu_{\mathrm{CF}} - \frac{1}{2} + \frac{1}{2}\right)$$

$$\nu_{\mathrm{CF}} = n$$

$$\nu = \frac{n+1}{2n+1}$$

$$B_{\mathrm{CF}} = -\,B/(2n+1)$$

$$\phi_{S_{\mathrm{F}}} = -\,\phi_{\mathrm{F}} = -\,\pi$$

$$\frac{S_{\mathrm{F}}}{2\pi} = \frac{e\,|B_{\mathrm{CF}}|}{\hbar}\left(\nu_{\mathrm{CF}} - \frac{1}{2} - \frac{1}{2}\right)$$

$$\nu_{\mathrm{CF}} = n + 1$$

Three competing pictures

Picture	Particle	Berry curvature
Halperin-Lee-Read	Newtonian	0
Son	Dirac	$\pi\delta(\boldsymbol{k})$
Ours	“Niuqian”	uniform $\frac{\hbar}{qB}$

Three competing pictures

Picture	Particle	Berry curvature
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Son	Dirac	$\pi\delta(\boldsymbol{k})$
Ours	“Niuqian”	uniform $\dfrac{\hbar}{qB}$

Microscopic wave function

dictates the interpretation

Rezayi-Read wave function for the composite Fermi-liquid:

$$\Psi_{\boldsymbol{k}}^{\mathrm{CF}}(z) = \hat{P}_{\mathrm{LLL}} \det \left[e^{\mathrm{i}\left(k_i z_j^* + k_i^* z_j\right)/2} \right] J(z)$$

$$= \hat{A} \prod_{i<j} (\underline{z_i + \mathrm{i}k_i} - z_j - \mathrm{i}k_j)^2 \prod_i e^{\mathrm{i}k_i^* z_i/2}$$

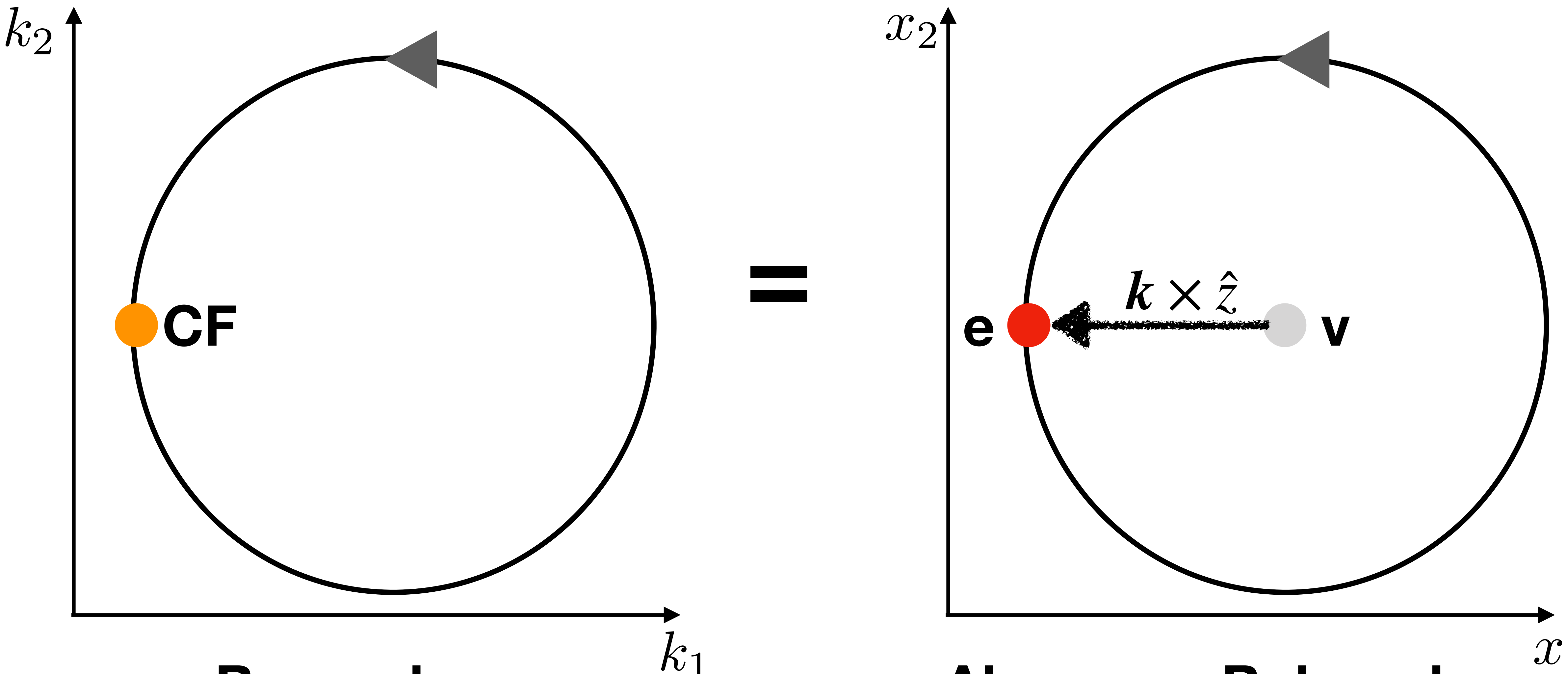
$z^{\mathrm{v}} = z + \mathrm{i}k$

vortex position

Dipole interpretation:

N. Read, Semicond. Sci. Tech. 9, 1859 (1994).

Berry phase of a CF dipole



Berry phase = **Aharanov-Bohm phase**

$$\Omega_z S_{\boldsymbol{k}} = eB \times \frac{S_{\boldsymbol{k}}}{(eB)^2}$$

JS, arXiv:1704.07712 (2017); Haldane, APS March meeting, 2016

CF dynamics from a WF

$$\Psi(z) = \hat{A} \prod_{i<j} (z_i - z_j)^{2p} \prod_i \boxed{e^{\frac{\mathrm{i}}{2} Z_i^* z_i}}$$

coherent state localized at Z_i

Introducing "momentum"

$$e^{\frac{\mathrm{i}}{2} Z_i^* z_i} \rightarrow e^{\frac{\mathrm{i}}{2} Z_i^* z_i} e^{\mathrm{i} \boldsymbol{k}_i \cdot \boldsymbol{z}_i} \sim e^{\frac{\mathrm{i}}{2} Z_i^* z_i + \frac{\mathrm{i}}{2} k_i z_i^*}$$

Projected to the LLL

$$\Psi_{Z^*,k}(z) = \hat{A} \prod_{i<j} (z_i + \mathrm{i}k_i - z_j - \mathrm{i}k_j)^{2p} \prod_i e^{\frac{\mathrm{i}}{2} Z_i^* z_i}$$

$$Z_i^* \rightarrow Z_i^*(t), k_i \rightarrow k_i(t)$$

$$\delta L = 0 \qquad L = \mathrm{i}\hbar \langle \Psi | \dot{\Psi} \rangle - \langle \Psi | \hat{V}_{ee} | \Psi \rangle$$

ALL CF wave functions can be expanded in $\Psi_{Z^*,k}(z)$

CF dynamics

for a CF-Wigner crystal

$$\dot{\boldsymbol{x}}_i = \frac{1}{\hbar}\frac{\partial V}{\partial \boldsymbol{k}_i} - \frac{\hbar}{eB}\dot{\boldsymbol{k}}_i \times \hat{z}$$

$$\hbar\dot{\boldsymbol{k}}_i = -\frac{\partial V}{\partial \boldsymbol{x}_i} + \sum_j eB^*_{ij}\dot{\boldsymbol{x}}_j \times \hat{z}$$

- **A uniform Berry curvature** $\Omega_z(\boldsymbol{p}) = \hbar/eB$;
- An emergent effective “magnetic field” (CS field) $\boldsymbol{B}^*_{ij}$.
- Note: **CF position is the vortex position**

JS and Wencheng Ji, PRB **97**, 125133 (2018).

Evidence for the Dirac CF?

PHYSICAL REVIEW LETTERS **121,** 147202 (2018)

Editors' Suggestion

Berry Phase and Model Wave Function in the Half-Filled Landau Level

Scott D. Geraedts,[1,2] Jie Wang,[1] E. H. Rezayi,[3] and F. D. M. Haldane[1]
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[2]*Department of Electrical Engineering, Princeton University, Princeton, New Jersey 08544, USA*
[3]*Department of Physics, California State University, Los Angeles, California 90032, USA*

(Received 23 December 2017; published 3 October 2018)

We construct model wave functions for the half-filled Landau level parametrized by "composite fermion occupation-number configurations" in a two-dimensional momentum space, which correspond to a Fermi sea with particle-hole excitations. When these correspond to a weakly excited Fermi sea, they have a large overlap with wave functions obtained by the exact diagonalization of lowest-Landau-level electrons interacting with a Coulomb interaction, allowing exact states to be identified with quasiparticle configurations. We then formulate a many-body version of the single-particle Berry phase for adiabatic transport of a single quasiparticle around a path in momentum space, and evaluate it using a sequence of exact eigenstates in which a single quasiparticle moves incrementally. In this formulation the standard free-particle construction in terms of the overlap between "periodic parts of successive Bloch wave functions" is reinterpreted as the matrix element of a "momentum boost" operator between the full Bloch states, which becomes the matrix elements of a Girvin-MacDonald-Platzman density operator in the many-body context. This allows the computation of the Berry phase for the transport of a single composite fermion around the Fermi surface. In addition to a phase contributed by the density operator, we find a phase of exactly π for this process.

Evidence for the Dirac CF?

PHYSICAL REVIEW LETTERS **121,** 147202 (2018)

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er 2018)

Our results are consistent with the theory of Son [10], in which the Berry phase arises from the Dirac nature of the composite fermions. However the composite fermions discussed in our work are single-component objects, and the relation to a two-component composite Dirac fermion is unclear. While the model wave function with a compact Fermi surface is unexpectedly close to being particle-hole symmetric, if a quasihole is moved inside the Fermi surface, this breaks down: the particle-hole conjugate states have less and less overlap and become orthogonal as the quasi-hole approaches the center of Fermi surface. By forming orthogonal linear combinations of particle-hole conjugate states, the Dirac cone is possible to be found.

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However…

- The definition of the Berry phase?
- Various numerical artifacts.
- Microscopic CFL wave function?

Issue #1: definition

Geraedts et al.'s defintion:

$$\phi_{\mathrm{B}} = -\arg\left\langle \Psi_{\boldsymbol{k}} \left| \hat{\rho}_{\boldsymbol{k}-\boldsymbol{k}'} \right| \Psi_{\boldsymbol{k}'} \right\rangle$$

$\Psi_{\boldsymbol{k}}$: CFL wave function with respect to a **k**-configuration

$\hat{\rho}_{\boldsymbol{k}-\boldsymbol{k}'} \equiv \sum_i e^{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\cdot \boldsymbol{r}_i}$: "momentum boost operator" to compensate the change of the momenta

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However, the Berry phase has an origin of the time-derivative term of the Schrödinger Lagrangian

$$L_0 = \mathrm{i}\hbar\langle \Psi_{\boldsymbol{k}} \,|\, \dot{\Psi}_{\boldsymbol{k}} \rangle$$

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However, the Berry phase has an origin of the time-derivative term of the Schrödinger Lagrangian

$$L_0 = \mathrm{i}\hbar\langle\Psi_{\boldsymbol{k}}|\dot{\Psi}_{\boldsymbol{k}}\rangle$$

Geraedts et al.'s phase is actually a **scattering phase**, which is **not a reliable predictor** for the Berry phase.

Issue #2: numerical artifacts

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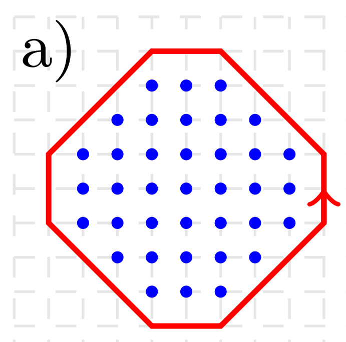
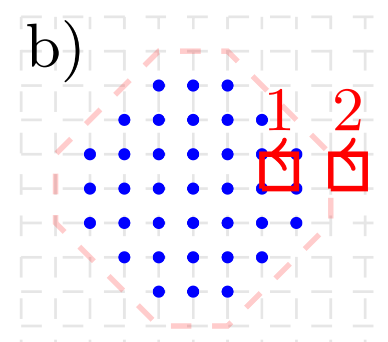
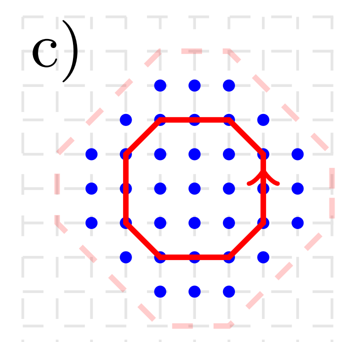
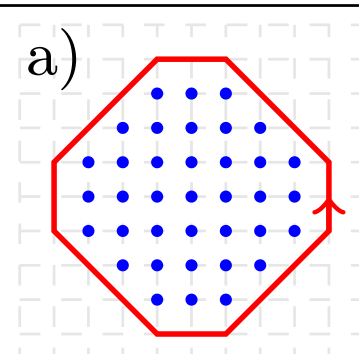
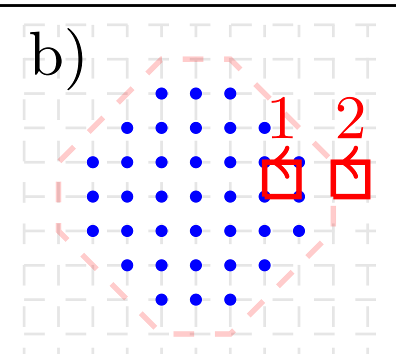
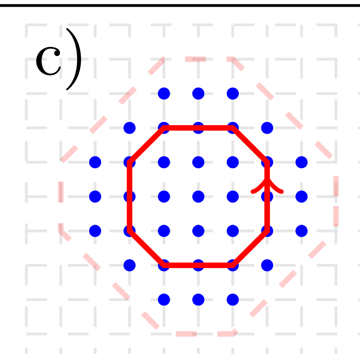
- small overlaps which further deteriorate when the system size is scaled up;

Path							
N_e		13	38	110	36(b1)	38(b2)	36
$\phi_{\mathrm{B}}^{\mathrm{v}}/\pi$	old	0.82	0.72	0.57	U.D.	U.D.	0.93
	[redacted]						
$\|D\|_{\min}$	old	0.65*	0.36*	0.18*	0.04*	0.01*	0.22*
	[redacted]						

Issue #2: numerical artifacts

- small overlaps which further deteriorate when the system size is scaled up;
- sensitive to the choice of paths: the paths vertical to the Fermi circle have nearly vanishing overlaps;

Path							
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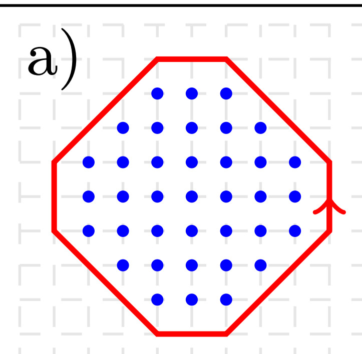
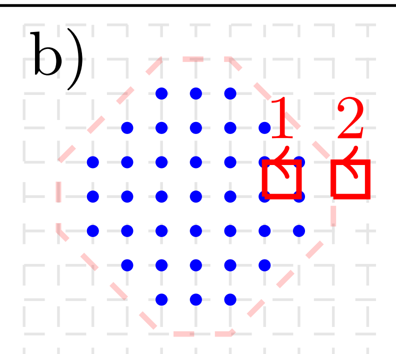
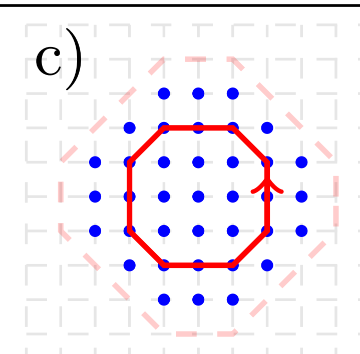
- extraneous $\pm\pi/2$ phases for each momentum change.

Path		a) 			b) 		c) 
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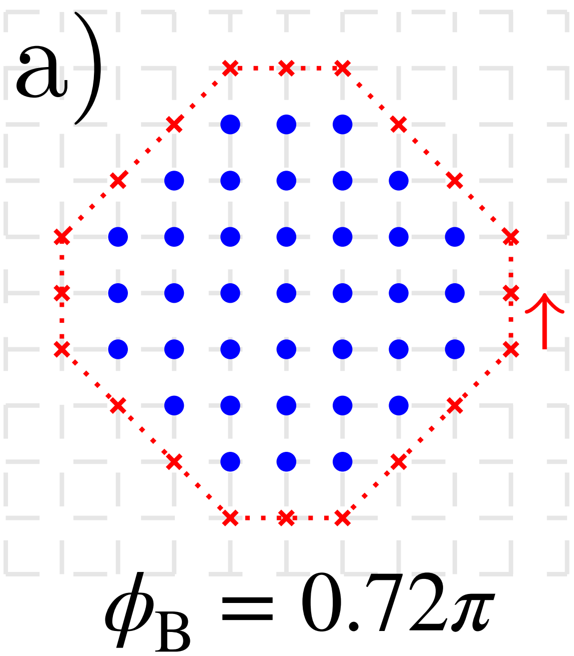

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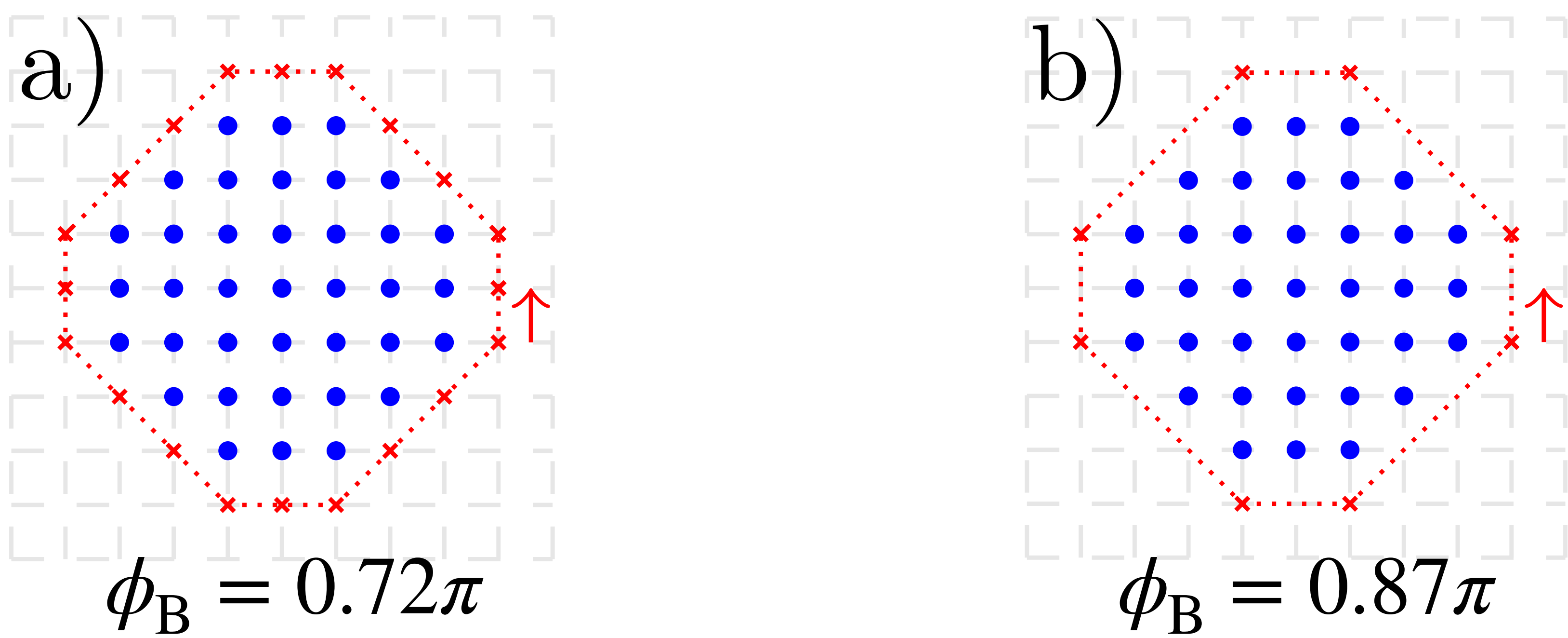

Issue #3: wave function

Standard CF wave function on a torus for $\nu = 1/m$:

$$\Psi^{\mathrm{CF}}_{\boldsymbol{k}}(z) = \hat{P}_{\mathrm{LLL}} \det\left[e^{\mathrm{i}\left(k_i z_j^* + k_i^* z_j\right)/2}\right] J(z) \sim \sum_P (-1)^P e^{\mathrm{i}k_{Pi}^* z_i/2} J\left(\{z_i + \mathrm{i}k_{Pi}\}\right)$$

$$J(z) = \tilde{\sigma}^m(Z) \prod_{i<j} \tilde{\sigma}^m(z_i - z_j),$$

$\tilde{\sigma}$: modified sigma function

Jain-Kamilla wave function (adopted by Geraedts et al.):

$$\Psi^{\mathrm{JK}}_{\boldsymbol{k}}(z) = \det\left[\psi_i(\boldsymbol{k}_j)\right] \tilde{\sigma}^m\left(Z + \mathrm{i}K\right) \prod_{i<j} \tilde{\sigma}^{m-2}(z_i - z_j),$$

$$\psi_i(\boldsymbol{k}_j) = e^{\mathrm{i}k_j^* z_i/2} \prod_{k \neq i} \tilde{\sigma}\left(z_i - z_k + \mathrm{i}m k_j - \mathrm{i}m\bar{k}\right),$$

Do they yield the same Berry curvature?

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- However, $\langle \Psi_{\boldsymbol{k}} | z_i | \Psi_{\boldsymbol{k}} \rangle$ does not define the position of an individual electron.
- Nevertheless, one can define the position of the electron *associated* with a momentum $\boldsymbol{k}_i$:

$$z_i = \sum_P (-1)^P \mathrm{Re} \langle \Psi_{\boldsymbol{k}} | \hat{\mathscr{P}} \hat{z}_i \varphi_{\boldsymbol{k}} \rangle / \langle \Psi_{\boldsymbol{k}} | \Psi_{\boldsymbol{k}} \rangle = \mathrm{Re} \langle \Psi_{\boldsymbol{k}} | \hat{z}_i | \varphi_{\boldsymbol{k}} \rangle / \langle \Psi_{\boldsymbol{k}} | \varphi_{\boldsymbol{k}} \rangle$$

  - $\varphi_{\boldsymbol{k}}$: unsymmetrized wave function

$$\varphi_{\boldsymbol{k}} = e^{\mathrm{i} k_i^* z_i / 2} J\left(\{ z_i + \mathrm{i} k_i \} \right) \qquad \Psi_{\boldsymbol{k}} = \sum_P (-1)^P \hat{\mathscr{P}} \varphi_{\boldsymbol{k}}$$

- The Berry connection can then be defined:

$$L_0 = -\frac{\mathrm{Im}\langle \Psi_{\boldsymbol{k}} | \dot{\Psi}_{\boldsymbol{k}} \rangle}{\langle \Psi_{\boldsymbol{k}} | \Psi_{\boldsymbol{k}} \rangle} \quad \Longrightarrow \quad L_0 = \boldsymbol{k}_1 \cdot \dot{\boldsymbol{z}}_1 + \boldsymbol{A}_{\boldsymbol{k}_1} \cdot \dot{\boldsymbol{k}}_1,$$

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Our definition

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Subtlety: electron vs. CF representation

- The definition of the Berry phase/connection depends on the definition of the "position".

- z_i is the position of an electron.

- **The position of a CF should be defined as** $z_i^{\mathrm{v}} = z_i + \mathrm{i}k_i$ [JS and Ji, PRB **97**, 125133 (2018); JS, arXiv:1704.07712 (2017)]

$$\boldsymbol{A}_{\boldsymbol{k}_1}^{\mathrm{v}} = \boldsymbol{A}_{\boldsymbol{k}_1}^{\mathrm{e}} - \boldsymbol{k}_1 \times \boldsymbol{n}$$

$$\phi_{\mathrm{B}}^{\mathrm{v}} = \phi_{\mathrm{B}}^{\mathrm{e}} + (\boldsymbol{k}_1 \times \boldsymbol{q}_1) \cdot \boldsymbol{n}$$

$$\Omega_z^{\mathrm{v}} = \Omega_z^{\mathrm{e}} + 2$$

Numerical evaluation

with the JK wave function

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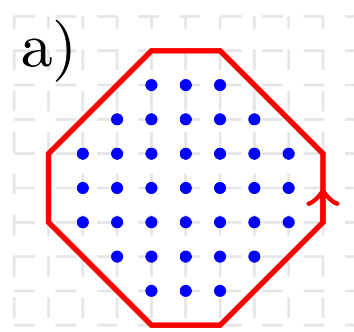
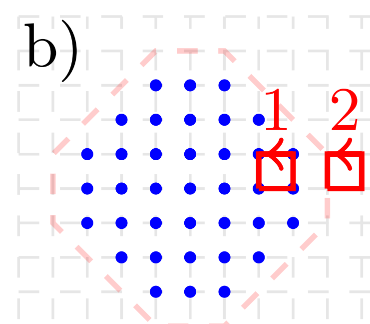
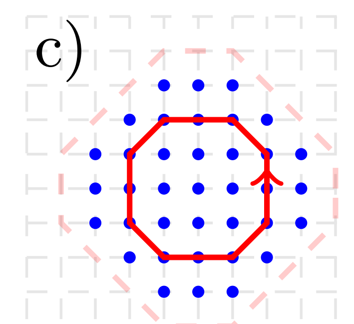
- overlaps are always close to 1;

Path		a) 			b) 		c) 
N_e		13	38	110	36(b1)	38(b2)	36
$\phi_{\mathrm{B}}^{\mathrm{v}}/\pi$	old	0.82	0.72	0.57	U.D.	U.D.	0.93
	new	1.11	1.03	1.01	0.75^*	0.05^*	0.61
$\lvert D\rvert_{\min}$	old	0.65^*	0.36^*	0.18^*	0.04^*	0.01^*	0.22^*
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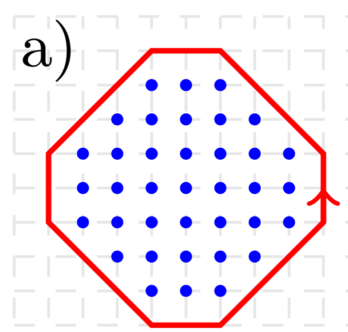
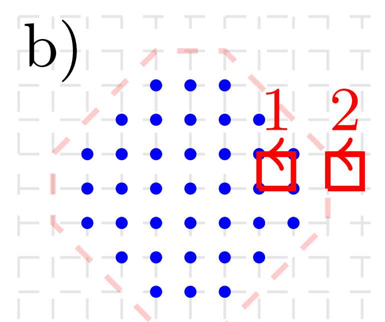
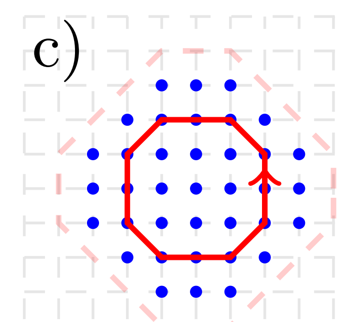
Path		a) 			b) 		c) 
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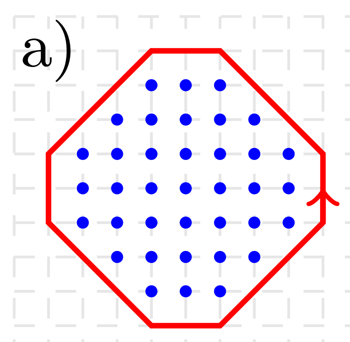
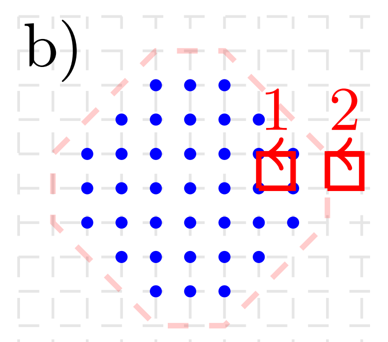
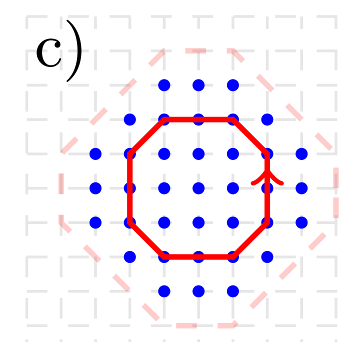
Path		a) 			b) 		c) 
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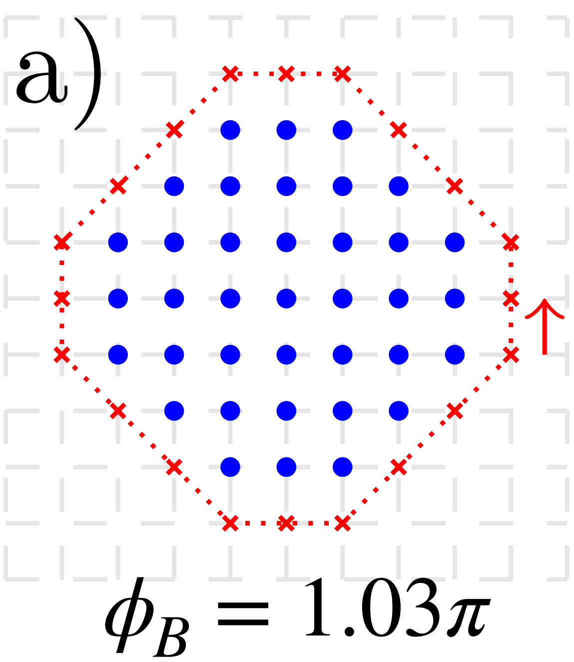

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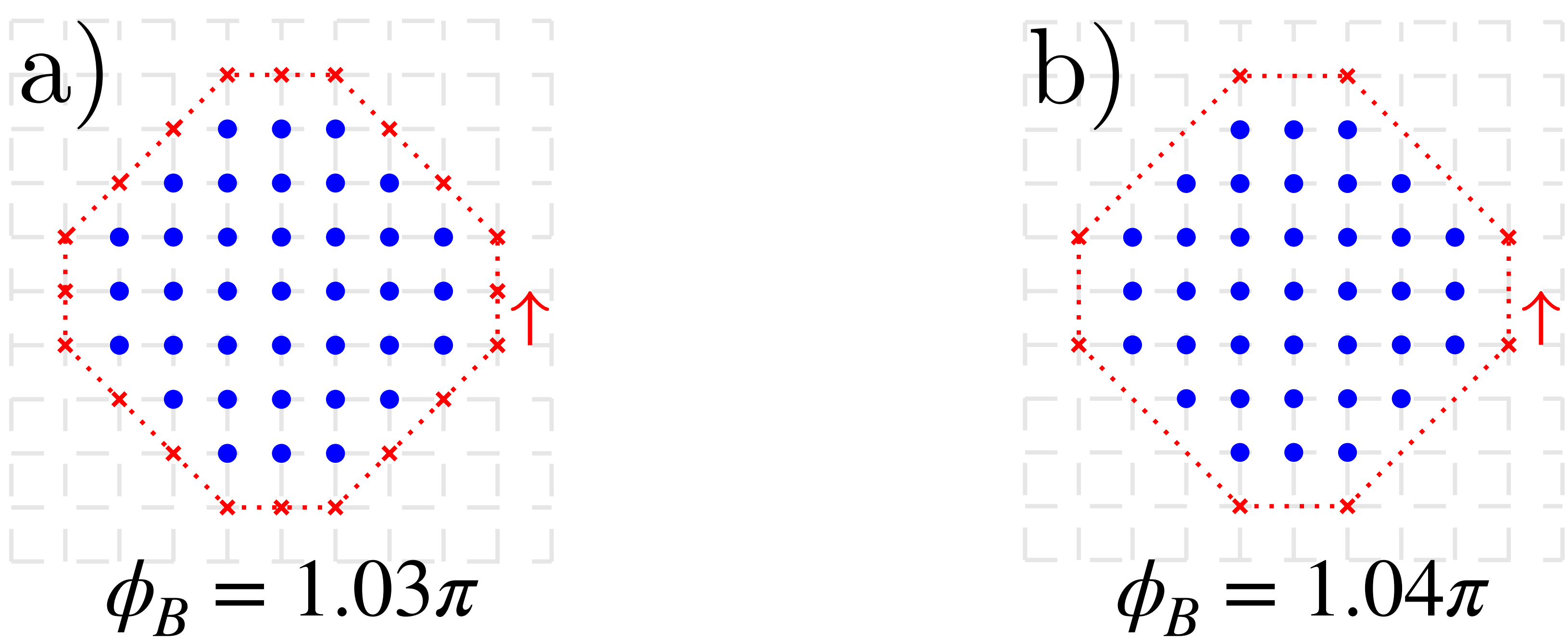
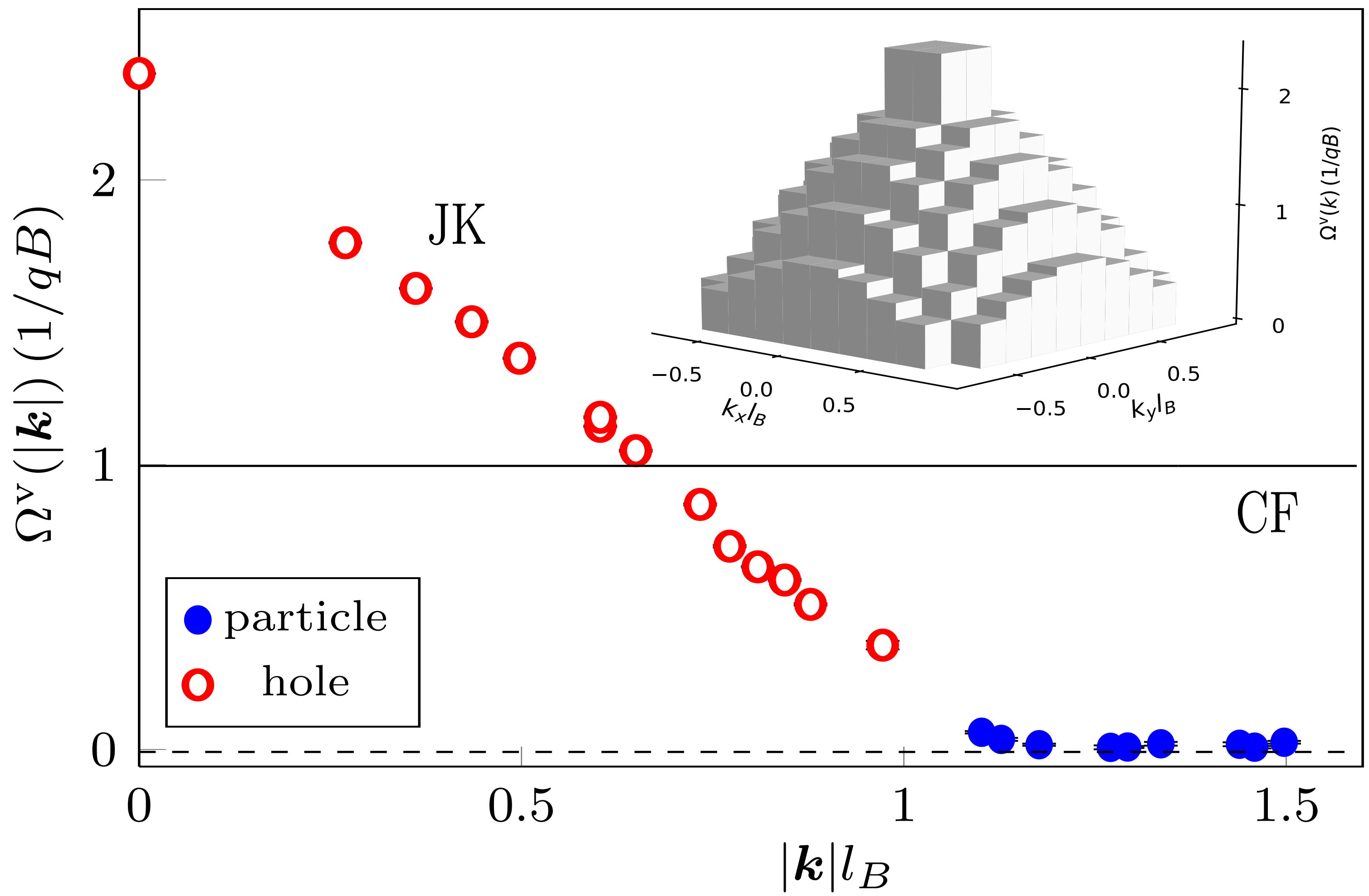


Berry curvature distribution



The Berry curvature vanishes outside the Fermi sea.
However, it is a continuous distribution inside
— NOT a massless Dirac Fermion.

JK & CF wave functions

yield different distributions of the Berry curvature because of different quasi-periodicities in the k-space

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CF wave function and the Segal-Bargmann transform

The Hilbert space of the LLL is a Segal-Bargmann space
(see Girvin and Jach, PRB **29**, 5617 (1984))

All CF wave functions can be expressed as projected Segal-Bargmann transforms

$$\Psi(z) = \left[e^{-z\cdot z'^*} \int \mathrm{d}\mu(\boldsymbol{\eta}) e^{\frac{1}{2}\left(\eta^* \cdot z + \eta \cdot z'^*\right)} J(\eta) \psi\left(\boldsymbol{\eta}\right) \right]_{z'^*=0}$$

Segal-Bargmann space

$$\mathrm{d}\mu(\boldsymbol{\eta}) \equiv \prod_i e^{-\left|\eta_i\right|^2/2} \mathrm{d}\eta_i \mathrm{d}\eta_i^* / 4\pi\mathrm{i}$$

$\leftarrow$ **Hidden Hilbert space**

Reproducing kernel

$$\int \mathrm{d}\mu(\boldsymbol{\eta}) e^{\frac{1}{2}\eta^* \cdot z} f(\eta) = f(z)$$

Berry curvature of the CF wave function

$$\phi_{\mathrm{B}} = -\frac{1}{2}\left[\arg\langle\Psi_{\boldsymbol{k}}|e^{-\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\cdot\hat{z}}|\varphi_{\boldsymbol{k}'}\rangle - (\boldsymbol{k}\rightleftharpoons\boldsymbol{k}')\right]$$

$$\langle\Psi_{\boldsymbol{k}}^{\mathrm{CF}}|e^{-\mathrm{i}\boldsymbol{q}\cdot\hat{z}}|\varphi_{\boldsymbol{k}+\boldsymbol{q}}^{\mathrm{CF}}\rangle = \sum_{P}(-1)^{P}\langle\varphi_{P\boldsymbol{k}}^{\mathrm{CF}}|e^{-\mathrm{i}\boldsymbol{q}\cdot\hat{z}}|\varphi_{\boldsymbol{k}+\boldsymbol{q}}^{\mathrm{CF}}\rangle$$

$$\varphi_{\boldsymbol{k}}^{\mathrm{CF}}(z) = \int \mathrm{d}\mu(\boldsymbol{\eta})e^{\frac{1}{2}\eta^{*}\cdot z}J(\eta)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\eta}}$$

Berry curvature of the CF wave function

$$\phi_{\mathrm{B}} = -\frac{1}{2}\left[\arg\langle\Psi_{\boldsymbol{k}}|e^{-\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\cdot\hat{z}}|\varphi_{\boldsymbol{k}'}\rangle - (\boldsymbol{k} \rightleftharpoons \boldsymbol{k}')\right]$$

$$\langle\Psi_{\boldsymbol{k}}^{\mathrm{CF}}|e^{-\mathrm{i}\boldsymbol{q}\cdot\hat{z}}|\varphi_{\boldsymbol{k}+\boldsymbol{q}}^{\mathrm{CF}}\rangle = \sum_{P}(-1)^{P}\langle\varphi_{P\boldsymbol{k}}^{\mathrm{CF}}|e^{-\mathrm{i}\boldsymbol{q}\cdot\hat{z}}|\varphi_{\boldsymbol{k}+\boldsymbol{q}}^{\mathrm{CF}}\rangle$$

$$\varphi_{\boldsymbol{k}}^{\mathrm{CF}}(z) = \int \mathrm{d}\mu(\boldsymbol{\eta})e^{\frac{1}{2}\eta^{*}\cdot z}J(\eta)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\eta}}$$

$$\langle\varphi_{P\boldsymbol{k}}^{\mathrm{CF}}|e^{-\mathrm{i}\boldsymbol{q}\cdot\hat{z}}|\varphi_{\boldsymbol{k}+\boldsymbol{q}}^{\mathrm{CF}}\rangle = \underbrace{\boxed{e^{-\frac{1}{2}q^{*}\left(q+k\right)}}}_{\text{transport}} \times \underbrace{\boxed{\langle\varphi_{P\boldsymbol{k}}^{\mathrm{CF}}|\varphi_{\boldsymbol{k}}^{\mathrm{CF}}\rangle}}_{\text{exchange}}$$

Berry curvature of the CF wave function

$$\phi_{\mathrm{B}} = -\frac{1}{2}\left[\arg\langle\Psi_{\boldsymbol{k}}|e^{-\mathrm{i}(\boldsymbol{k}'-\boldsymbol{k})\cdot\hat{z}}|\varphi_{\boldsymbol{k}'}\rangle - (\boldsymbol{k} \rightleftharpoons \boldsymbol{k}')\right]$$

$$\langle\Psi^{\mathrm{CF}}_{\boldsymbol{k}}|e^{-\mathrm{i}\boldsymbol{q}\cdot\hat{z}}|\varphi^{\mathrm{CF}}_{\boldsymbol{k}+\boldsymbol{q}}\rangle = \sum_{P}(-1)^{P}\langle\varphi^{\mathrm{CF}}_{P\boldsymbol{k}}|e^{-\mathrm{i}\boldsymbol{q}\cdot\hat{z}}|\varphi^{\mathrm{CF}}_{\boldsymbol{k}+\boldsymbol{q}}\rangle$$

$$\varphi^{\mathrm{CF}}_{\boldsymbol{k}}(z) = \int \mathrm{d}\mu(\boldsymbol{\eta})e^{\frac{1}{2}\eta^{*}\cdot z}J(\eta)e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\eta}}$$

$$\langle\varphi^{\mathrm{CF}}_{P\boldsymbol{k}}|e^{-\mathrm{i}\boldsymbol{q}\cdot\hat{z}}|\varphi^{\mathrm{CF}}_{\boldsymbol{k}+\boldsymbol{q}}\rangle = \underbrace{\boxed{e^{-\frac{1}{2}q^{*}\left(q+k\right)}}}_{\text{transport}} \times \underbrace{\boxed{\langle\varphi^{\mathrm{CF}}_{P\boldsymbol{k}}|\varphi^{\mathrm{CF}}_{\boldsymbol{k}}\rangle}}_{\text{exchange}}$$

$$\Omega^{\mathrm{e}}_{z}(\boldsymbol{k}) = -1 \qquad \Omega^{\mathrm{v}}_{z}(\boldsymbol{k}) = +1$$

for the standard CF wave function

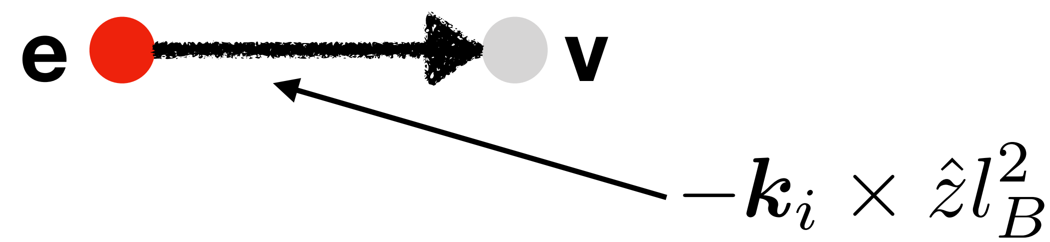
Conclusion and Outlook

- No, the composite Fermion **is not a massless Dirac Fermion**, at least for the two wave functions.
- The CF with respect to the standard CF theory **is a "Niuqian" particle with a uniform Berry curvature.**
- The structure of the hidden Hilbert space?
  - Is it finite? Topology?
  - An explicit construction?
- A proper field theory for CFs? — a field theory defined in the Segal-Bargmann space?

$$\left\{\psi(z), \psi^{\dagger}(z')\right\} = e^{\frac{1}{2}zz'^{*}}$$

Thank you very much for your attention!

CF: A Dipole



N. Read, Semicond. Sci. Tech. 9, 1859 (1994)
Pasquier, Haldane, Nuclear Physics B 516, 719 (1998)

The “momentum” is actually the displacement from the electron to the quantum vortices.

$$\sum_j \begin{bmatrix} e\Delta B^*_{ij}\hat{z}\times & 0 \\ 0 & -eB\delta_{ij}\hat{z}\times \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}^v_j \\ \dot{\boldsymbol{x}}^e_j \end{bmatrix} = \begin{bmatrix} \boldsymbol{\nabla}_{\boldsymbol{x}^v_i} V \\ \boldsymbol{\nabla}_{\boldsymbol{x}^e_i} V \end{bmatrix}$$

The electron is only coupled to the external (real) magnetic field
The quantum vortices are only coupled to the emergent (CS) field

CF Position: Electron vs. Vortices?

The position of a CF is assigned to its constituent quantum vortices

$$\boldsymbol{x}_i \equiv \left\langle \hat{\boldsymbol{\xi}}_i \right\rangle - \boldsymbol{k}_i \times \hat{z} l_B^2 \qquad \hat{\xi}_i \equiv \hat{r}_i - \boldsymbol{R}_i^0$$

$$\Psi(\{\boldsymbol{r}_i\}) = \mathcal{A} \prod_{i<j} (\underline{z_i} - z_j + \underline{i k_i l_B^2} - i k_j l_B^2)^m \prod_i \phi_{\boldsymbol{R}_i}(\boldsymbol{r}_i)$$

$$\Rightarrow \quad z_i^v = z_i + i k_i l_B^2$$

In case we use the electron position:

$$\sum_j \begin{bmatrix} eB_{ij}^* \hat{z}\times & \frac{\Delta B_{ij}^*}{B} \\ -\frac{\Delta B_{ij}^*}{B} & -\frac{1}{eB}\left(\frac{\Delta B_{ij}^*}{B}\right)\hat{z}\times \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{x}}_j^e \\ \dot{\boldsymbol{p}}_j \end{bmatrix} = - \begin{bmatrix} \frac{\partial V}{\partial \boldsymbol{x}_i^e} \\ \frac{\partial V}{\partial \boldsymbol{p}_i} \end{bmatrix}$$

Case II: CF Fermi Liquid

Rezayi-Read WF: Phys. Rev. Lett. 72, 900 (1994)

$$\Psi_{CFS} = \hat{P}_{LLL} e^{-\frac{|B|}{4}\sum_i |z_i|^2} \prod_{i<j} (z_j - z_j)^2 \Psi_F\left(\{\boldsymbol{x}_i\}\right)$$

Adding an electron into the CF Fermi sea:

$$\Psi_{\boldsymbol{k}}\left(\boldsymbol{x}, \{\boldsymbol{x}_i\}\right) \propto e^{i\boldsymbol{k}\cdot\boldsymbol{x}/2} \Psi_a\left(z + ik/B, \{z_i\}\right),$$

$$\Psi_a(z, \{z_i\}) \propto \exp(-|B||z|^2/4) \prod_i (z - z_i)^2 \Psi_{CFS}$$

$$\rho_a\left(\boldsymbol{x}, \boldsymbol{x}'\right) = \frac{1}{S} e^{-\frac{|B|}{4}\left|\boldsymbol{x}-\boldsymbol{x}'\right|^2 + \frac{iB}{2}\left(\boldsymbol{x}\times\boldsymbol{x}'\right)\cdot\hat{z}},$$

$$\left\langle \Psi_{\boldsymbol{k}} \middle| \Psi_{\boldsymbol{k}'} \right\rangle = \delta_{\boldsymbol{k}\boldsymbol{k}'}$$

Wave-packet dynamics can be constructed using the basis.

Sundaram and Niu, Phys. Rev. B **59**, 14915 (1999)

CF Dynamics

$$\boldsymbol{x} \equiv \boldsymbol{x}^v = \boldsymbol{x}^e - \boldsymbol{k}_i \times \hat{z} l_B^2$$

$$\dot{\boldsymbol{x}}^e = \frac{\partial V}{\partial \boldsymbol{p}} + \frac{1}{eB}\dot{\boldsymbol{p}} \times \hat{z}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial V}{\partial \boldsymbol{x}^e}$$

$$\dot{\boldsymbol{x}} = \frac{\partial V}{\partial \boldsymbol{p}} - \frac{1}{eB}\dot{\boldsymbol{p}} \times \hat{z}$$

$$\dot{\boldsymbol{p}} = -\frac{\partial V}{\partial \boldsymbol{x}}$$

$$\boldsymbol{\Omega}(\boldsymbol{k}) = \frac{1}{eB}\hat{z}$$ **for all *k***

It is consistent with the dynamics of guiding centers:
Haldane, in *The Quantum Hall effect*, ed. by Prange and Girvin (1990)

$$[R_i^\alpha, R_j^\beta] = i\,\frac{q}{e}\,\epsilon^{\alpha\beta\gamma}\,\hat{z}^\gamma$$

CF Dynamics

$$\dot{\boldsymbol{x}} = \frac{\partial h_{cf}}{\partial \boldsymbol{k}} - \dot{\boldsymbol{k}} \times \boldsymbol{\Omega}(\boldsymbol{k}),$$
$$\dot{\boldsymbol{k}} = \boldsymbol{E}_{cf} + \dot{\boldsymbol{x}} \times \boldsymbol{B}_{cf}$$

$$\boldsymbol{\Omega}(\boldsymbol{k}) = \frac{1}{eB}\hat{z}$$

CFs follow Niu's dynamics instead of Newton's

Sundaram-Niu, Phys. Rev. B **59**, 14915 (1999),

Xiao-Chang-Niu, Rev. Mod. Phys. **82**, 1959 (2010).

CF Momentum Manifold

The CF momentum manifold inherits its characteristics from the Landau level

- A finite phase volume: $S = 2\pi|B|$
- Particle-hole symmetry
- A non-zero Chern number $C = \mathrm{sign}(B) = -C_L$

An effective theory starting from a Landau level, instead of a band of free electrons/Dirac fermions?

Four-band Toy Model

$$H_{cf} = \epsilon_0 \begin{bmatrix} \Lambda\left[(\boldsymbol{p}\cdot\boldsymbol{\sigma})\sigma_\tau(\boldsymbol{p}\cdot\boldsymbol{\sigma}) + \beta\right] & \epsilon_p \boldsymbol{p}\cdot\boldsymbol{\sigma} + \frac{\beta}{2} \\ \epsilon_p \boldsymbol{p}\cdot\boldsymbol{\sigma} + \frac{\beta}{2} & -\Lambda\sigma_\tau \end{bmatrix}$$

$$\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3) \qquad \boldsymbol{p} \equiv (p_1, p_2, \epsilon_p)$$

$$\beta \equiv 2B_{cf}/|B| \qquad \sigma_\tau \equiv (\tau + \sigma_3)/2$$

The Hamiltonian defines an effective CF model on a disc-shape momentum manifold with a uniformly distributed Berry curvature

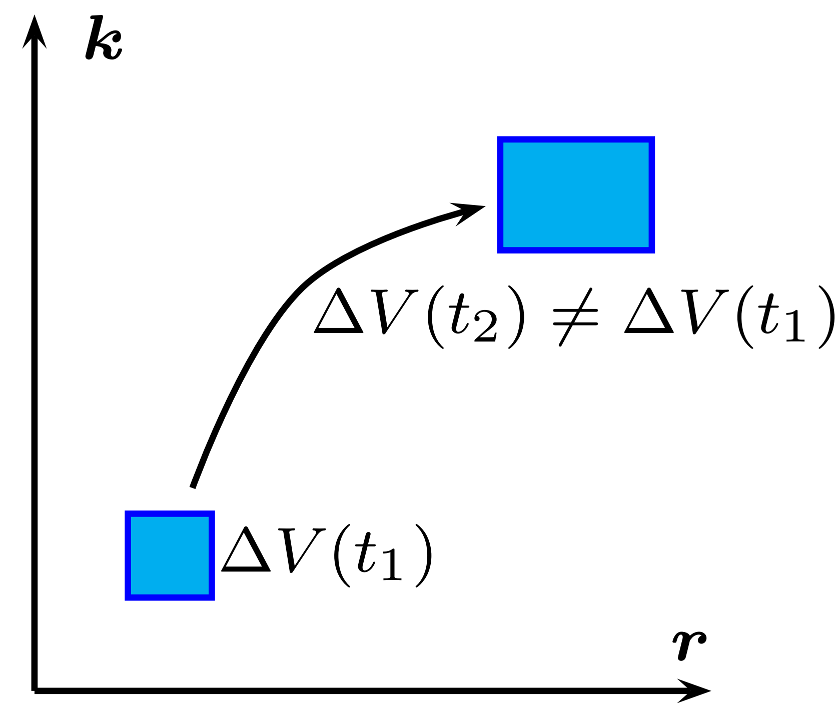
Compared to the Dirac Theory

- Both theories predict the π-Berry phase for the half filling.
- The CF conductance at half filling is protected in our theory, but probably not in the Dirac theory — Wang, Cooper, Halperin, and Stern, preprint, arXiv:1701.00007 (2017).
- Our theory is based on existing wave functions, while the Dirac theory may imply new constructions of ground/excited state wave-functions, which are deemed to be revolutionary.

Berry curvature correction to the density of states

Liouville's theorem breaks down for the Sundaram-Niu dynamics

$$\Delta V(t) = \frac{\Delta V(0)}{1 - (q/\hbar)\boldsymbol{B}\cdot\boldsymbol{\Omega}(\boldsymbol{k})}$$



The phase space has a measure:

$$D(\boldsymbol{k}) = 1 - \frac{q}{\hbar}B\Omega_z(\boldsymbol{k})$$

$$\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2} \to \int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2} D(\boldsymbol{k})$$

Xiao, JS, and Niu, PRL 95, 137204 (2005)

Symmetry out of asymmetry

The phase space area of a half-filled CFL Fermi-sea can accommodate

$$n_e = \int_{\mathrm{FS}_0} \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^2} \left(1 - \frac{q}{\hbar} B_{cf} \Omega(\boldsymbol{k})\right) = \frac{eB}{2h} - \frac{eB_{cf}}{2h}$$

Quantization condition

$$n_e = N_L \frac{e|B_{cf}|}{h}$$

$$\nu = \frac{n}{2n+1}$$

$$B_{cf} = B/(2n+1)$$

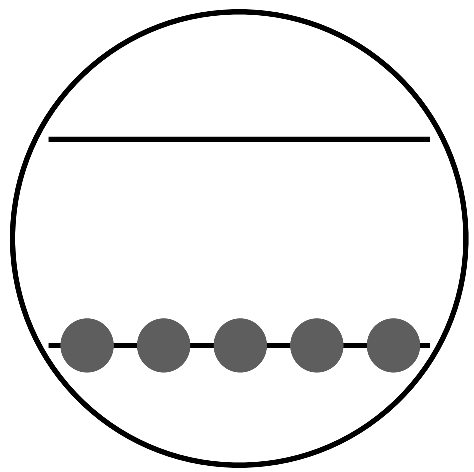
$$N_L = n$$

$$\nu = \frac{n+1}{2n+1}$$

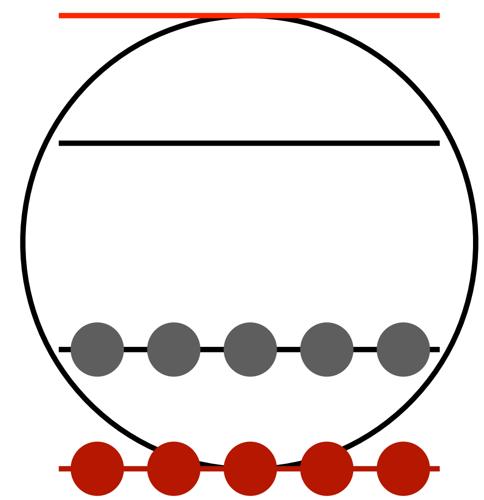
$$B_{cf} = -B/(2n+1)$$

$$N_L = n+1$$

CF Spectrum



$\nu = 1/3$
$CB_{cf} > 0$



$\nu = 2/3$
$CB_{cf} < 0$

- The apparent asymmetry is due to the necessity of filling an extraneous LL at the band bottom.
- Similar asymmetry (between valleys) arises in gapped graphene-like systems (e.g., BN, MoS2).

JS, arXiv:1704.07712 (2017)